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Fuzzy hyper BCK-implicative ideals of hyper BCK-algebras

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Abstract

The fuzzification of (weak, strong, reflexive) hyper BCK-implicative ideals in hyper BCK-algebras is considered. It is shown that every fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal is a fuzzy (weak, strong, reflexive) hyper BCK-ideal. We have discussed the properties of (fuzzy) weak hyper BCK-implicative ideals, (fuzzy) hyper BCK-implicative ideals, (fuzzy) strong hyper BCK-implicative ideals and (fuzzy) reflexive hyper BCK-implicative ideals and also their relations are given. Characterization of fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals is given. The hyper homomorphic pre-image of a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal is discussed. Lastly the properties of product of fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals are discussed.

Keywords: Hyper BCK-algebra; (fuzzy) hyper BCK-implicative ideal; (fuzzy) weak hyper BCK-implicative ideal; (fuzzy) strong hyper BCK-implicative ideal; (fuzzy) reflexive hyper BCK-implicative ideal.

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1 Introduction

In 1966, Imai and Iseki [7] introduced the notion of BCK-algebra. In the same year, Iseki introduced another notion called BCI-algebra. Liu et al. [13] discussed the concept of BCI-implicative ideals in BCI-algebras. Dudek [2] introduced the class of medial BCI-algebras. In 1983, Komori [11] introduced the notion of BCC-algebras as a new class of algebras. Then Dudek [3, 5] studied BCC-algebras and discussed the number of subalgebras of finite BCC-algebras. Dudek in [4] also gave the construction of BCC-algebras. After the introduction of the concept of fuzzy sets by Zadeh [16], various researchers discussed the idea of fuzzification of ideals in BCK/BCI/BCC-algebras. Khalid and Ahmad [10] considered the fuzzification of H-ideals in BCI-algebras. Mustafa [15] introduced the concept of fuzzy implicative ideals in BCK-algebras. Zhan and Jun [17] discussed generalized fuzzy ideals in BCI-algebras. Dudek and Jun [6] applied the idea of fuzzy sets to ideals in BCC-algebras. Marty [14], in 1934 introduced the hyper structure theory at the 8th Congrass of Scandinavian Mathematicians. Jun et al. [9] applied the hyper structures to BCK-algebras by introducing the concept of a hyper BCK-algebras, which is a generalization of BCK-algebras. In this paper, we introduce the concept of fuzzification of (weak, strong, reflexive) hyper BCK-implicative ideals in hyper BCK-algebras and discuss some of their properties.

2 Preliminaries

Let H be a non-empty set endowed with a hyper operation "$\circ$", that is, $\circ$ is a function from $H \times H$ to $P(H) - \emptyset$. For two subset A and B of H, denote by $A \circ B$ the set $\bigcup\{a \circ b \mid a \in A, b \in B\}$. We shall use $x \circ y$ instead of $x \circ \{y\}$, $\{x\} \circ y$ or $\{x\} \circ \{y\}$.

Definition 2.1. [9] By a hyper BCK-algebra we mean a non-empty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:
$(HK1)$ $(x \circ z) \circ (y \circ z) \ll x \circ y$
$(HK2)$ $(x \circ y) \circ z = (x \circ z) \circ y$
$(HK3)$ $x \circ H \ll \{x\}$
$(HK4)$ $x \ll y$ and $y \ll x$ imply $x = y$
for all $x, y, z \in H$, where $x \ll y$ is defined by $0 \in x \circ y$ and for every $A, B \subseteq H$, $A \ll B$ is defined by $\forall\, a \in A$, $\exists\, b \in B$ such that $a \ll b$. In such case we call "$\ll$" the hyper order in H.

Proposition 2.2. [9] *In any hyper BCK-algebra H, the following hold:*

(i) $x \circ 0 = \{x\}$	*(vi)* $A \circ \{0\} = \{0\}$ *implies* $A = \{0\}$
(ii) $x \circ y \ll x$	*(vii)* $0 \ll x$
(iii) $0 \circ A = \{0\}$	*(viii)* $0 \circ x = \{0\}$
(iv) $A \ll A$	*(ix)* $0 \circ 0 = \{0\}$
(v) $A \subseteq B$ *implies* $A \ll B$	*(x)* $y \ll z$ *implies* $x \circ z \ll x \circ y$

for all $x, y, z \in H$ and for all non-empty subsets A and B of H.

Let I be a non-empty subset of hyper BCK-algebra H and $0 \in I$. Then I is called a hyper BCK-subalgebra of H if $x \circ y \subseteq I$, for all $x, y \in I$, a weak hyper BCK-ideal of H if $x \circ y \subseteq I$ and $y \in I$ imply $x \in I$, for all $x, y \in H$, a hyper BCK-ideal of H if $x \circ y \ll I$ and $y \in I$ imply $x \in I$, for all $x, y \in H$, a strong hyper BCK-ideal of H if $x \circ y \cap I \neq \emptyset$ and $y \in I$ imply $x \in I$, for all $x, y \in H$. I is said to be reflexive if $x \circ x \subseteq I$ for all $x \in H$.

Lemma 2.3. [9] *Let H be a hyper BCK-algebra. Then*
- *any reflexive hyper BCK-ideal of H is a strong hyper BCK-ideal of H.*
- *any strong hyper BCK-ideal of H is a hyper BCK-ideal of H.*
- *any hyper BCK-ideal of H is a weak hyper BCK-ideal of H.*

Lemma 2.4. [8] *Let I be a reflexive hyper BCK-ideal of a hyper BCK-algebra H. Then $x \circ y \cap I \neq \emptyset$ implies $x \circ y \ll I$, $\forall x, y \in H$.*

Proposition 2.5. [8] *Let A be a subset of a hyper BCK-algebra H. If I is a hyper BCK-ideal of H such that $A \ll I$ then $A \subseteq I$.*

Definition 2.6. Let H be a hyper BCK-algebra. A non-empty subset $I \subseteq H$ containing 0 is called
- a weak hyper BCK-implicative ideal of H if
$$((x \circ y) \circ y) \circ z \subseteq I \text{ and } z \in I \text{ imply } x \circ (y \circ (y \circ x)) \subseteq I.$$
- a hyper BCK-implicative ideal of H if
$$((x \circ y) \circ y) \circ z \ll I \text{ and } z \in I \text{ imply } x \circ (y \circ (y \circ x)) \subseteq I.$$
- a strong hyper BCK-implicative ideal of H if
$$(((x \circ y) \circ y) \circ z) \cap I \neq \emptyset \text{ and } z \in I \text{ imply } x \circ (y \circ (y \circ x)) \subseteq I.$$

Theorem 2.7. *Every (weak, strong, reflexive) hyper BCK-implicative ideal of a hyper BCK-algebra H is a (weak, strong, reflexive) hyper BCK-ideal of H.*

Proof. Suppose that I is a hyper BCK-implicative ideal of H. Then for any $x, y, z \in H$
$$((x \circ y) \circ y) \circ z \ll I \text{ and } z \in I \text{ imply } x \circ (y \circ (y \circ x)) \subseteq I.$$

Putting $y = 0$ and $z = y$ we get

$$((x \circ 0) \circ 0) \circ y \ll I \text{ and } y \in I \text{ imply } x \circ (0 \circ (0 \circ x)) \subseteq I.$$

$$\Rightarrow (x \circ y) \ll I \text{ and } y \in I \Rightarrow x \in I.$$

Hence I is a hyper BCK-ideal of H. $\square$

The converse of theorem 2.7 is not true in general. It can be observed by the following example

Example 2.8. Let $H = \{0, 1, 2, 3\}$ be a hyper BCK-algebra defined by the following table:

$\circ$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,1\}$

Take $I = \{0, 1\}$. Then I is a hyper BCK-ideal of H but it is not a hyper BCK-implicative ideal of H because
$((2 \circ 3) \circ 3) \circ 1 = \{0\} \ll I$ and $1 \in I$ but $2 \circ (3 \circ (3 \circ 2)) = \{2\} \nsubseteq I$.
It can be observed from the above example that I is a weak hyper BCK-ideal of H but it not a weak hyper BCK-implicative ideal of H because
$((2 \circ 3) \circ 3) \circ 1 = \{0\} \subseteq I$ and $1 \in I$ but $2 \circ (3 \circ (3 \circ 2)) = \{2\} \nsubseteq I$.
Also I is a strong hyper BCK-ideal of H but it is not a strong hyper BCK-implicative ideal of H because
$((2 \circ 3) \circ 3) \circ 1 = \{0\} \cap I \neq \emptyset$ and $1 \in I$ but $2 \circ (3 \circ (3 \circ 2)) = \{2\} \nsubseteq I$.
Moreover it is clear that I is a reflexive hyper BCK-ideal of H but it is not a reflexive hyper BCK-implicative ideal of H.

Theorem 2.9. *Let H be a hyper BCK-algebra. Then*
(i) Every hyper BCK-implicative ideal of H is a weak hyper BCK-implicative ideal of H.
(ii) Every strong hyper BCK-implicative ideal of H is a hyper BCK-implicative ideal of H.
(iii) Every reflexive hyper BCK-implicative ideal of H is a strong hyper BCK-implicative ideal of H.

Proof. (i) Suppose that I is a hyper BCK-implicative ideal of H.
For any $x, y, z \in H$, let $((x \circ y) \circ y) \circ z \subseteq I$ and $z \in I$. Then $((x \circ y) \circ y) \circ z \subseteq I$ implies $((x \circ y) \circ y) \circ z \ll I$ (by Proposition 2.2(v)), which along with $z \in I$ implies $x \circ (y \circ (y \circ x)) \subseteq I$. Hence I is a weak hyper BCK-implicative ideal of H.

(ii) Suppose that I is a strong hyper BCK-implicative ideal of H. Let $((x \circ y) \circ y) \circ z \ll I$ and $z \in I$. Then for all $a \in ((x \circ y) \circ y) \circ z$, $\exists\, b \in I$ such that $a \ll b$. This implies $0 \in a \circ b$ and thus $(a \circ b) \cap I \neq \emptyset$. By Theorem 2.7, I is also a strong hyper BCK-ideal of H, therefore $(a \circ b) \cap I \neq \emptyset$ along with $b \in I$ implies $a \in I$, that is $((x \circ y) \circ y) \circ z \subseteq I$. Therefore $(((x \circ y) \circ y) \circ z) \cap I \neq \emptyset$, which along with $z \in I$ implies $x \circ (y \circ (y \circ x)) \subseteq I$. Hence I is a hyper BCK-implicative ideal of H.

(iii) Suppose that I is a reflexive hyper BCK-implicative ideal of H. For any $x, y, z \in H$, let $(((x \circ y) \circ y) \circ z) \cap I \neq \emptyset$ and $z \in I$. Being a reflexive hyper BCK-implicative ideal, I is also a reflexive hyper BCK-ideal of H (by Theorem 2.7), therefore by Lemma 2.4, $(((x \circ y) \circ y) \circ z) \cap I \neq \emptyset \Rightarrow ((x \circ y) \circ y) \circ z \ll I$, which along with $z \in I$ implies $x \circ (y \circ (y \circ x)) \subseteq I$. Hence I is a strong hyper BCK-implicative ideal of H. □

The converse of Theorem 2.9 may not be true. It can be observed by the following examples:

Example 2.10. Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra defined by the following table:

$\circ$	$\{0\}$	$\{1\}$	$\{2\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{0, 1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$

Take $I = \{0, 2\}$. Then I is a weak hyper BCK-implicative ideal of H but it is not a hyper BCK-implicative ideal of H because

$$((1 \circ 0) \circ 0) \circ 2 = \{0, 1\} \ll I \text{ and } 2 \in I \text{ but } 1 \circ (0 \circ (0 \circ 1)) = \{1\} \nsubseteq I.$$

Example 2.11. Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra defined by the following table:

$\circ$	$\{0\}$	$\{1\}$	$\{2\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1, 2\}$	$\{0,1,2\}$

Take $I = \{0,1\}$. Then I is a hyper BCK-implicative ideal of H but it is not a strong hyper BCK-implicative ideal of H because

$$(((2 \circ 0) \circ 0) \circ 1) \cap I = \{1,2\} \cap I \neq \emptyset \text{ and } 1 \in I \text{ but } 2 \circ (0 \circ (0 \circ 2)) = \{2\} \nsubseteq I.$$

Zadeh [16] defined fuzzy set μ in H as a function

$$\mu : H \to [0,1]$$

Definition 2.12. [8] A fuzzy set μ of a hyper BCK-algebra H is called

- a fuzzy weak hyper BCK-ideal of H if for all $x, y \in H$,
$$\mu(0) \geq \mu(x) \geq min\ \{inf_{a \in x \circ y}\ \mu(a),\ \mu(y)\}$$
- a fuzzy hyper BCK-ideal of H if $x \ll y$ implies $\mu(x) \geq \mu(y)$ and for all $x, y \in H$,
$$\mu(x) \geq min\ \{inf_{a \in x \circ y}\ \mu(a),\ \mu(y)\}$$
- a fuzzy strong hyper BCK-ideal of H if for all $x, y \in H$,
$$inf_{a \in x \circ x}\ \mu(a) \geq \mu(x) \geq min\ \{sup_{b \in x \circ y}\ \mu(b),\ \mu(y)\}$$
- a fuzzy reflexive hyper BCK-ideal of H if for all $x, y \in H$,
$$inf_{a \in x \circ x}\ \mu(a) \geq \mu(y) \text{ and } \mu(x) \geq min\ \{sup_{b \in x \circ y}\ \mu(b),\ \mu(y)\}$$

Theorem 2.13. [8] *Let H be a hyper BCK-aglebra. Then*

- *Every fuzzy hyper BCK-ideal of H is a fuzzy weak hyper BCK-ideal of H.*
- *Every fuzzy strong hyper BCK-ideal of H is a fuzzy hyper BCK-ideal of H.*
- *Every fuzzy reflexive hyper BCK-ideal of H is a fuzzy strong hyper BCK-ideal of H.*

3 Fuzzy hyper BCK-implicative ideals

Now we introduce the notions of fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals in hyper BCK-algebras and discuss some of their properties.

Definition 3.1. Let H be hyper BCK-algebra . A fuzzy set μ in H is called

- a fuzzy weak hyper BCK-implicative ideal of H if for all $x, y, z \in H$,

$\mu(0) \geq \mu(x)$ and for all $t \in x \circ (y \circ (y \circ x))$,

$$\mu(t) \geq min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\}$$

- a fuzzy hyper BCK-implicative ideal of H if for all $x, y, z \in H$,

$x \ll y$ implies $\mu(x) \geq \mu(y)$ and for all $t \in x \circ (y \circ (y \circ x))$,

$$\mu(t) \geq min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\}$$

- a fuzzy strong hyper BCK-implicative ideal of H if for all $x, y, z \in H$,

$inf_{a \in x \circ x}\ \mu(a) \geq \mu(x)$ and for all $t \in x \circ (y \circ (y \circ x))$,

$$\mu(t) \geq min\ \{sup_{b \in ((x \circ y) \circ y) \circ z}\ \mu(b),\ \mu(z)\}$$

- a fuzzy reflexive hyper BCK-implicative ideal of H if for all $x, y, z \in H$,

$inf_{a \in x \circ x}\ \mu(a) \geq \mu(y)$ and for all $t \in x \circ (y \circ (y \circ x))$,

$$\mu(t) \geq min\ \{sup_{b \in ((x \circ y) \circ y) \circ z}\ \mu(b),\ \mu(z)\}$$

Theorem 3.2. *Let H be a hyper BCK-algebra. Then every fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H is a fuzzy (weak, strong, reflexive) hyper BCK-ideal of H.*

Proof. Let μ be a fuzzy hyper BCK-implicative ideal of H. Then for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$ we have,

$$\mu(t) \geq min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\}$$

Putting $y = 0$ and $z = y$ we get,

$$\mu(x) \geq min\ \{inf_{a \in ((x \circ 0) \circ 0) \circ y}\ \mu(a),\ \mu(y)\}$$

which gives,

$$\mu(x) \geq min\ \{inf_{a \in x \circ y}\ \mu(a),\ \mu(y)\}$$

Thus μ is a fuzzy hyper BCK-ideal of H. □

The converse of Theorem 3.2 may not be true. It can be observed by considering the hyper BCK-algebra $H = \{0, 1, 2, 3\}$ defined by the table given in example (2.8). Define a fuzzy set μ in H by:

$$\mu(0) = \mu(1) = 1,\ \mu(2) = 0.5,\ \mu(3) = 0.3$$

Then μ is a fuzzy hyper BCK-ideal of H but it is not a fuzzy hyper BCK-implicative ideal of H because for $2 \in (2 \circ (3 \circ (3 \circ 2)))$

$$\mu(2) = 0.5 < 1 = min\ \{inf_{a \in ((2 \circ 3) \circ 3) \circ 0}\ \mu(a),\ \mu(0)\}$$

From above example it can be observed that μ is a fuzzy weak hyper BCK-ideal of H but it is not a fuzzy weak hyper BCK-implicative ideal of H.

Also μ is a fuzzy strong hyper BCK-ideal of H but it is not a fuzzy strong hyper BCK-implicative ideal of H because for $2 \in (2 \circ (3 \circ (3 \circ 2)))$

$$\mu(2) = 0.5 < 1 = min\ \{sup_{a \in ((2 \circ 3) \circ 3) \circ 0}\ \mu(a),\ \mu(0)\}$$

Moreover it is clear that μ is a fuzzy reflexive hyper BCK-ideal of H but it is not a fuzzy reflexive hyper BCK-implicative ideal of H.

Theorem 3.3. *Let H be a hyper BCK-algebra. Then*
(i) Every fuzzy hyper BCK-implicative ideal of H is a fuzzy weak hyper BCK-implicative ideal of H.
(ii) Every fuzzy Strong hyper BCK-implicative ideal of H is a fuzzy hyper BCK-implicative ideal of H.
(iii) Every fuzzy reflexive hyper BCK-implicative ideal of H is a fuzzy strong hyper BCK-implicative ideal of H.

Proof. (i) Let μ be a fuzzy hyper BCK-implicative ideal of H. Since every fuzzy hyper BCK-implicative ideal is a fuzzy hyper BCK-ideal (By Theorem 3.2) and every fuzzy hyper BCK-ideal is a fuzzy weak hyper BCK-ideal (By Theorem 2.13), therefore μ is a fuzzy weak hyper BCK-ideal of H. Hence μ satisfies $\mu(0) \geq \mu(x)$ for all $x \in H$. Also being a fuzzy hyper BCK-implicative ideal, for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$, μ satisfies:

$$\mu(t) \geq min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\}$$

Hence μ is a fuzzy weak hyper BCK-implicative ideal of H.

(ii) Suppose that μ is a fuzzy strong hyper BCK-implicative ideal of H. Since every fuzzy strong hyper BCK-implicative ideal is a fuzzy strong hyper BCK-ideal (by Theorem 3.2) and every fuzzy strong hyper BCK-ideal is a fuzzy hyper BCK-ideal (by Theorem 2.13), therefore

μ is a fuzzy hyper BCK-ideal of H. Hence for any $x, y \in H$, if $x \ll y$ then $\mu(x) \geq \mu(y)$.

Also being a fuzzy strong hyper BCK-implicative ideal, for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$, μ satisfies

$$\mu(t) \geq min\ \{sup_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\}$$

Since $sup_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a) \geq \mu(b)$, for all $b \in ((x \circ y) \circ y) \circ z$, therefore we get,

$$\mu(t) \geq min\ \{\mu(b),\ \mu(z)\}, \text{ for all } b \in ((x \circ y) \circ y) \circ z$$

Since $\mu(b) \geq inf_{c \in ((x \circ y) \circ y) \circ z}\ \mu(c)$ for all $b \in ((x \circ y) \circ y) \circ z$, therefore we have,
$\mu(t) \geq min\ \{\mu(b),\ \mu(z)\} \geq min\ \{inf_{c \in ((x \circ y) \circ y) \circ z}\ \mu(c),\ \mu(z)\}$, that is

$$\mu(t) \geq min\ \{inf_{c \in ((x \circ y) \circ y) \circ z}\ \mu(c),\ \mu(z)\}$$

Hence μ is a fuzzy hyper BCK-implicative ideal of H.

(iii) Let μ be a fuzzy reflexive hyper BCK-implicative ideal of H. Then μ satisfies

$$inf_{a \in x \circ x}\ \mu(a) \geq \mu(y), \text{ for all } x, y \in H$$

$$\Rightarrow inf_{a \in x \circ x}\ \mu(a) \geq \mu(x), \text{ for all } x \in H$$

Hence the first condition for μ to be a fuzzy strong hyper BCK-implicative ideal of H is satisfied. Also being a fuzzy reflexive hyper BCK-implicative ideal, for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$, μ satisfies

$$\mu(t) \geq min\ \{sup_{b \in ((x \circ y) \circ y) \circ z}\ \mu(b),\ \mu(z)\}$$

Hence μ is a fuzzy strong hyper BCK-implicative ideal of H. □

The converse of Theorem 3.3 may not be true. Consider the hyper BCK-algebra $H = \{0, 1, 2\}$ defined by the table given in example (2.10). Define a fuzzy set μ in H by:

$$\mu(0) = \mu(2) = 1,\ \mu(1) = 0$$

Then μ is a fuzzy weak hyper BCK-implicative ideal of H but it is not a fuzzy hyper BCK-implicative ideal of H because:

$$1 \ll 2 \text{ but } \mu(1) = 0 < 1 = \mu(2)$$

Example 3.4. Let $H = \{0, 1, 2\}$ be a hyper BCK-algebra defined by the following table:

$\circ$	$\{0\}$	$\{1\}$	$\{2\}$
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

Define a fuzzy set μ in H by:

$$\mu(0) = \mu(1) = 1,\ \mu(2) = 0$$

Then μ is a fuzzy hyper BCK-implicative ideal of H but it is not a fuzzy strong hyper BCK-implicative ideal of H because for $2 \in (2 \circ (2 \circ (2 \circ 2)))$

$$\mu(2) = 0 < 1 = min\ \{sup_{a \in (((2 \circ 2) \circ 2) \circ 0)}\ \mu(a),\ \mu(0)\}$$

Let μ be a fuzzy set in a hyper BCK-algebra H. Then the set defined by $\mu_t = \{x \in H : \mu(x) \geq t\}$, where $t \in [0, 1]$, is called a level subset of H.

Theorem 3.5. *Let μ be a fuzzy set in a hyper BCK-algebra H. Then μ is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H if and only if for all $t \in [0, 1]$, $\mu_t \neq \emptyset$ is a (weak, strong, reflexive) hyper BCK-implicative ideal of H.*

Proof. Suppose that μ is a fuzzy hyper BCK-implicative ideal of H. Since $\mu_t \neq \emptyset$, so for any $x \in \mu_t$, $\mu(x) \geq t$. Since every fuzzy hyper BCK-implicative ideal is also a fuzzy weak hyper BCK-implicative ideal (by Theorem 3.3(i)), so μ is also a fuzzy weak hyper BCK-implicative ideal of H. Thus $\mu(0) \geq \mu(x) \geq t$, for all $x \in H$, which implies $0 \in \mu_t$.

Let $((x \circ y) \circ y) \circ z \ll \mu_t$ and $z \in \mu_t$, for some $x, y, z \in H$. Then for all $a \in ((x \circ y) \circ y) \circ z$, $\exists\ b \in \mu_t$ such that $a \ll b$. So $\mu(a) \geq \mu(b) \geq t$, for all $a \in ((x \circ y) \circ y) \circ z$. Thus $inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a) \geq t$. Also $\mu(z) \geq t$, as $z \in \mu_t$. Therefore for all $v \in x \circ (y \circ (y \circ x))$, μ satisfies

$$\mu(v) \geq min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\} \geq min\ \{t, t\} = t$$
$$\Rightarrow v \in \mu_t, \text{ for all } v \in x \circ (y \circ (y \circ x))$$
$$\Rightarrow x \circ (y \circ (y \circ x)) \subseteq \mu_t$$

Hence μ_t is hyper BCK-implicative ideal of H.

Conversely suppose that $\mu_t \neq \emptyset$ is a hyper BCK-implicative ideal of H for all $t \in [0,1]$. Let $x \ll y$ for some $x, y \in H$ and put $\mu(y) = t$. Then $y \in \mu_t$. So $x \ll y \in \mu_t \Rightarrow x \ll \mu_t$. Being a hyper BCK-implicative ideal, μ_t is also a hyper BCK-ideal of H (by Theorem (2.7)) therefore by Proposition 2.5, $x \in \mu_t$. Hence $\mu(x) \geq t = \mu(y)$. That is $x \ll y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in H$.

Moreover for any $x, y, z \in H$, let $d = min\ \{inf_{c \in ((x \circ y) \circ y) \circ z}\ \mu(c),\ \mu(z)\}$. Then $\mu(z) \geq d \Rightarrow z \in \mu_d$ and for all $e \in ((x \circ y) \circ y) \circ z$, $\mu(e) \geq inf_{c \in ((x \circ y) \circ y) \circ z}\ \mu(c) \geq d$, which implies $e \in \mu_d$. Thus $((x \circ y) \circ y) \circ z \subseteq \mu_d$. By Proposition 2.2(v), $((x \circ y) \circ y) \circ z \subseteq \mu_d \Rightarrow ((x \circ y) \circ y) \circ z \ll \mu_d$, which along with $z \in \mu_d$ implies $x \circ (y \circ (y \circ x)) \subseteq \mu_d$. Hence for all $u \in x \circ (y \circ (y \circ x))$, we get

$$\mu(u) \geq d = min\ \{inf_{c \in ((x \circ y) \circ y) \circ z}\ \mu(c),\ \mu(z)\} \quad .$$

Thus μ is a fuzzy hyper BCK-implicative ideal of H. □

Theorem 3.6. *If μ is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H then the set $A = \{x \in H \mid \mu(x) = \mu(0)\}$ is a (weak, strong, reflexive) hyper BCK-implicative ideal of H.*

Proof. Suppose that μ is a fuzzy hyper BCK-implicative ideal of H. Clearly $0 \in A$. Let $((x \circ y) \circ y) \circ z \ll A$ and $z \in A$ for any $x, y, z \in H$. Then for all $a \in ((x \circ y) \circ y) \circ z$, $\exists\ b \in A$ such that $a \ll b$. Therefore $\mu(a) \geq \mu(b) = \mu(0)$. But being a fuzzy hyper BCK-implicative ideal, μ is also a fuzzy weak hyper BCK-implicative ideal of H (by Theorem 3.3(i)), so μ satisfies $\mu(0) \geq \mu(v)$, for all $v \in H$. This implies $\mu(0) \geq \mu(a)$, for all $a \in ((x \circ y) \circ y) \circ z$. Therefore $\mu(a) = \mu(0)$, for all $a \in ((x \circ y) \circ y) \circ z$, that is, $inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a) = \mu(0)$. Also $\mu(z) = \mu(0)$. Being a fuzzy hyper BCK-implicative ideal, for all $t \in x \circ (y \circ (y \circ x))$, μ satisfies

$$\mu(t) \geq min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\} = min\ \{\mu(0),\ \mu(0)\} = \mu(0)$$

Since $\mu(0) \geq \mu(v)$, for all $v \in H$, therefore $\mu(t) = \mu(0)$, for all $t \in x \circ (y \circ (y \circ x))$. Thus $x \circ (y \circ (y \circ x)) \subseteq A$.

Hence A is a hyper BCK-implicative ideal of H. □

The transfer principle for fuzzy sets described in [12] suggest the following theorem.

Theorem 3.7. *For any subset A of a hyper BCK-algebra H, let μ be a fuzzy set in H defined by:*

$$\mu(x) = \begin{cases} t & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

for all $x \in H$, where $t \in (0,1]$. Then A is a (weak, strong, reflexive) hyper BCK-implicative ideal of H if and only if μ is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H.

Proof. Suppose that A is a hyper BCK-implicative ideal of H. Let $x \ll y$ for some $x, y \in H$ and put $\mu(y) = t$. Then $y \in \mu_t$. So $x \ll y \in \mu_t \Rightarrow x \ll \mu_t$. Being a hyper BCK-implicative ideal, μ_t is also a hyper BCK-ideal of H (by Theorem (2.7)) therefore by Proposition 2.5, $x \in \mu_t$. Hence $\mu(x) \geq t = \mu(y)$. That is $x \ll y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in H$

Moreover for any $x, y, z \in H$,
If $((x \circ y) \circ y) \circ z \ll A$ and $z \in A$ then $x \circ (y \circ (y \circ x)) \subseteq A$. Since A is a hyper BCK-implicative ideal of H, so by Proposition 2.5, $((x \circ y) \circ y) \circ z \subseteq A$. Thus $\mu(a) = t$, for all $a \in ((x \circ y) \circ y) \circ z$ which implies $inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a) = t$. Also $\mu(z) = t$. Since $x \circ (y \circ (y \circ x)) \subseteq A$, for all $u \in x \circ (y \circ (y \circ x))$, we have

$$\mu(u) = t = min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\}$$

If $((x \circ y) \circ y) \circ z \not\ll A$ and $z \notin A$ then

$$min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\} = 0 \leq \mu(u), \text{ for all } u \in x \circ (y \circ (y \circ x))$$

If $((x \circ y) \circ y) \circ z \not\ll A$ and $z \in A$ (OR) If $((x \circ y) \circ y) \circ z \ll A$ and $z \notin A$

Then in both of these cases we have

$$min\ \{inf_{a \in ((x \circ y) \circ y) \circ z}\ \mu(a),\ \mu(z)\} = 0 \leq \mu(u), \text{ for all } u \in x \circ (y \circ (y \circ x))$$

Hence μ is a fuzzy hyper BCK-implicative ideal of H.

Conversely suppose that μ is a fuzzy hyper BCK-implicative ideal of H. Then by Theorem 3.5, for all $t \in (0,1]$, $\mu_t = A$ is a hyper BCK-implicative ideal of H. □

For a family $\{\mu_i \mid i \in I\}$ of fuzzy sets in a non-empty set X, define the join $\vee_{i\in I}\ \mu_i$ and meet $\wedge_{i\in I}\ \mu_i$ as follows:

$$(\vee_{i\in I}\ \mu_i)(x) = sup_{i\in I}\ \mu_i(x)$$
$$(\wedge_{i\in I}\ \mu_i)(x) = inf_{i\in I}\ \mu_i(x)$$

for all $x \in X$, where I is any indexing set.

Theorem 3.8. *The family of fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals of a hyper BCK-algebra H is a completely distributive lattice with respect to join and meet.*

Proof. Let $\{\mu_i \mid i \in I\}$ be a family of fuzzy hyper BCK-implicative ideals of H. Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering in $[0, 1]$, it is sufficient to show that $\vee_{i\in I}\ \mu_i$ and $\wedge_{i\in I}\ \mu_i$ are fuzzy hyper BCK-implicative ideals of H.

For any $x, y \in H$, if $x \ll y$ then

$$(\vee_{i\in I}\ \mu_i)(x) = sup_{i\in I}\ \mu_i(x) \geq sup_{i\in I}\ \mu_i(y) = (\vee_{i\in I}\ \mu_i)(y)$$
$$\Rightarrow (\vee_{i\in I}\ \mu_i)(x) \geq (\vee_{i\in I}\ \mu_i)(y)$$

Moreover, for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$, we have

$$(\vee_{i\in I}\ \mu_i)(t) = sup_{i\in I}\ \mu_i(t) \geq sup_{i\in I}\ [min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ \mu_i(a),\ \mu_i(z)\}]$$
$$= min\ \{sup_{i\in I}\ (inf_{a\in((x\circ y)\circ y)\circ z}\ \mu_i(a)),\ sup_{i\in I}\ (\mu_i(z))\}$$
$$= min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ (sup_{i\in I}\ \mu_i(a)),\ sup_{i\in I}\ (\mu_i(z))\}$$
$$= min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ ((\vee_{i\in I}\ \mu_i)(a)),\ (\vee_{i\in I}\ \mu_i)(z)\}$$
$$\Rightarrow (\vee_{i\in I}\ \mu_i)(t) \geq min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ ((\vee_{i\in I}\ \mu_i)(a)),\ (\vee_{i\in I}\ \mu_i)(z)\}$$

Hence $\vee_{i\in I}\ \mu_i$ is a fuzzy hyper BCK-implicative ideal of H.

Now we prove that $\wedge_{i\in I}\ \mu_i$ is a fuzzy hyper BCK-implicative ideal of H.
For any $x, y \in H$ we have, if $x \ll y$ then

$$(\wedge_{i\in I}\ \mu_i)(x) = inf_{i\in I}\ \mu_i(x) \geq inf_{i\in I}\ \mu_i(y) = (\wedge_{i\in I}\ \mu_i)(y)$$
$$\Rightarrow (\wedge_{i\in I}\ \mu_i)(x) \geq (\wedge_{i\in I}\ \mu_i)(y)$$

Moreover, for any $x, y, z \in H$ and for all $t \in x \circ (y \circ (y \circ x))$, we have

$$(\wedge_{i\in I}\ \mu_i)(t) = inf_{i\in I}\ \mu_i(t) \geq inf_{i\in I}\ [min\ \{inf_{b\in((x\circ y)\circ y)\circ z}\ \mu_i(b),\ \mu_i(z)\}]$$
$$= min\ \{inf_{i\in I}\ (inf_{b\in((x\circ y)\circ y)\circ z}\ \mu_i(b)),\ inf_{i\in I}\ (\mu_i(z))\}$$

$$= min\ \{inf_{b\in((x\circ y)\circ y)\circ z}\ (inf_{i\in I}\ \mu_i(b)),\ inf_{i\in I}\ (\mu_i(z))\}$$
$$= min\ \{inf_{b\in((x\circ y)\circ y)\circ z}\ ((\wedge_{i\in I}\ \mu_i)(b)),\ (\wedge_{i\in I}\ \mu_i)(z)\}$$
$$\Rightarrow (\wedge_{i\in I}\ \mu_i)(t) \geq min\ \{inf_{b\in((x\circ y)\circ y)\circ z}\ ((\wedge_{i\in I}\ \mu_i)(b)),\ (\wedge_{i\in I}\ \mu_i)(z)\}$$

Hence $\wedge_{i\in I}\ \mu_i$ is a fuzzy hyper BCK-implicative ideal of H.

Thus the family of fuzzy hyper BCK-implicative ideals of H is a completely distributive lattice with respect to join and meet. □

Let X and Y be hyper BCK-algebras. A mapping $f : X \to Y$ is called a hyper homomorphism if

(i) $f(0) = 0$

(ii) $f(x \circ y) = f(x) \circ f(y)$, for all $x, y \in X$.

Theorem 3.9. *Let $f : X \to Y$ be an onto hyper homomorphism from a hyper BCK-algebra X to a hyper BCK-algebra Y. If ν is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of Y then the hyper homomorphic pre-image μ of ν under f is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of X.*

Proof. Suppose that ν is a fuzzy hyper BCK-implicative ideal of Y. Since μ is a hyper homomorphic pre-image of ν under f then μ is defined by $\mu = \nu \circ f$ that is $\mu(x) = \nu(f(x))$ for all $x \in X$.

For any $x, y \in X$ and $f(x), f(x) \in Y$
If $x \ll y$ then $0 \in x \circ y$, which implies $f(0) \in f(x \circ y)$

$$\Rightarrow 0 \in f(x) \circ f(y) \Rightarrow f(x) \ll f(y)$$
$$\Rightarrow \nu(f(x)) \geq \nu(f(y)) \Rightarrow \mu(x) \geq \mu(y)$$

that is, $x \ll y \Rightarrow \mu(x) \geq \mu(y)$, for all $x, y \in X$

Now for all $t \in x \circ (y \circ (y \circ x))$, $f(t) \in f(x \circ (y \circ (y \circ x))) = f(x) \circ (f(y) \circ (f(y) \circ f(x)))$, where $x, y \in X$ and $f(x), f(y) \in Y$, we have

$$\mu(t) = \nu(f(t)) \geq min\ \{inf_{f(a)\in((f(x)\circ f(y))\circ f(y))\circ z'}\ \nu(f(a)),\ \nu(z')\}$$

where $z' \in Y$. Since $f : X \to Y$ is an onto hyper homomorphism, so for $z' \in Y$, $\exists\ z \in X$

such that $f(z) = z'$. Hence we get

$$\mu(t) \geq min\ \{inf_{f(a)\in((f(x)\circ f(y))\circ f(y))\circ f(z)=f(((x\circ y)\circ y)\circ z)}\ \nu(f(a)),\ \nu(f(z))\}$$

$$\Rightarrow \mu(t) \geq min\ \{inf_{a\in((x\circ y)\circ y)\circ z}\ \mu(a),\ \mu(z)\} \text{ for all } x, y, z \in X$$

Hence μ is a fuzzy hyper BCK-implicative ideal of X. □

4 Product of fuzzy hyper BCK-implicative ideals

Definition 4.1. [1] Let $(H_1, \circ_1, 0_1)$ and $(H_2, \circ_2, 0_2)$ are hyper BCK-algebras and $H = H_1 \times H_2$. We define a hyper operation " $\circ$ " on H by

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 \circ a_2,\ b_1 \circ b_2)$$

for all $(a_1, b_1), (a_2, b_2) \in H$, where for $A \subseteq H_1$ and $B \subseteq H_2$ by (A, B) we mean

$$(A, B) = \{(a, b) : a \in A, b \in B\}$$

and $0 = (0_1, 0_2)$ and a hyper order " $\ll$ " on H by

$$(a_1, b_1) \ll (a_2, b_2) \Leftrightarrow a_1 \ll a_2 \text{ and } b_1 \ll b_2$$

Thus $(H, \circ, 0)$ is a hyper BCK-algebra.

Let μ and ν be fuzzy sets in hyper BCK-algebras H_1 and H_2 respectively. Then $\mu \times \nu$, the product of μ and ν of $H = H_1 \times H_2$ is defined as

$$(\mu \times \nu)((x, y)) = min\ \{\mu(x), \nu(y)\}$$

From now on, let H_1 and H_2 are hyper BCK-algebras and let $H = H_1 \times H_2$.

Definition 4.2. Let μ be a fuzzy set in H. Then fuzzy sets μ_1 and μ_2 on H_1 and H_2 respectively, are defined as

$$\mu_1(x) = \mu((x, 0)), \quad \mu_2(y) = \mu((0, y))$$

Theorem 4.3. *Let μ be a fuzzy set in H. If μ is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H, then $\mu = \mu_1 \times \mu_2$, where μ_1 and μ_2 are fuzzy sets on H_1 and H_2 respectively.*

Proof. Suppose that μ is a fuzzy hyper BCK-implicative ideal of H.
Then for any $(x, u),\ (y, v),\ (z, w) \in H$, where $x, y, z \in H_1$ and $u, v, w \in H_2$ and for all

$(a,b) \in (x,u) \circ ((y,v) \circ ((y,v) \circ (x,u))) = (x \circ (y \circ (y \circ x)),\ u \circ (v \circ (v \circ u)))$, we have

$$\mu((a,b)) \geq min\ \{inf_{(c,d) \in (((x,u) \circ (y,v)) \circ (y,v)) \circ (z,w)}\ \mu((c,d)),\ \mu((z,w))\}$$

Putting $y = v = z = d = 0$ and $w = u$, we get

$$\begin{aligned} \mu((x,u)) &\geq min\ \{inf_{(c,0) \in (((x,u) \circ (0,0)) \circ (0,0)) \circ (0,u)}\ \mu((c,0)),\ \mu((0,u))\} \\ \Rightarrow \mu((x,u)) &\geq min\ \{inf_{(c,0) \in (x,\ u \circ u)}\ \mu((c,0)),\ \mu((0,u))\} \\ \Rightarrow \mu((x,u)) &\geq min\ \{\mu_1(x),\ \mu_2(u)\} \\ \Rightarrow \mu((x,u)) &\geq (\mu_1 \times \mu_2)((x,u)) \\ \Rightarrow \mu_1 \times \mu_2 &\subseteq \mu \qquad (1) \end{aligned}$$

Conversely, since $(x,0) \ll (x,u)$ and $(0,u) \ll (x,u)$

$$\Rightarrow \mu((x,0)) \geq \mu((x,u)) \text{ and } \mu((0,u)) \geq \mu((x,u))$$

Thus we have

$$\begin{aligned} (\mu_1 \times \mu_2)((x,u)) &= min\ \{\mu_1(x),\ \mu_2(u)\} = min\ \{\mu(x,0),\ \mu(0,u)\} \\ &\geq min\ \{\mu(x,u),\ \mu(x,u)\} = \mu(x,u) \\ &\Rightarrow (\mu_1 \times \mu_2)((x,u)) \geq \mu(x,u) \\ &\Rightarrow \mu \subseteq \mu_1 \times \mu_2 \qquad (2) \end{aligned}$$

Hence from (1) and (2) we have, $\mu_1 \times \mu_2 = \mu$ □

Theorem 4.4. *Let $\mu = \mu_1 \times \mu_2$ be a fuzzy set in H. Then $\mu = \mu_1 \times \mu_2$ is a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H if and only if μ_1 and μ_2 are fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals of H_1 and H_2 respectively.*

Proof. Let μ be a fuzzy hyper BCK-implicative ideal of H and let $x_1 \ll x_2$ for some $x_1, x_2 \in H_1$. Then $(x_1, 0) \ll (x_2, 0)$ which implies $\mu((x_1, 0)) = \mu_1(x_1) \geq \mu((x_2, 0)) = \mu_1(x_2)$, that is, $\mu_1(x_1) \geq \mu_1(x_2)$

Moreover for any $x_1, y_1, z_1 \in H_1$, let $t = min\ \{inf_{a \in ((x_1 \circ y_1) \circ y_1) \circ z_1}\ \mu_1(a),\ \mu_1(z_1)\}$

Then for all $b \in ((x_1 \circ y_1) \circ y_1) \circ z_1$, $\mu_1(b) \geq inf_{a \in ((x_1 \circ y_1) \circ y_1) \circ z_1}\ \mu_1(a) \geq t$ and $\mu_1(z_1) \geq t$

$\Rightarrow \mu((b,0)) \geq t$ and $\mu((z_1,0)) \geq t$, for all $(b,0) \in (((x_1,0) \circ (y_1,0)) \circ (y_1,0)) \circ (z_1,0)$

$\Rightarrow (b,0) \in \mu_t$ and $(z_1,0) \in \mu_t$, for all $(b,0) \in (((x_1,0) \circ (y_1,0)) \circ (y_1,0)) \circ (z_1,0)$

$$\Rightarrow (((x_1,0)\circ(y_1,0))\circ(y_1,0))\circ(z_1,0)\subseteq \mu_t \text{ and } (z_1,0)\in\mu_t$$

Since by Theorem 3.5, $\mu_t \neq \emptyset$ is a hyper BCK-implicative ideal of H and so is a weak hyper BCK-implicative ideal of H (by Theorem 2.9(i)). Thus

$$((((x_1,0)\circ(y_1,0))\circ(y_1,0))\circ(z_1,0))\subseteq \mu_t \text{ and } (z_1,0)\in\mu_t \text{ imply}$$
$$(x_1,0)\circ((y_1,0)\circ((y_1,0)\circ(x_1,0)))\subseteq\mu_t$$

Therefore $\mu((s,0)) \geq t$, for all $(s,0) \in (x_1,0)\circ((y_1,0)\circ((y_1,0)\circ(x_1,0))) = (x_1\circ(y_1\circ(y_1\circ x_1)),\ 0)$

$$\Rightarrow \mu_1(s)\geq t = min\ \{inf_{a\in((x_1\circ y_1)\circ y_1)\circ z_1}\ \mu_1(a),\ \mu_1(z_1)\},$$
$$\text{for all } s\in x_1\circ(y_1\circ(y_1\circ x_1))$$

Hence μ_1 is a fuzzy hyper BCK-implicative ideal of H_1.

Similarly we can prove that μ_2 is a fuzzy hyper BCK-implicative ideal of H_2.

Conversely suppose that μ_1 and μ_2 are fuzzy hyper BCK-implicative ideals of H_1 and H_2 respectively.

For any $(x,u),(y,v)\in H$, where $x,y\in H_1$ and $u,v\in H_2$, let $(x,u)\ll(y,v)$
Since $(x,u)\ll(y,v)\Leftrightarrow x\ll y$ and $u\ll v$

$$\Rightarrow \mu_1(x)\geq\mu_1(y) \text{ and } \mu_2(u)\geq\mu_2(v)$$
$$\Rightarrow min\ \{\mu_1(x),\ \mu_2(u)\}\geq min\ \{\mu_1(y),\ \mu_2(v)\}$$
$$\Rightarrow (\mu_1\times\mu_2)((x,u))\geq(\mu_1\times\mu_2)((y,v))$$
$$\Rightarrow \mu((x,u))\geq\mu((y,v))$$
$$\text{Thus } (x,u)\ll(y,v)\Rightarrow\mu((x,u))\geq\mu((y,v))$$

Moreover for any $(x,u),\ (y,v),\ (z,w)\in H$, where $x,y,z\in H_1$ and $u,v,w\in H_2$ and for all $(a,b)\in(x,u)\circ((y,v)\circ((y,v)\circ(x,u))) = (x\circ(y\circ(y\circ x)),\ u\circ(v\circ(v\circ u)))$, we have

$$\mu((a,b)) = (\mu_1\times\mu_2)((a,b)) = min\ \{\mu_1(a),\ \mu_2(b)\}$$

$$\geq min\ [min\ \{inf_{c\in((x\circ y)\circ y)\circ z}\ \mu_1(c),\ \mu_1(z)\},\ min\ \{inf_{d\in((u\circ v)\circ v)\circ w}\ \mu_2(d),\ \mu_2(w)\}]$$

$$= min\ [min\ \{inf_{c\in((x\circ y)\circ y)\circ z}\ \mu_1(c),\ inf_{d\in((u\circ v)\circ v)\circ w}\ \mu_2(d)\},\ min\ \{\ \mu_1(z),\ \mu_2(w)\}]$$

$$= min\ [inf_{c\in((x\circ y)\circ y)\circ z,\ d\in((u\circ v)\circ v)\circ w}\ \{min\ \{\mu_1(c),\ \mu_2(d)\}\},\ min\ \{\ \mu_1(z),\ \mu_2(w)\}]$$

$$= min\ \{inf_{(c,d)\in(((x\circ y)\circ y)\circ z,\ ((u\circ v)\circ v)\circ w)}\ (\mu_1\times\mu_2)((c,d)),\ (\mu_1\times\mu_2)((z,w))\}$$

$$= min\ \{inf_{(c,d)\in(((x\circ y)\circ y)\circ z,\ ((u\circ v)\circ v)\circ w)}\ \mu((c,d)),\ \mu((z,w))\}$$

$$\Rightarrow \mu((a,b))\geq min\ \{inf_{(c,d)\in(((x,u)\circ(y,v))\circ(y,v))\circ(z,w)}\ \mu((c,d)),\ \mu((z,w))\}$$

Hence μ is a fuzzy hyper BCK-implicative ideal of H. $\square$

5 CONCLUSION

Every (fuzzy) reflexive hyper BCK-implicative ideal of a hyper BCK-algebra H is a (fuzzy) strong hyper BCK-implicative ideal of H and every (fuzzy) strong hyper BCK-implicative ideal of H is a (fuzzy) hyper BCK-implicative ideal of H, each of which in turn is a (fuzzy) weak hyper BCK-implicative ideal of H. Moreover a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal of H is a fuzzy (weak, strong, reflexive) hyper BCK-ideal of H. The hyper homomorphic pre-image of a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal is also a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal in any onto hyper homomorphism of two hyper BCK-algebras. The product of two fuzzy (weak, strong, reflexive) hyper BCK-implicative ideals is also a fuzzy (weak, strong, reflexive) hyper BCK-implicative ideal.

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