BIFURCATION OF SOLUTIONS TO SEMILINEAR ELLIPTIC PROBLEMS ON S^2 WITH A SMALL HOLE

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Abstract. In this paper we prove the existence of non-positive or non-radial solutions to semilinear elliptic problems on $\mathbf{S}^2$ with a small hole. When the hole is sufficiently small, we prove that the multiplicity of eigenvalues to the corresponding linearized problem is 1 or 2. Thus, by using the result, we show those eigenvalues are bifurcation points, and the corresponding bifurcating solutions are not positive except for a bifurcating solution which is corresponding to the first eigenvalue. Moreover if the multiplicity of a eigenvalue is 2, then the corresponding bifurcating solution is not radially symmetric.

1 Introduction

We investigate the existence of non-trivial solutions to

$$\begin{cases} \Delta_{\mathbf{S}^N} u + \lambda u + |u|^{p-1}u = 0 & \text{in } B_{\theta_0}, \\ u = 0 & \text{on } \partial B_{\theta_0}, \end{cases} \tag{1.1}$$

where $\Delta_{\mathbf{S}^N}$ is the Laplace–Beltrami operator on the N-dimensional unit sphere $\mathbf{S}^N$ $(N \geq 2)$ and $1 < p < \infty$. Here B_{θ_0} is a geodesic ball on $\mathbf{S}^N$ with the geodesic radius θ_0. In addition the origin of B_{θ_0} is at the North Pole $(0, 0, ..., 0, 1)$ in the $(N+1)$-dimensional Euclidean space $\mathbf{R}^{N+1}$. In this paper we consider a classical solution to (1.1) (in fact we shall prove the existence of a solution $u \in C^{2,\alpha}(B_{\theta_0})$ to (1.1) with some $\alpha \in (0, 1)$).

When $(N-2)p < N+2$ and $\lambda < \lambda_1$ (λ_1 is the first eigenvalue of $\Delta_{\mathbf{S}^N}$ on B_{θ_0} with the homogeneous Dirichlet boundary condition), we can prove the existence of a solution to (1.1) by using the mountain pass lemma (e.g., see Theorem 6.2 in Chapter II of Struwe [17]). In fact, by the Rellich-Kondrachov theorem (e.g., see Theorem 2.34 in Aubin [2]), the compactness of $H^1_0(B_{\theta_0}) \hookrightarrow L^2(B_{\theta_0})$ is guaranteed, and we can apply the mountain pass lemma.

In the case of $N \geq 3$ and $p \geq (N+2)/(N-2)$, the compactness of $H^1_0(B_{\theta_0}) \hookrightarrow L^2(B_{\theta_0})$ is lost, and hence we need other approaches to prove the existence of solutions. The first result on this problem is by Bandle, Brillard and Flucher [5]. For $N \geq 3$, $p = (N+2)/(N-2)$ and $\lambda = 0$, they proved the following result: there exists some $\theta_c \in [0, \pi)$ such that (1.1) has a positive and radial solution if and only if $\theta_0 \in (\theta_c, \pi)$ (the *radial solution* means a solution depending only on the geodesic distance from the North Pole). Additionally if $N \geq 4$, then $\theta_c = 0$. On the other hand, if $N = 3$, then $\theta_c \neq 0$. Later Bandle and Peletier [8] investigated the case $N = 3$, $p = 5$ and $\lambda = 0$ in detail, and they showed that $\theta_c = \pi/2$. Moreover the author of this paper [14] also focused attention on the case $N = 3$, $p = 5$ and $\lambda = 0$. Namely, instead of the Dirichlet boundary condition, the author assumed the Robin boundary condition and clarified the structure of positive and radial solutions to (1.1) under

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$N = 3$ and $p = 5$. In addition, for $p > (N+2)/(N-2)$ with $N \geq 3$, a solution to (1.1) seems to exist, but it seems difficult to investigate the structure of solutions to (1.1).

The case $\lambda \neq 0$ is also studied. Bandle and Benguria [6] proved that, for $N = 3$, $p = 5$ and $\lambda > -3/4$, there exists a unique, positive and radial solution to (1.1). Here $\lambda = -3/4$ is the second eigenvalue of

$$\Delta_{\mathbf{S}^3} w + 4\lambda w = 0 \qquad \text{in } \mathbf{S}^3$$

which is the linearized equation of $\Delta_{\mathbf{S}^3} u + \lambda(|u|^4 u - u) = 0$ around $u \equiv 1$. On the other hand, Brezis and Peletier [9] showed the existence of positive and radial solutions for negatively large λ (they proved that many positive and radial solutions exist). Furthermore Bandle, Kabeya and Ninomiya [7] investigated the bifurcation structure for $\lambda < -3/4$ in detail. In studies above positive solutions are only treated, and no one investigates the structure of *non-positive* or *non-radial* solutions. Thus we focus our attention on those kinds of solutions.

Linearizing (1.1) around $u \equiv 0$, we obtain

$$\begin{cases} \Delta_{\mathbf{S}^N} w + \lambda w = 0 & \text{in } B_{\theta_0}, \\ w = 0 & \text{on } \partial B_{\theta_0}. \end{cases} \tag{1.2}$$

In this paper we only consider the case $N = 2$ and shall prove that eigenvalues of (1.2) are bifurcation points of (1.1). For the purpose we shall apply the *Lyapunov–Schmidt reduction method* and construct bifurcating solutions (for the Lyapunov–Schmidt reduction method, e.g., see Section 5.3 in Ambrosetti and Prodi [1]). To apply the method, we are required to know the multiplicity of eigenvalues for (1.2). Moreover, to see the positivity and the radial symmetry of the bifurcating solutions, we need to know profiles of eigenfunctions. Hence we shall investigate eigenvalues and eigenfunctions.

Let the polar coordinates

$$\begin{cases} y_1 = \sin\varphi \sin\theta \\ y_2 = \sin\varphi \cos\theta \\ y_3 = \cos\theta \end{cases}$$

with $(y_1, y_2, y_3) \in \mathbf{S}^2 \subset \mathbf{R}^3$, $\theta \in (0, \theta_0)$ and $\varphi \in [0, 2\pi]$. Then the operator $\Delta_{\mathbf{S}^2} + \lambda$ is expressed as

$$\Delta_{\mathbf{S}^2} w + \lambda w = \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\frac{\partial w}{\partial\theta} \sin\theta \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 w}{\partial\varphi^2} + \lambda w.$$

Additionally, for convenience below, we define $\nu \geq 0$ satisfying

$$\lambda := \nu(\nu+1). \tag{1.3}$$

Solutions to (1.2) are expressed by using the separation of variables. Namely let

$$w(\theta, \varphi) = P(x)\Phi(\varphi)$$

with

$$x = \cos\theta.$$

Here, by considering the regularity of solutions, $|P(1)| < \infty$, $\Phi(0) = \Phi(2\pi)$ and $\Phi'(0) = \Phi'(2\pi)$ must be satisfied. Functions $P(x)$ and $\Phi(\varphi)$ satisfy

$$(1-x^2)\frac{d^2P}{dx^2} - 2x\frac{dP}{dx} + \left\{ \nu(\nu+1) - \frac{m^2}{1-x^2} \right\} P = 0, \tag{1.4}$$

and

$$\frac{d^2\Phi}{d\varphi^2} + m^2\Phi = 0. \tag{1.5}$$

From the periodicity of $\Phi(\varphi)$, m is a non-negative integer, and any solutions to (1.5) are expressed as $\Phi(\varphi) = C_1 \cos m\varphi + C_2 \sin m\varphi$.

On the other hand, (1.4) is known as the *associated Legendre equation.* The equation (1.4) has two kinds of solutions $P = P_\nu^m(x)$ and $Q_\nu^m(x)$ such that $|P_\nu^m(1)| < \infty$ and $|Q_\nu^m(x)| \to \infty$ as $x \to 1$, respectively. In addition these solutions are linearly independent. From the condition $|P(1)| < \infty$, we have only to treat $P = P_\nu^m(x)$. Each $P_\nu^m(x)$ satisfies

$$P_\nu^m(1) = \begin{cases} 1 & (m = 0), \\ 0 & (m = 1, 2, ...), \end{cases}$$

and

$$\mathrm{sgn}(P_\nu^m(x)) = \ (-1)^m \quad \text{near } x = 1, \tag{1.6}$$

where $\mathrm{sgn}(a)$ denotes the signature of a (see (4.1) and (4.3) in Appendix).

We assume $\nu = j \geq 0$ which is an integer. Then, from (1.6) and

$$P_j^m(-x) = (-1)^{j+m} P_j^m(x)$$

(e.g., p.131 in Moriguchi, Udagawa and Hitotsumatsu [15]), it follows that

$$P_j^m(-1) = \begin{cases} (-1)^j & (m = 0), \\ 0 & (m = 1, 2, ...). \end{cases}$$

Hence $\lambda = j(j+1)$ and $C_1 P_j^m(\cos\theta)\cos m\varphi + C_2 P_j^m(\cos\theta)\sin m\varphi$ are an eigenvalue and an eigenfunction of $\Delta_{\mathbf{S}^2} w + \lambda w = 0$ on $\mathbf{S}^2$, respectively.

On the other hand, to solve the eigenvalue problem (1.2), we are required to find solutions to (1.4) satisfying the boundary condition

$$P(\cos\theta_0) = 0. \tag{1.7}$$

For any fixed $m = 0, 1, 2...$, there exist infinitely many $\lambda = \nu(\nu+1)$ satisfying (1.4), (1.7) and $P(1) = 1$ or 0 (e.g., see Chapter 10.6 in Ince [13], which is a general result on the Sturm–Liouville equations). But, in general, it seems difficult to investigate the multiplicity of eigenvalues for (1.2). In fact, for any m and n ($m \neq n$), it is not known whether $P_\nu^m(\cos\theta_0) = P_\nu^n(\cos\theta_0) = 0$ holds or not (partial results are obtained by Baginski [3], [4]). Hence, for any $\theta_0 \in (0, \pi)$, we does not see the multiplicity of eigenvalues for (1.2).

Thus, in this paper, we set

$$\theta_0 = \pi - \epsilon$$

and only consider a sufficiently small $\epsilon > 0$, that is, $B_{\pi-\epsilon}$ is $\mathbf{S}^2$ with a small hole. Then we can exactly prove the multiplicity of eigenvalues λ. Hereafter we use the notation

$$(a)_k := a(a+1)(a+2)...(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}, \tag{1.8}$$

where $k \geq 0$ is an integer and $\Gamma(x)$ is the *gamma function* ($(a)_0 = 1$ and $(1)_k = (k-1)!$). The result on eigenvalues of (1.2) is as follows:

Theorem 1.1 *Assume $N = 2$, $\theta_0 = \pi - \epsilon$, $1 < p < \infty$, and arbitrarily fix an integer $j \geq 0$. Then there exist $(j+1)$ positive values $\{\nu_{j,\epsilon}^m\}_{m=0}^j$ such that each $\lambda_{j,\epsilon}^m := \nu_{j,\epsilon}^m(\nu_{j,\epsilon}^m + 1)$ is an eigenvalue of (1.2). Moreover, as $\epsilon \to 0$, it holds that*

$$\lambda_{j,\epsilon}^m = \begin{cases} j(j+1) + \dfrac{2j+1}{2|\log\epsilon|} + o\left(\dfrac{1}{|\log\epsilon|}\right) & (m = 0), \\ j(j+1) + (2j+1)c_{j,m}\epsilon^{2m} + o(\epsilon^{2m}) & (1 \leq m \leq j), \end{cases} \tag{1.9}$$

where $c_{j,m} = (j+m)!/[4^m m!(m-1)!(j-m)!]$.

The asymptotic formula (1.9) implies that, for sufficiently small $\epsilon > 0$, it holds that

$$j(j+1) < \lambda_{j,\epsilon}^j < \lambda_{j,\epsilon}^{j-1} < \lambda_{j,\epsilon}^{j-2} < \ldots < \lambda_{j,\epsilon}^0,$$

and each $\lambda_{j,\epsilon}^m$ is located near $j(j+1)$. In addition the eigenspace corresponding to $\lambda_{j,\epsilon}^0$ is spanned by $P_{\nu_{j,\epsilon}^0}^0(\cos\theta)$. On the other hand, for $1 \leq m \leq j$, the eigenspace corresponding to $\lambda_{j,\epsilon}^m$ is spanned by $P_{\nu_{j,\epsilon}^m}^m(\cos\theta)\cos m\varphi$ and $P_{\nu_{j,\epsilon}^m}^m(\cos\theta)\sin m\varphi$. Therefore the multiplicity of eigenvalues of (1.2) is 1 or 2.

By Theorem 1.1, we can apply the Lyapunov–Schmidt reduction method, and the following result is proved:

Theorem 1.2 *Assume the same assumptions as in Theorem 1.1. Let $\mu := \lambda - \lambda_{j,\epsilon}^m$. Then the following statements hold:*

(i) *For $m = 0$, there exist a constant $\delta_j > 0$ and a non-trivial solution*

$$u_{j,\epsilon}^0(\,\cdot\,,\,\cdot\,;\mu + \lambda_{j,\epsilon}^0) := |\mu|^{\frac{1}{p-1}}\{t_{j,\epsilon}^0(\mu)v_* + l_{j,\epsilon}^0(\,\cdot\,,\,\cdot\,;\mu)\}$$

to (1.1) for $\mu \in (-\delta_j, 0)$, where $t_{j,\epsilon}^0(\mu) \in \mathbf{R}$ and $l_{j,\epsilon}^0(\,\cdot\,,\,\cdot\,;\mu) \in C^{2,\alpha}(B_{\pi-\epsilon})$ are of class C^1 with respect to μ. Here $t_{j,\epsilon}^0(0) = 1$, $l_{j,\epsilon}^0(\,\cdot\,,\,\cdot\,;0) \equiv 0$ and

$$v_* = M_0 P_{\nu_{j,\epsilon}^0}^0(\cos\theta),$$

with some constant $M_0 > 0$.

(ii) *Arbitrarily fix $t_* \in \mathbf{R}$ and $s_* \in \mathbf{R}$ such that the condition $t_*^2 + s_*^2 = 1$. For $1 \leq m \leq j$, there exist a constant $\delta_j > 0$ and a non-trivial solution*

$$u_{j,\epsilon}^m(\,\cdot\,,\,\cdot\,;\mu + \lambda_{j,\epsilon}^m) := |\mu|^{\frac{1}{p-1}}\{v(t_{j,\epsilon}^m(\mu), s_{j,\epsilon}^m(\mu)) + l_{j,\epsilon}^m(\,\cdot\,,\,\cdot\,;\mu)\}$$

to (1.1) for $\mu \in (-\delta_j, 0)$, where $t_{j,\epsilon}^m(\mu), s_{j,\epsilon}^m(\mu) \in \mathbf{R}$ and $l_{j,\epsilon}^m(\,\cdot\,,\,\cdot\,;\mu) \in C^{2,\alpha}(B_{\pi-\epsilon})$ are of class C^1 with respect to μ. Here $t_{j,\epsilon}^m(0) = t_*$, $s_{j,\epsilon}^m(0) = s_*$, $l_{j,\epsilon}^m(\,\cdot\,,\,\cdot\,;0) \equiv 0$ and

$$v(t,s) = M_m\left\{tP_{\nu_{j,\epsilon}^m}^m(\cos\theta)\cos m\varphi + sP_{\nu_{j,\epsilon}^m}^m(\cos\theta)\sin m\varphi\right\} \quad (t, s \in \mathbf{R})$$

with some constant $M_m > 0$.

Since $l_{j,\epsilon}^m(\,\cdot\,,\,\cdot\,;\mu) \to 0$ uniformly as $\mu \to 0$ and $P_{\nu_{0,\epsilon}^0}^0(\cos\theta)$ (the first eigenfunction of (1.2)) is positive on $(0, \pi - \epsilon)$, the bifurcating solution $u_{0,\epsilon}^0(\mu)$ is also positive. On the other hand, $u_{j,\epsilon}^m$ with $j \neq 0$ are not positive on $(0, \pi - \epsilon)$. Especially, for each $m \neq 0$, $u_{j,\epsilon}^m(\mu)$ is not radially symmetric since eigenfunctions $P_{\nu_{j,\epsilon}^m}^m(\cos\theta)\cos m\varphi$ and $P_{\nu_{j,\epsilon}^m}^m(\cos\theta)\sin m\varphi$ are not radially symmetric.

In Section 2 we investigate zeros of associated Legendre functions and show Theorem 1.1. In Section 3, by using this result, we prove Theorem 1.2.

2 Zeros of associated Legendre functions and Proof of Theorem 1.1 In this section we prove Theorem 1.1. In arguments below, we set $N = 2$ and $\theta_0 = \pi - \epsilon$. Since our aim in this paper is to investigate (1.1) with a sufficiently small $\epsilon > 0$, it suffices to investigate zeros of $P_\nu^m(x)$ near $x = -1$. In fact the following proposition holds:

Proposition 2.1 *Assume j and m be integers satisfying $0 \le m \le j$. If $j < \nu < j+1$, then $P_\nu^m(x)$ has $j - m + 1$ zeros in $(-1, 1)$. Moreover let $z_j^m(\nu)$ be the smallest zero of $P_\nu^m(x)$ in $(-1, 1)$. Then $z_j^m(\nu) \in C^1((j, j+1))$ and $z_j^m(\nu) \searrow -1$ as $\nu \searrow j$. Furthermore it holds that*

$$z_j^m(\nu) = \begin{cases} -1 + 2(1 + o(1)) \exp\left(-\dfrac{1}{\nu - j}\right) & (m = 0), \\ -1 + (d_{j,m} + o(1))(\nu - j)^{\frac{1}{m}} & (1 \le m \le j), \end{cases} \tag{2.1}$$

where $d_{j,m} = 2[m!(m-1)!(j-m)!/(j+m)!]^{1/m}$ and $o(1) \to 0$ as $\nu \searrow j$.

By the monotonicity of $z_j^m(\nu)$, we obtain a unique solution $\nu = \nu_{j,\epsilon}^m$ to $z_j^m(\nu) = \cos(\pi - \epsilon)$ for any sufficiently small $\epsilon > 0$. Moreover (1.9) follows from (2.1). Therefore, to show Theorem 1.1, we prove Proposition 2.1. Before the proof of Proposition 2.1, we state some preliminaries.

First we define

$$\psi(x) := \frac{d}{dx} \log \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

For $\Gamma(x)$ and $\psi(x)$, the following lemma holds:

Lemma 2.1 (Theorem 2.1.1 in [10]) *Functions $\Gamma(x)$ and $\psi(x)$ are analytic on $\mathbf{R}$ except for non-positive integers, that is, $x = 0, -1, -2, \ldots$. Moreover, for an integer $k \ge 0$, it holds that*

$$\lim_{x \to -k} (x + k)\Gamma(x) = \frac{(-1)^k}{k!} \quad \text{and} \quad \lim_{x \to -k} (x + k)\psi(x) = -1.$$

Second we state some properties of $P_\nu^m(x)$. For $P_\nu^m(x)$, the following two lemmas hold:

Lemma 2.2 *It holds that $P_0^0(x) \equiv 1$ and $P_m^m(x) \equiv 0$ in $(-1, 1)$ $(m \ge 1)$. Moreover if $m > \nu$, then $P_\nu^m(x) > 0$ in $(-1, 1)$ (ν is not an integer) or $P_\nu^m(x) \equiv 0$ in $(-1, 1)$ (ν is an integer).*

Lemma 2.3 *Assume that $\nu \ge 0$ and ν is not an integer. If m is an integer satisfying $0 \le m < \nu$, then it holds that, for any $x \in (0, 1)$,*

$$\begin{aligned} P_\nu^m(-1 + 2x) = {} & \frac{(-1)^m \Gamma(\nu + m + 1)}{\Gamma(\nu - m + 1) m!} x^{\frac{m}{2}} (1 - x)^{\frac{m}{2}} \\ & \times \left\{ \frac{(-1)^{m+1}}{\Gamma(-\nu)\Gamma(\nu + 1)} \left[P_\nu^m(1 - 2x) \log x \right. \right. \\ & + \sum_{k=0}^{\infty} \frac{(-\nu + m)_k (\nu + m + 1)_k}{(m + 1)_k k!} \{\psi(-\nu + m + k) \\ & \left. + \psi(\nu + m + k + 1) - \psi(k + 1) - \psi(m + k + 1)\} x^k \right] \\ & \left. + \frac{(m - 1)! m!}{\Gamma(-\nu + m)\Gamma(\nu + m + 1)} x^{-m} \sum_{k=0}^{m-1} \frac{(-\nu)_k (\nu + 1)_k}{(1 - m)_k k!} x^k \right\} \end{aligned}$$

with $(-1)! := 0$.

Concerning proofs of Lemmas 2.2 and 2.3, see Appendix. Lemma 2.3 implies that if ν is not an integer, then $P_\nu^m(x)$ tends to ∞ or $-\infty$ as $x \to -1$.

Next we prove a result on the number of zeros of $P_\nu^m(x)$. For integers $j \geq 0$ and $m = 0, 1, ..., j$, it is known that $P_j^m(x)$ has $j - m$ zeros in $(-1, 1)$ (e.g., see p.246 in Sansone [16]). On the other hand, when ν is not an integer, the following lemma holds:

Lemma 2.4 *Let j and m be integers satisfying $0 \leq m \leq j$. If $\nu \in (j, j+1)$ is fixed, then $P_\nu^m(x)$ has $j - m + 1$ zeros in $(-1, 1)$.*

Proof. Let $m = 0$. Then, from Lemma 2.3, we see that, as $x \to 0$, the leading term of $P_\nu^0(-1 + 2x)$ is $\log x$. Hence, from $P_\nu^0(1) = 1$ and

$$\Gamma(y)\Gamma(-y+1) = \frac{\pi}{\sin \pi y}, \tag{2.2}$$

(e.g., see p.1 in [15] or Theorem 2.2.3 in Beals and Wong in [10]) it holds that, as $x \to 0$,

$$\begin{aligned} P_\nu^0(-1+2x) &= -\frac{1}{\Gamma(-\nu)\Gamma(\nu+1)} P_\nu^0(1-2x) \log x + o(|\log x|) \\ &= -\frac{\sin \nu\pi}{\pi} |\log x| + o(|\log x|). \end{aligned} \tag{2.3}$$

On the other hand, let $1 \leq m \leq j$. Then, from Lemma 2.3, we see that, as $x \to 0$, the leading term of $P_\nu^m(-1+2x)$ is $x^{-m/2}$. Hence, from (2.2), it holds that, as $x \to 0$,

$$\begin{aligned} P_\nu^m(-1+2x) &= \frac{(-1)^m (m-1)!(1-x)^{\frac{m}{2}}}{\Gamma(\nu-m+1)\Gamma(-\nu+m)} x^{-\frac{m}{2}} + o(x^{-\frac{m}{2}}) \\ &= (-1)^{m+1} \frac{(m-1)! \sin(\nu-m)\pi}{\pi} x^{-\frac{m}{2}} + o(x^{-\frac{m}{2}}). \end{aligned} \tag{2.4}$$

Hence, from (2.3), (2.4) and

$$\operatorname{sgn}(\sin(\nu-m)\pi) = (-1)^{j-m} \qquad \text{for } \nu \in (j, j+1),$$

it holds that

$$\operatorname{sgn}(P_\nu^m(-1+2x)) = (-1)^{j+1} \qquad \text{near } x = 0. \tag{2.5}$$

Now the number of zeros for $P_\nu^m(x)$ $(x \in (-1,1))$ is denoted by $\sharp(P_\nu^m)$, and we recall $\sharp(P_j^m) = j - m$. We apply the Sturm–Liouville theorem (e.g., see pp.224–225 in [13]) for (1.4), and, from $\nu \in (j, j+1)$, we see that $\sharp(P_\nu^m) \geq \sharp(P_j^m) = j - m$. Similarly, we compare $P_\nu^m(x)$ and $P_{j+1}^m(x)$, and hence it is proved that $\sharp(P_\nu^m) \leq \sharp(P_{j+1}^m) = j - m + 1$. Thus, for $\nu \in (j, j+1)$, $\sharp(P_\nu^m) = j - m$ or $j - m + 1$.

From (1.6), $\operatorname{sgn}(P_\nu^m(x)) = (-1)^m$ near $x = 1$. If $\sharp(P_\nu^m) = j - m$, then $\operatorname{sgn}(P_\nu^m(x)) = (-1)^j$ near $x = -1$, which is inconsistent with (2.5). Therefore $\sharp(P_\nu^m) = j - m + 1$, and Lemma 2.4 is proved. ■

Now we show Proposition 2.1.

Proof of Proposition 2.1. Fix an integer $j \geq 0$ and an integer $m \in [0, j]$. The number of zeros is already known by Lemma 2.4. Thus it suffices to prove the asymptotic formula (2.1). The following arguments are divided into three steps.

Step 1. We prove that the smallest zero $z_j^m(\nu)$ is of class C^1 with respect to $\nu \in (j, j+1)$.

Let $x = z_j^m(\nu)$ be the smallest zero of $P_\nu^m(x)$ for $\nu \in (j, j+1)$. By Lemma 2.4, the number of zeros for $P_\nu^m(x)$ ($x \in (-1,1)$ and $\nu \in (j, j+1)$) is identically $j - m + 1 \geq 1$. Hence $z_j^m(\nu)$ always exists for $\nu \in (j, j+1)$.

Arbitrarily fix $\nu = \nu_0 \in (j, j+1)$ with $z_0 := z_j^m(\nu_0)$, and we prove that $z_j^m(\nu)$ is of class C^1 near ν_0. In fact, by differentiability with respect to a parameter, $P_\nu^m(x)$ is differentiable with respect to x and ν. If $(P_{\nu_0}^m)_x(z_0) = 0$ holds, then, by the uniqueness of a solution to (1.4), $P_{\nu_0}^m(z_0) = 0$ and $(P_{\nu_0}^m)_x(z_0) = 0$ imply $P_{\nu_0}^m(x) \equiv 0$, and it is a contradiction to $P_{\nu_0}^m(x) \not\equiv 0$. Thus $(P_{\nu_0}^m)_x(z_0) \neq 0$ holds.

Hence, from the implicit function theorem, there exists an implicit function $x = z(\nu)$ such that $P_\nu^m(z(\nu)) = 0$ and $z(\nu)$ is of class C^1 near $\nu = \nu_0$ (for the implicit function theorem, e.g., see Theorem 2.3 in Chapter 2 of [1]). By the uniqueness of the implicit function $z(\nu)$ near $\nu = \nu_0$, $z_j^m(\nu) \equiv z(\nu)$ holds near $\nu = \nu_0$. Therefore, from the arbitrariness of $\nu_0 \in (j, j+1)$, $z_j^m(\nu) \in C^1((j, j+1))$ holds. Step 1 is finished.

Step 2. We prove that $z_j^m(\nu) \searrow -1$ as $\nu \searrow j$.

First we prove that $z_j^m(\nu) \to -1$ as $\nu \searrow j$. We recall the *Gauss hypergeometric function*, and $P_\nu^m(x)$ is expressed by using the function (see (4.1) and (4.3) in Appendix). Since the series (4.1) uniformly converges in any closed interval of $(-1, 1]$, $P_\nu^m(x)$ is of class C^1 as ν in $x \in [-1+\delta, 1]$ with arbitrarily fixed $\delta \in (0, 2)$. Hence, when ν varies sufficiently near j, the number of zeros of $P_\nu^m(x)$ in $[-1+\delta, 1]$ is equal to the number of zeros of $P_j^m(x)$ in $[-1+\delta, 1]$.

Assume that $z_j^m(\nu) \not\to -1$ as $\nu \searrow j$. Then we can take $\delta > 0$ such that

$$z_j^m(\nu) \in [-1+\delta, 1] \qquad \text{as } \nu \searrow j \tag{2.6}$$

and all of zeros of $P_j^m(\nu)$ are contained in $[-1+\delta, 1]$. Since the number of zeros of $P_\nu^m(x)$ does not varies near $\nu = j$, there exists $j - m$ zeros of $P_\nu^m(x)$ in $[-1+\delta, 1)$ near $\nu = j$.

On the other hand, by Lemma 2.4, $P_\nu^m(x)$ has $j - m + 1$ zeros in $(-1, 1)$ if $\nu \in (j, j+1)$. Thus $z_j^m(\nu) \in (-1, -1+\delta)$ for $\nu > j$ near j, and it is inconsistent with (2.6). Therefore $z_j^m(\nu) \to -1$ as $\nu \searrow j$.

Next we show that $z_j^m(\nu) \searrow -1$ as $\nu \searrow j$. Arbitrarily fix a zero z_0 of $P_\nu^m(x)$. By the uniqueness of a solution to (1.4), $P_\nu^m(x) \equiv 0$ if $(P_\nu^m)_x(z_0) = 0$. Hence if $P_\nu^m(x) \not\equiv 0$, then it holds that

$$(P_\nu^m)_x(z_0) \neq 0.$$

On the other hand, a solution ν of $P_\nu^m(z_0) = 0$ is simple (e.g., see p.241 in [13]), that is,

$$(P_\nu^m)_\nu(z_0) \neq 0.$$

Thus, since

$$0 = \frac{d}{d\nu}\left[P_\nu^m(z_j^m(\nu))\right] = (P_\nu^m)_\nu(z_j^m(\nu)) + (P_\nu^m)_x(z_j^m(\nu)) \cdot (z_j^m)_\nu(\nu),$$

it holds that

$$(z_j^m)_\nu(\nu) = -\frac{(P_\nu^m)_\nu(z_j^m(\nu))}{(P_\nu^m)_x(z_j^m(\nu))} \neq 0 \qquad \text{for } \nu \in (j, j+1). \tag{2.7}$$

Since $z_j^m(\nu) \to -1$ as $\nu \searrow j$, there exists some $\nu = \nu_a > j$ such that $(z_j^m)_\nu(\nu_a) > 0$. If there exists some $\nu = \nu_b \in (j, \nu_a)$ such that $(z_j^m)_\nu(\nu_b) < 0$, then, by the continuity of

$(z_j^m)_\nu(\nu)$, there exists some $\nu_c \in [\nu_b, \nu_a]$ such that $(z_j^m)_\nu(\nu_c) = 0$, and it is a contradiction to (2.7). Therefore Step 2 is finished.

Step 3. The asymptotic formula (2.1) is proved.

We define

$$\zeta_j^m(\nu) := \frac{1 + z_j^m(\nu)}{2}.$$

Since $z_j^m(\nu) \searrow -1$ as $\nu \searrow j$, it follows that $\zeta_j^m(\nu) \searrow 0$ as $\nu \searrow j$.

We assume $m = 0$. Then, from Lemma 2.3, it holds that ,

$$-P_\nu^0(1 - 2\zeta_j^0(\nu)) \log \zeta_j^0(\nu) = \psi(-\nu) + \psi(\nu+1) - 2\psi(1) + R_j^0 \zeta_j^0(\nu) \qquad \text{as } \nu \searrow j, \tag{2.8}$$

where

$$R_j^0 = \sum_{k=1}^{\infty} \frac{(-\nu)_k (\nu+1)_k}{(k-1)^2 k!} \{\psi(-\nu+k) + \psi(\nu+k+1) - 2\psi(k+1)\} (\zeta_j^0(\nu))^{k-1}. \tag{2.9}$$

Now we prove

$$|(\nu - j) R_j^0| \le C_0 \qquad \text{as } \nu \searrow j, \tag{2.10}$$

where $C_0 > 0$ is independent of ν near j.

To prove (2.10), we first show that

$$|(\nu - j)\psi(-\nu + k)| \le k + C \qquad \text{for } \nu \in \left(j, j + \frac{1}{2}\right), \tag{2.11}$$

where $C > 0$ is independent of ν and $k \ge 1$. In fact, from Lemma 2.1, we obtain

$$\begin{aligned} \lim_{\nu \searrow j} (\nu - j)\psi(-\nu + k) &= -\lim_{s \nearrow -(j-k)} \{s + (j-k)\}\psi(s) \\ &= \begin{cases} 1 & (k \le j) \\ 0 & (k > j) \end{cases} \end{aligned} \tag{2.12}$$

with an integer k. For $k \le j$, (2.11) immediately follows from (2.12).

On the other hand, we assume $k > j$. From the following equality (e.g., see p.34 in [10])

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \tag{2.13}$$

it follows that

$$\psi(-\nu + k) = \psi(-\nu + j + 1) + \sum_{l=1}^{k-j-1} \frac{1}{-\nu + k - l} \tag{2.14}$$

with $\sum_{l=1}^{0} (-\nu + k - l)^{-1} = 0$. Here $(\nu - j)\psi(-\nu + j + 1) \to 0$ as $\nu \searrow j$. Moreover, from $l \in [1, k - j - 1]$, it holds that

$$\frac{\nu - j}{-\nu + k - l} \le \frac{\nu - j}{-\nu + j + 1} < 1 \qquad \text{for } \nu \in \left(j, j + \frac{1}{2}\right).$$

Hence we obtain (2.11) for $k > j$.

Similarly, from (2.13), it follows that

$$\psi(\nu+k+1)=\psi(\nu+1)+\sum_{l=0}^{k-1}\frac{1}{\nu+k-l}, \tag{2.15}$$

$$\psi(k+1)=\psi(1)+\sum_{l=0}^{k-1}\frac{1}{k-l}. \tag{2.16}$$

Since (2.15) and (2.16) do not have singularity as $\nu\searrow j$, it holds that, for $\nu\in(j,j+2-1)$,

$$|(\nu-j)\psi(\nu+k+1)|\le k+C, \tag{2.17}$$

$$|(\nu-j)\psi(k+1)|\le k+C, \tag{2.18}$$

where $C>0$ is independent of ν and $k\ge 1$.

Let

$$a_k(\nu):=\left|\frac{(-\nu)_k(\nu+1)_k}{(k-1)!k!}\{\psi(-\nu+k)+\psi(\nu+k+1)-2\psi(k+1)\}(\zeta_j^0(\nu))^{k-1}\right|. \tag{2.19}$$

Then, from (2.11), (2.17) and (2.18), it follows that

$$|(\nu-j)a_k(\nu)|^{\frac{1}{k}}\le\left|\frac{(-\nu)_k(\nu+1)_k}{(k-1)!k!}\right|^{\frac{1}{k}}\cdot 4^{\frac{1}{k}}(k+C)^{\frac{1}{k}}\zeta_j^0(\nu). \tag{2.20}$$

Now we prove that

$$\left|\frac{(-\nu)_k(\nu+1)_k}{(k-1)!k!}\right|^{\frac{1}{k}}\to 1 \qquad \text{as } k\to\infty \tag{2.21}$$

and the asymptotic formula (2.21) is uniform with respect to ν sufficiently near j. For the purpose we use the following result (e.g., see Corollary 2.1.4 in [10])

$$\lim_{k\to\infty}\frac{(a)_k}{(b)_k}k^{b-a}=\frac{\Gamma(b)}{\Gamma(a)}, \tag{2.22}$$

where $a,b\ne 0,-1,-2,\ldots$. From (2.22), we obtain

$$\lim_{k\to\infty}\frac{(-\nu)_k(\nu+1)_k}{(k-1)!k!}=\frac{\Gamma(1)}{\Gamma(-\nu)\Gamma(\nu+1)},$$

and hence (2.21) holds. Moreover, since $|\Gamma(-\nu)|\to\infty$ as $\nu\searrow j$, (2.21) is uniform with respect to ν near j.

Therefore, from (2.20), $(k+C)^{1/k}\to 1$ $(k\to\infty)$ and $\zeta_j^0(\nu)\searrow 0$ $(\nu\searrow j)$, we obtain

$$|(\nu-j)a_k(\nu)|\le c^k \qquad \text{for sufficiently near } \nu=j, \tag{2.23}$$

where a constant $c\in(0,1)$ is independent of ν (near j) and k. Hence, by (2.9), (2.19), (2.23) and the majorant test, it holds that

$$|R_j^0(\nu-j)|\le\sum_{k=1}^{\infty}|(\nu-j)a_k(\nu)|\le C_0$$

with some $C_0 > 0$ which is independent of ν near j.

Hence, from $P_\nu^0(1) = 1$, (2.8), (2.10) and (2.12), the singularity coming from $\log \zeta_j^0(\nu)$ must be canceled by $\psi(-\nu)$ as $\nu \searrow j$. Namely we obtain

$$(\nu - j) \log \zeta_j^0(\nu) = -1 + o(1) \qquad \text{as } \nu \searrow j,$$

where $o(1) \to 0$ as $\nu \searrow$. Therefore, from $z_j^m(\nu) = -1 + 2\zeta_j^m(\nu)$, (2.1) holds for $m = 0$.

Next we assume $1 \leq m \leq j$. The proof is similar to that of the case $m = 0$. Namely, from Lemma 2.3 and $P_\nu^m(-1 + 2\zeta_j^m(\nu)) \equiv 0$, it follows that, as $\nu \searrow j$,

$$\frac{m!(m-1)!}{\Gamma(-\nu+m)\Gamma(\nu+m+1)}(1 + L_j^m \zeta_j^m(\nu))(\zeta_j^m(\nu))^{-m} + \frac{(-1)^{m+1} P_\nu^m(1 - 2\zeta_j^m(\nu))}{\Gamma(-\nu)\Gamma(\nu+1)} \log \zeta_j^m(\nu)$$
$$= \frac{(-1)^m}{\Gamma(-\nu)\Gamma(\nu+1)m!}\{\psi(-\nu+m) + \psi(\nu+m+1) - \psi(1) - \psi(m+1)\} + R_j^m \zeta_j^m(\nu),$$

where

$$L_j^m = \sum_{k=1}^{m-1} \frac{(-\nu)_k(\nu+1)_k}{(1-m)_k k!}(\zeta_j^m(\nu))^{k-1},$$

and

$$R_j^m = \sum_{k=1}^{\infty} \frac{(-\nu+m)_k(\nu+m+1)_k}{(m+1)_k k!}$$
$$\times \{\psi(-\nu+m+k) + \psi(\nu+m+k+1) - \psi(k+1) - \psi(m+k+1)\}(\zeta_j^m(\nu))^{k-1}.$$

By similar arguments to the case $m = 0$, we obtain

$$|L_j^m| \leq C, \tag{2.24}$$

$$|(\nu - j)R_j^m| \leq C, \tag{2.25}$$

where $C > 0$ is independent of ν near j (we apply (2.12), (2.13) and (2.22)).

Now we remark that, as $\nu \searrow j$, it holds that $(\zeta_j^m(\nu))^m \log \zeta_j^m(\nu) = o(1)$. Hence, from (2.12), (2.24), (2.25) and $P_\nu^m(1) = 0$, the singularity coming from $(\zeta_j^m(\nu))^{-m}$ must be canceled by $\psi(-\nu+m)$. Namely we obtain, as $\nu \searrow j$,

$$(\nu-j)(1+o(1))(\zeta_j^m(\nu))^{-m} = \frac{(-1)^m\Gamma(-\nu+m)\Gamma(\nu+m+1)}{m!(m-1)!\Gamma(-\nu)\Gamma(\nu+1)}(\nu-j)\psi(-\nu+m) + o(1), \tag{2.26}$$

where $o(1) \to 0$ as $\nu \searrow j$. From Lemma 2.1, it holds that

$$\lim_{\nu \searrow j} \frac{\Gamma(-\nu+m)}{\Gamma(-\nu)} = \frac{(-1)^{j-m+1}}{(j-m)!} \times \frac{j!}{(-1)^{j+1}} = \frac{(-1)^{-m} j!}{(j-m)!}. \tag{2.27}$$

Thus, from (2.12), (2.26) and (2.27) and $\Gamma(j+m+1) = (j+m)!$, it holds that

$$\lim_{\nu \searrow j} (\nu - j)(\zeta_j^m(\nu))^{-m} = \frac{(j+m)!}{m!(m-1)!(j-m)!}$$

Therefore, from $z_j^m(\nu) = -1 + 2\zeta_j^m(\nu)$, we obtain (2.1) with $1 \leq m \leq j$. Now all of steps are finished, and Proposition 2.1 is completely shown. ■

Theorem 1.1 follows from Proposition 2.1.

Proof of Theorem 1.1. We prove the existence of ν satisfying

$$P_\nu^m(\cos(\pi-\epsilon))=0$$

and investigate its behavior as $\epsilon\to 0$. Here we take $z_j^m(\nu)$ which is the same definition as in Proposition 2.1.

Since $z_j^m(\nu)\searrow -1$ as $\nu\searrow j$ (see Proposition 2.1), the equation

$$z_j^m(\nu)=\cos(\pi-\epsilon) \tag{2.28}$$

has a unique solution $\nu=\nu_{j,\epsilon}^m$ for a sufficiently small $\epsilon>0$. Moreover $\lambda_{j,\epsilon}^m:=\nu_{j,\epsilon}^m(\nu_{j,\epsilon}^m+1)$ is an eigenvalue of (1.2).

Next if $m\ge\nu$, then, from Lemma 2.2, $P_\nu^m(\cos\theta)$ does not have a zero in $(0,\pi-\epsilon)$ or $P_\nu^m(\cos\theta)\equiv 0$. Therefore, for $\nu\in(j,j+1)$, there exist exactly $(j+1)$ eigenvalues.

Finally we show (1.9). From (2.28), it follows that

$$z_j^m(\nu_{j,\epsilon}^m)=-1+\frac{1}{2}\epsilon^2+O(\epsilon^4)\quad\text{as }\epsilon\to 0.$$

Thus, from Proposition 2.1, it holds that

$$\frac{1}{2}\epsilon^2+O(\epsilon^4)=\begin{cases}2(1+o(1))\exp\left(-\dfrac{1}{\nu_{j,\epsilon}^m-j}\right) & (m=0),\\ (d_{j,m}+o(1))(\nu_{j,\epsilon}^m-j)^{\frac{1}{m}} & (1\le m\le j),\end{cases}$$

where $o(1)\to 0$ as $\epsilon\to 0$. Hence, as $\epsilon\to 0$, we obtain

$$\nu_{j,\epsilon}^m=\begin{cases}j+\dfrac{1}{2|\log\epsilon|}+o\left(\dfrac{1}{|\log\epsilon|}\right) & (m=0),\\ j+c_{j,m}\epsilon^{2m}+o(\epsilon^{2m}) & (1\le m\le j),\end{cases}$$

where $c_{j,m}=(2d_{j,m})^{-m}=(j+m)!/[4^m m!(m-1)!(j-m)!]$. Recall $\lambda=\nu(\nu+1)$ (see (1.3)), and (1.9) is shown. Now Theorem 1.1 is proved. ■

3 Proof of Theorem 1.2 In Section 2, we investigated eigenvalues $\{\lambda_{j,\epsilon}^m\}$ of (1.2). In this section we prove that $\{\lambda_{j,\epsilon}^m\}$ are bifurcation points of (1.1).

We use the Lyapunov–Schmidt reduction method, that is, we consider our problem by dividing $C^{2,\alpha}(B_{\pi-\epsilon})$ into the eigenspace corresponding to $\lambda_{j,\epsilon}^m$ and its orthogonal complement.

We introduce the Banach spaces

$$\mathcal{X}:=\{u\in C^{2,\alpha}(B_{\pi-\epsilon})\mid u=0\text{ on }\partial B_{\pi-\epsilon}\},$$
$$\mathcal{Y}:=C^\alpha(B_{\pi-\epsilon}).$$

Then the following function

$$f(\lambda,u):=\Delta_{\mathbf{S}^2}u+\lambda u+|u|^{p-1}u$$

satisfies $f\in C^1(\mathbf{R}\times\mathcal{X};\mathcal{Y})$ for $1<p<\infty$. We see that $u\equiv 0$ is a solution to $f(\lambda,u)=0$. In arguments below, we show that there exists a non-trivial solution $u\in\mathcal{X}$ to $f(\lambda,u)=0$ near $\lambda=\lambda_{j,\epsilon}^m$ and $u\equiv 0$.

For convenience let $\lambda_* := \lambda^m_{j,\epsilon}$, and we define

$$\begin{aligned} L &:= f_u(\lambda_*, 0) = \Delta_{\mathbf{S}^2} + \lambda_*, \\ \mathcal{V} &:= \mathrm{Ker}(L), \\ \mathcal{R} &:= \mathrm{Ran}(L), \end{aligned}$$

where Ker(L) and Ran(L) denote the kernel of L and the range of L, respectively. From Theorem 1.1, the dimension of $\mathcal{V}$ is 1 or 2. Moreover $L : \mathcal{X} \to \mathcal{Y}$ is continuous, and $\mathcal{V}$ and $\mathcal{R}$ are closed subspaces of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Moreover we define the following inner product

$$\langle u, v\rangle := \int_0^{\pi-\epsilon}\int_0^{2\pi} uv \sin\theta d\theta d\varphi \qquad (u, v \in \mathcal{Y}).$$

For this inner product, the orthogonal complement $\mathcal{W}$ of $\mathcal{V}$ is defined, and it holds that

$$\mathcal{X} = \mathcal{V} \oplus \mathcal{W}.$$

We remark that $L : \mathcal{W} \to \mathcal{R}$ is one-to-one and onto.

On the other hand, as for $\mathcal{R}$, the following lemma holds:

Lemma 3.1 *Let $u \in \mathcal{Y}$. Then u belongs to $\mathcal{R} \subset \mathcal{Y}$ if and only if*

$$\langle u, v\rangle = 0 \qquad \text{for any } v \in \mathcal{V}.$$

Lemma 3.1 is proved in Appendix. Lemma 3.1 implies that $\mathcal{Y}$ is expressed as

$$\mathcal{Y} = \mathcal{R} \oplus \mathcal{V}. \tag{3.1}$$

By (3.1), we define orthogonal projections

$$\begin{aligned} Q &: \mathcal{Y} \to \mathcal{R}, \\ P &: \mathcal{Y} \to \mathcal{V}. \end{aligned}$$

From preliminaries above we show the existence of non-trivial solutions to $f(\lambda, u) = 0$. Let $\mu := \lambda - \lambda_*$, and we seek a solution whose form is

$$u = |\mu|^{\frac{1}{p-1}}(v + w) \qquad \text{with } v \in \mathcal{V} \text{ and } w \in \mathcal{W}. \tag{3.2}$$

Namely we solve

$$f(\mu + \lambda_*, |\mu|^{\frac{1}{p-1}}(v + w)) = 0,$$

We define

$$\begin{aligned} h(\mu, v, w) &:= \frac{1}{|\mu|^{\frac{1}{p-1}}} f(\mu + \lambda_*, |\mu|^{\frac{1}{p-1}}(v + w)) \\ &= Lw + \mu(v + w) + |\mu||v + w|^{p-1}(v + w) \end{aligned}$$

For $\mu \neq 0$, $h(\mu, v, w) = 0$ is equivalent to $f(\mu + \lambda_*, \mu^{1/(p-1)}(v + w)) = 0$.

In fact we can find a non-trivial solution to $h(\mu, v, w) = 0$ for $\mu < 0$, where

$$h(\mu, v, w) = Lw + \mu(v + w) - \mu|v + w|^{p-1}(v + w).$$

On the other hand, by arguments below, we cannot find a non-trivial solution to $f(\mu + \lambda_*, u) = 0$ with $\mu > 0$ (in fact, for $\mu > 0$, we only obtain a trivial solution by the implicit function theorem in proofs of Proposition 3.2 or 3.3 below). Therefore we only consider the case $\mu \le 0$.

By orthogonal projections Q and P, $h(\mu, v, w) = 0$ is equivalent to the following simultaneous equations

$$Qh(\mu, v, w) = 0, \tag{3.3}$$

$$Ph(\mu, v, w) = 0. \tag{3.4}$$

Arguments below are divided into the following two steps:

(s1) we show that there exists a function $l(\mu, v)$ such that $Qh(\mu, v, l(\mu, v)) = 0$,

(s2) we solve $Ph(\mu, v, l(\mu, v)) = 0$ and find a non-trivial solution $v = v(\mu)$.

Step (s1). Let

$$J(\mu, v, w) := Qh(\mu, v, w) = Lw + \mu w - \mu Q|v + w|^{p-1}(v + w). \tag{3.5}$$

For $J(\mu, v, w)$, the following proposition holds:

Proposition 3.1 *Assume $v_* \in \mathcal{V}$. Then there exists $l \in C^1((-\delta_0, 0] \times \mathcal{V}_*; \mathcal{W}_0)$ such that $J(\mu, v, w) = 0$ implies $w = l(\mu, v)$, where $\delta_0 > 0$ is some constant, $\mathcal{V}_* \subset \mathcal{V}$ and $\mathcal{W}_0 \subset \mathcal{W}$ are neighborhoods of $v_* \in \mathcal{V}$ and $0 \in \mathcal{W}$, respectively. Namely $l(0, v_*) = 0$ holds.*

Proof. From (3.5), we obtain

$$J(0, v_*, 0) = 0,$$

and

$$J_w(0, v_*, 0)[\xi] = L\xi \qquad \text{for any } \xi \in \mathcal{W}.$$

Therefore, by the implicit function theorem, Proposition 3.1 holds. ■

Here we remark that v_* is arbitrarily fixed, and $l(\mu, v)$ depends on $v_* \in \mathcal{V}$. In addition, for $l(\mu, v)$, the following Lemma 3.2 holds:

Lemma 3.2 *For $l(\mu, v)$ defined in Proposition* 3.1*, it holds that*

$$l_v(0, v_*) = 0.$$

Proof. From $J(\mu, v, l(\mu, v)) = 0$ and (3.5), it holds that

$$Ll(\mu, v) + \mu l(\mu, v) - \mu Q|v + l(\mu, v)|^{p-1}(v + l(\mu, v)) = 0. \tag{3.6}$$

Differentiating (3.6) by v, we obtain

$$Ll_v(\mu, v)[\xi] + \mu l_v(\mu, v)[\xi] - \mu p Q|v + l(\mu, v)|^{p-1}(\xi + l_v(\mu, v)[\xi]) = 0 \qquad \text{for any } \xi \in \mathcal{V}.$$

Hence, by substituting $\mu = 0$ and $v = v_*$, we obtain

$$Ll_v(0, v_*)[\xi] = 0 \qquad \text{for any } \xi \in \mathcal{V}. \tag{3.7}$$

From (3.7) and Proposition 3.1, it follows that $l_v(0, v_*)[\xi] \in \mathcal{V} \cap \mathcal{W} = \{0\}$ for any $\xi \in \mathcal{V}$. Therefore, from the arbitrariness of $\xi \in \mathcal{V}$, Lemma 3.2 is shown. ■

Lemma 3.2 is required in arguments below.

Step (s2). Next we consider (3.4). Hereafter let $w = l(\mu, v)$ which is defined in Proposition 3.1. Let

$$K(\mu, v) := Ph(\mu, v, l(\mu, v)) = \mu v - \mu P|v + l(\mu, v)|^{p-1}(v + l(\mu, v)). \tag{3.8}$$

We show the existence of $v = v(\mu) \not\equiv 0$ satisfying $K(\mu, v) = 0$ by dividing a proof into two cases, that is, the dimension of $\mathcal{V}$ is 1 or 2.

First we consider the case that the dimension of $\mathcal{V}$ is 1. Then, from Theorem 1.1, it follows that

$$\mathcal{V} = \{tP_\nu^0(\cos\theta) \mid t \in \mathbf{R}\}, \tag{3.9}$$

where $\nu_{j,\epsilon}^m$ is abbreviated to ν. Then the following proposition holds.

Proposition 3.2 *Assume* $1 < p < \infty$ *and* $\nu = \nu_{j,\epsilon}^0$. *Let* $M_0 > 0$ *satisfy*

$$\int_0^{\pi-\epsilon} \left\{|P_\nu^0(\cos\theta)|^2 - M_0^{p-1}|P_\nu^0(\cos\theta)|^{p+1}\right\} \sin\theta d\theta = 0. \tag{3.10}$$

Then there exist a constant $\delta > 0$ *and a* C^1*-function* $t(\mu)$ *such that*

$$K(\mu, t(\mu)M_0P_\nu^m(\cos\theta)) = 0 \qquad \text{for } \mu \leq 0,$$

where $t(0) = 1$ *and* $|t(\mu) - 1| + |\mu| < \delta$.

Proof. Let

$$v_* := M_0P_\nu^0(\cos\theta).$$

Then, by Proposition 3.1, there exists an implicit function $w = l(\mu, v)$ satisfying (3.3) and $l(0, v_*) = 0$. From (3.8) and (3.9), it follows that

$$\begin{aligned} K(\mu, tv_*) &= \mu a_\nu^0 N(\mu, t)P_\nu^0(\cos\theta) \\ N(\mu, t) &:= \langle P_\nu^0(\cos\theta), h(\mu, tv_*, l(\mu, tv_*))\rangle \\ &= \langle P_\nu^0(\cos\theta), tv_* - |tv_* + l(\mu, tv_*)|^{p-1}(tv_* + l(\mu, tv_*)[v_*])\rangle. \end{aligned}$$

Here $(a_\nu^0)^{-1} = 2\pi \int_0^{\pi-\epsilon} |P_\nu^0(\cos\theta)|^2 \sin\theta d\theta$. We remark $N(\mu, t) = 0$ $(\mu \neq 0)$ is equivalent to $K(\mu, tv_*) = 0$. We show that there exists some $t(\mu)$ satisfying $N(\mu, t(\mu)) = 0$ by the implicit function theorem. For the purpose it suffices to prove that $N(0, 1) = 0$ and $N_t(0, 1) \neq 0$.

From (3.10) and $l(0, v_*) = 0$, we obtain

$$\begin{aligned} N(0, 1) &= 2\pi M_0 \int_0^{\pi-\epsilon} \left\{|P_\nu^0(\cos\theta)|^2 - M_0^{p-1}|P_\nu^0(\cos\theta)|^{p+1}\right\} \sin\theta d\theta \\ &= 0. \end{aligned}$$

Moreover, from direct calculation, it follows that

$$N_t(\mu, t) = \langle P_\nu^0(\cos\theta), v_* - p|tv_* + l(\mu, tv_*)|^{p-1}(v_* + l_v(\mu, tv_*)[v_*])\rangle.$$

Hence, from $l_v(0, v_*)$ (see Lemma 3.2) and (3.10), we obtain

$$\begin{aligned} N_t(0, 1) &= 2\pi M_0 \int_0^{\pi-\epsilon} \left\{|P_\nu^0(\cos\theta)|^2 - pM_0^{p-1}|P_\nu^0(\cos\theta)|^{p+1}\right\} \sin\theta d\theta \\ &= -2\pi(p-1)M_0^p \int_0^{\pi-\epsilon} |P_\nu^0(\cos\theta)|^{p+1} \sin\theta d\theta < 0. \end{aligned}$$

Therefore, by the implicit function theorem, there exist a constant $\delta > 0$ and a C^1-function $t(\mu)$ such that

$$N(\mu, t(\mu)) = 0 \qquad \text{for } \mu \leq 0,$$

where $t(0) = 1$ and $|t(\mu)-1|+|\mu| < \delta$. Since $K(\mu, t(\mu)v_*) = 0$ is equivalent to $N(\mu, t(\mu)) = 0$ $(\mu \neq 0)$ and $t(0) = 1$, Proposition 3.2 is proved. ■

Second we consider the case that the dimension of $\mathcal{V}$ is 2. Then, from Theorem 1.1, it follows that

$$\mathcal{V} = \{P_\nu^m(\cos\theta)(t\cos m\varphi + s\sin m\varphi) \mid t, s \in \mathbf{R}\} \qquad \text{with } 1 \leq m \leq j. \tag{3.11}$$

Then, for $K(\mu, v)$ defined in (3.8), the following lemma holds:

Proposition 3.3 *Assume $1 < p < \infty$, $1 \leq m \leq j$ and $\nu = \nu_{j,\epsilon}^m$. Let $M_m > 0$ satisfy*

$$\begin{aligned}\int_0^{\pi-\epsilon} \Big\{ &|P_\nu^m(\cos\theta)|^2 \\ &-M_m^{p-1}\left[\frac{1}{\pi}\int_0^{2\pi} |\cos m\varphi|^{p+1} d\varphi\right] |P_\nu^m(\cos\theta)|^{p+1}\Big\} \sin\theta d\theta = 0.\end{aligned} \tag{3.12}$$

Then there exist a constant $\delta > 0$ and C^1-functions $t(\mu), s(\mu)$ such that

$$K(\mu, t(\mu)M_m P_\nu^m(\cos\theta)\cos m\varphi + s(\mu)M_m P_\nu^m(\cos\theta)\cos m\varphi) = 0 \quad \text{for } \mu \leq 0,$$

where $0 \leq m \leq j$, $t(0) = t_*$ $s(0) = s_*$ and $|t(\mu) - t_*| + |s(\mu) - s_*| + |\mu| < \delta$. Here $t_*, s_* \in \mathbf{R}$ is arbitrarily taken such that $t_*^2 + s_*^2 = 1$, $t_* \neq 0$ and $s_* \neq 0$.

Proof. Let

$$v(t, s) := M_m\left(tP_\nu^m(\cos\theta)\cos m\varphi + sP_\nu^m(\cos\theta)\sin m\varphi\right).$$

Moreover we arbitrary fix t_* and s_* satisfying $t_*^2 + s_*^2 = 1$. In addition let $v_* := v(t_*, s_*)$. Then, by Proposition 3.1, there exists an implicit function $l(\mu, v)$ for v_*.

From (3.8) and (3.11), $K(\mu, v)$ is expressed as

$$K(\mu, v) = \mu a_\nu^m\{N_1(\mu, t)P_\nu^m(\cos\theta)\cos m\varphi + N_2(\mu, t)P_\nu^m(\cos\theta)\sin m\varphi\},$$

and

$$N_1(\mu, t, s) := \langle P_\nu^m(\cos\theta)\cos m\varphi, v - |v + l(\mu, v)|^{p-1}(v + l(\mu, v))\rangle, \tag{3.13}$$

$$N_2(\mu, t, s) := \langle P_\nu^m(\cos\theta)\sin m\varphi, v - |v + l(\mu, v)|^{p-1}(v + l(\mu, v))\rangle, \tag{3.14}$$

where $(a_\nu^m)^{-1} = \int_0^{\pi-\epsilon} |P_\nu^m(\cos\theta)|^2 \sin\theta d\theta \int_0^{2\pi} |\cos m\varphi|^2 d\varphi = \pi \int_0^{\pi-\epsilon} |P_\nu^m(\cos\theta)|^2 \sin\theta d\theta$. The equation $K(\mu, v) = 0$ $(\mu \neq 0)$ is equivalent to

$$N_1(\mu, t, s) = N_2(\mu, t, s) = 0. \tag{3.15}$$

Thus it suffices to show the existence of non-trivial solutions $t = t(\mu)$ and $s = s(\mu)$ to (3.15), and we prove it by using the implicit function theorem.

From direct calculation, it follows that

$$
\begin{aligned}
&N_1(0,t_*,s_*)\\
&= M_m \int_0^{\pi-\epsilon} |P_\nu^m(\cos\theta)|^2 \sin\theta d\theta \int_0^{2\pi} (t_*\cos^2 m\varphi + s_*\sin m\varphi\cos m\varphi)d\varphi\\
&\quad - M_m^p \int_0^{\pi-\epsilon} |P_\nu^m(\cos\theta)|^{p+1}\sin\theta d\theta\\
&\quad \times \int_0^{2\pi} |t_*\cos m\varphi + s_*\sin m\varphi|^{p-1}(t_*\cos^2 m\varphi + s_*\sin m\varphi\cos m\varphi)d\varphi.
\end{aligned} \tag{3.16}
$$

We see that $\int_0^{2\pi}\cos^2 m\varphi d\varphi = \pi$ and $\int_0^{2\pi}\sin m\varphi\cos m\varphi d\varphi = 0$. Moreover let

$$\cos\beta := t_* \quad \text{and} \quad \sin\beta := s_*.$$

In addition, since $|\cos m\varphi|^{p-1}\cos m\varphi\sin m\varphi$ is odd and periodic, it holds that

$$\int_0^{2\pi} |\cos m\varphi|^{p-1}\cos m\varphi\sin m\varphi d\varphi = \int_{-\pi}^{\pi} |\cos m\varphi|^{p-1}\cos m\varphi\sin m\varphi d\varphi = 0. \tag{3.17}$$

Hence, from (3.17), it holds that

$$
\begin{aligned}
&\int_0^{2\pi} |t_*\cos m\varphi + s_*\sin m\varphi|^{p-1}(t_*\cos^2 m\varphi + s_*\sin m\varphi\cos m\varphi)d\varphi\\
&= \int_0^{2\pi} |\cos(m\varphi-\beta)|^{p-1}\cos(m\varphi-\beta)\cos m\varphi d\varphi\\
&= \int_0^{2\pi} |\cos m\varphi|^{p-1}\cos m\varphi\cos(m\varphi+\beta)d\varphi\\
&= \int_0^{2\pi} |\cos m\varphi|^{p-1}\cos m\varphi(t_*\cos m\varphi - s_*\sin m\varphi)d\varphi\\
&= t_* D_p\pi + s_*\int_0^{2\pi} |\cos m\varphi|^{p-1}\cos m\varphi\sin m\varphi d\varphi\\
&= t_* D_p \pi
\end{aligned}
$$

with

$$D_p := \frac{1}{\pi}\int_0^{2\pi} |\cos m\varphi|^{p+1}d\varphi = \frac{1}{\pi}\int_0^{2\pi}|\cos\varphi|^{p+1}d\varphi \quad (m\geq 1).$$

Thus, from (3.12) and (3.16), we obtain

$$
\begin{aligned}
N_1(0,t_*,s_*) &= t_* M_m\pi \int_0^{\pi-\epsilon} \left\{|P_\nu^m(\cos\theta)|^2 - M_m^{p-1}D_p|P_\nu^m(\cos\theta)|^{p+1}\right\}\sin\theta d\theta\\
&= 0.
\end{aligned}
$$

Similarly, since it holds that

$$\int_0^{2\pi} |t_*\cos m\varphi + s_*\sin m\varphi|^{p-1}(t_*\cos m\varphi + s_*\sin m\varphi)\sin m\varphi d\varphi = s_* D_p\pi,$$

we obtain

$$
\begin{aligned}
N_2(0,t_*,s_*) &= s_* M_m\pi \int_0^{\pi-\epsilon} \left\{|P_\nu^m(\cos\theta)|^2 - M_m^{p-1}D_p|P_\nu^m(\cos\theta)|^{p+1}\right\}\sin\theta d\theta\\
&= 0.
\end{aligned}
$$

Next, from direct calculation, it holds that

$$\begin{aligned}(N_1)_t(\mu,t,s) =&\langle P_\nu^m(\cos\theta)\cos m\varphi, M_m P_\nu^m(\cos\theta)\cos m\varphi \\ &- pM_m|v+l(\mu,v)|^{p-1}(1+l_v(\mu,v))P_\nu^m(\cos\theta)\cos m\varphi\rangle.\end{aligned}$$

Thus it holds that

$$\begin{aligned}(N_1)_t(0,t_*,s_*) =& M_m\pi\int_0^{\pi-\epsilon}|P_\nu^m(\cos\theta)|^2\sin\theta d\theta \\ &- pM_m^p\pi\int_0^{\pi-\epsilon}|P_\nu^m(\cos\theta)|^{p+1}\sin\theta d\theta \\ &\times\int_0^{2\pi}|t_*\cos m\varphi+s_*\sin m\varphi|^{p-1}\cos^2 m\varphi d\varphi.\end{aligned} \tag{3.18}$$

Here, from (3.17), it follows that

$$\begin{aligned}&\int_0^{2\pi}|t_*\cos m\varphi+s_*\sin m\varphi|^{p-1}\cos^2 m\varphi d\varphi \\ &=\int_0^{2\pi}|\cos m\varphi|^{p-1}\cos^2(m\varphi+\beta)d\varphi \\ &=\int_0^{2\pi}|\cos m\varphi|^{p-1}(t_*\cos m\varphi - s_*\sin m\varphi)^2 d\varphi \\ &= t_*^2\int_0^{2\pi}|\cos m\varphi|^{p-1}\cos^2 m\varphi d\varphi + s_*^2\int_0^{2\pi}|\cos m\varphi|^{p-1}\sin^2 m\varphi d\varphi.\end{aligned}$$

Since

$$\begin{aligned}&\int_0^{2\pi}|\cos m\varphi|^{p-1}\sin^2 m\varphi d\varphi \\ &=\left[-\frac{1}{mp}|\cos m\varphi|^{p-1}\cos m\varphi\sin m\varphi\right]_0^{2\pi}+\frac{1}{p}\int_0^{2\pi}|\cos m\varphi|^{p+1}d\varphi \\ &=\frac{1}{p}\int_0^{2\pi}|\cos m\varphi|^{p+1}d\varphi,\end{aligned}$$

we obtain

$$\int_0^{2\pi}|t_*\cos m\varphi+s_*\sin m\varphi|^{p-1}\cos^2 m\varphi d\varphi = D_p\pi\left(t_*^2+\frac{s_*^2}{p}\right).$$

Hence, from (3.18), it holds that

$$\begin{aligned}(N_1)_t(0,t_*,s_*) =& M_m\pi\int_0^{\pi-\epsilon}|P_\nu^m(\cos\theta)|^2\sin\theta d\theta \\ &-\left(pt_*^2+s_*^2\right)M_m^p D_p\pi\int_0^{\pi-\epsilon}|P_\nu^m(\cos\theta)|^{p+1}\sin\theta d\theta.\end{aligned}$$

From (3.12) and $pt_*^2+s_*^2>1$, we obtain $(N_1)_t(0,t_*,s_*)<0$.

On the other hand, it holds that

$$\begin{aligned}(N_1)_s(\mu,t,s) =&\langle P_\nu^m(\cos\theta)\cos m\varphi, M_m P_\nu^m(\cos\theta)\sin m\varphi \\ &- p|v+l(\mu,v)|^{p-1}(1+l_v(\mu,v))M_m P_\nu^m(\cos\theta)\sin m\varphi\rangle.\end{aligned}$$

Thus, by similar calculation above, we obtain

$$\begin{aligned}(N_1)_s(0,t_*,s_*) = {}& M_m\pi\int_0^{\pi-\epsilon}|P_\nu^m(\cos\theta)|^2\sin\theta d\theta\int_0^{2\pi}\cos m\varphi\sin m\varphi d\varphi\\&- pM_m^pD_p\pi\int_0^{\pi-\epsilon}|P_\nu^m(\cos\theta)|^{p+1}\sin\theta d\theta\\&\times\int_0^{2\pi}|t_*\cos m\varphi+s_*\sin m\varphi|^{p-1}\cos m\varphi\sin m\varphi d\varphi\\={}&0.\end{aligned}$$

Similarly, since (3.12) and $t_*^2+ps_*^2>1$, it follows that

$$\begin{aligned}(N_2)_t(0,t_*,s_*) &= 0,\\(N_2)_s(0,t_*,s_*) &= M_m\pi\int_0^{\pi-\epsilon}|P_\nu^m(\cos\theta)|^2\sin\theta d\theta\\&\quad-\left(t_*^2+ps_*^2\right)M_m^pD_p\pi\int_0^{\pi-\epsilon}|P_\nu^m(\cos\theta)|^{p+1}\sin\theta d\theta\\&<0.\end{aligned}$$

Therefore, by the implicit function theorem, there exists a constant $\delta>0$ and C^1-functions $t(\mu)$ and $s(\mu)$ such that, for $|t(\mu)-t_*|+|s(\mu)-s_*|+|\mu|<\delta$, it holds that

$$N_1(\mu,t(\mu),s(\mu)) = N_2(\mu,t(\mu),s(\mu)) = 0 \tag{3.19}$$

with $t(0)=t_*$ and $s(0)=s_*$. Since $K(\mu,t(\mu)v(t(\mu),s(\mu)))=0$ is equivalent to (3.19) with $\mu\neq 0$, Proposition 3.3 is proved. ■

Proposition 3.3 is considered in the case of $t_*\neq 0$ and $s_*\neq 0$. If $t_*=0$ (or $s_*=0$), then $(N_1)_t(0,0,1)=0$ (or $(N_2)_s(0,1,0)=0$), and hence the argument in Proposition 3.3 is not valid (we cannot apply the implicit function theorem). Thus we show Proposition 3.3 in the case of $t_*=0$ or $s_*=0$ by another method. Namely we prove that $N_1(\mu,0,s)\equiv 0$ ($N_2(\mu,t,0)\equiv 0$) holds if $t_*=0$ ($s_*=0$), and we follows the same argument as in the proof of Proposition 3.2.

We prepare for a proof of the case $t_*=0$ or $s_*=0$. For $J(\mu,v,w)$ defined in (3.5), we remark that

$$J(\mu,-v,-w) = -Lw-\mu w-\mu Q|v+w|^{p-1}(-v-w) = -J(\mu,v,w). \tag{3.20}$$

From Proposition 3.1 and (3.20), it holds that

$$J(\mu,-v,-l(\mu,v)) = J(\mu,v,l(\mu,v)) = 0.$$

Hence we extend $l(\mu,v)$ for $-v$ ($v\in\mathcal{V}_*$) by

$$l(\mu,-v) := -l(\mu,v). \tag{3.21}$$

and then $J(\mu,-v,l(\mu,-v))=0$ holds. Thus, from (3.8) and (3.21), it follows that

$$K(\mu,-v) = Ph(\mu,-v,l(\mu,-v)) = -Ph(\mu,v,l(\mu,v)) = -K(\mu,v).$$

Hence if $K(\mu,v)=0$ holds for the extended $l(\mu,v)$, then $K(\mu,-v)=0$ also holds.

Now we prove the following lemma.

Lemma 3.3 *Assume the same assumption as in Proposition* 3.3 *and extend* $l(\mu, v)$ *by* (3.21). *If* $t_* = 0$ ($s_* = 0$) *for* $N_1(\mu, t, s)$ *and* $N_2(\mu, t, s)$ *as in* (3.13) *and* (3.14), *then it holds that* $N_1(\mu, 0, s) \equiv 0$ ($N_2(\mu, t, 0) \equiv 0$) *near* $\mu = 0$ *and* $s = 1$ ($\mu = 0$ *and* $t = 1$).

Proof. First we assume $t_* = 0$. Let $v_1 = v_1(\theta, \varphi; s) := sM_m P_\nu^m(\cos\theta)\sin m\varphi$. Then, from $\int_0^{2\pi} \cos m\varphi \sin m\varphi d\varphi = 0$, it follows that

$$\begin{aligned} N_1(\mu, 0, s) &= sM_m \int_0^{\pi-\epsilon} |P_\nu^m(\cos\theta)|^2 \sin\theta d\theta \int_0^{2\pi} \cos m\varphi \sin m\varphi d\varphi \\ &\quad - M_m^p \int_0^{\pi-\epsilon} \left\{ \int_0^{2\pi} V_1(\theta, \varphi; s) \cos m\varphi d\varphi \right\} P_\nu^m(\cos\theta) \sin\theta d\theta \\ &= -M_m^p \int_0^{\pi-\epsilon} \left\{ \int_0^{2\pi} V_1(\theta, \varphi; s) \cos m\varphi d\varphi \right\} P_\nu^m(\cos\theta) \sin\theta d\theta \end{aligned}$$

with

$$V_1(\theta, \varphi; s) := |v_1(\theta, \varphi; s) + l(\mu, v_1(\theta, \varphi; s))|^{p-1}(v_1(\theta, \varphi; s) + l(\mu, v_1(\theta, \varphi; s)))$$

The function $v_1(\theta, \varphi; s)$ is odd and periodic with respect to φ. Moreover, from (3.21), it holds that

$$\begin{aligned} l(\mu, v_1(\theta, -\varphi; s)) &= l(\mu, sM_m P_\nu^m(\cos\theta) \sin m(-\varphi)) \\ &= l(\mu, -sM_m P_\nu^m(\cos\theta) \sin m\varphi) \\ &= -l(\mu, v_1(\theta, \varphi; s)). \end{aligned}$$

Hence $V_1(\theta, \varphi; s)$ is odd and periodic with respect to φ, and it holds that

$$\int_0^{2\pi} V_1(\theta, \varphi; s) \cos m\varphi d\varphi = \int_{-\pi}^{\pi} V_1(\theta, \varphi; s) \cos m\varphi d\varphi = 0.$$

Therefore we obtain

$$N_1(\mu, 0, s) \equiv 0 \qquad \text{near } (\mu, s) = (0, 1).$$

Next we assume $s_* = 0$. Let $v_2 = v_2(\theta, \varphi; t) := tM_m P_\nu^m(\cos\theta)\cos m\varphi$. Then, from $\int_0^{2\pi} \cos m\varphi \sin m\varphi d\varphi = 0$, it follows that

$$N_2(\mu, t, 0) = -M_m^p \int_0^{\pi-\epsilon} \left\{ \int_0^{2\pi} V_2(\theta, \varphi; s) \sin m\varphi d\varphi \right\} P_\nu^m(\cos\theta) \sin\theta d\theta$$

with

$$V_2(\theta, \varphi; t) := |v_2(\theta, \varphi; s) + l(\mu, v_2(\theta, \varphi; s))|^{p-1}(v_2(\theta, \varphi; s) + l(\mu, v_2(\theta, \varphi; s)))$$

Since $v_2(\theta, \varphi; t)$ is even and periodic with respect to φ, $l(\mu, v_2(\theta, \varphi; t))$ is also even and periodic with respect to φ. Therefore, since $\cos m\varphi \sin m\varphi$ and $V_2(\theta, \varphi; t) \sin m\varphi$ are odd and periodic with respect to φ, we obtain

$$N_2(\mu, t, 0) \equiv 0 \qquad \text{near } (\mu, t) = (0, 1).$$

Lemma 3.3 is proved. ■

Lemma 3.3 implies that if $\mathcal{V}$ is restricted to $\mathcal{V}_c := \{tP_\nu^m(\cos\theta)\cos m\varphi \mid t \in \mathbf{R}\}$ (or $\mathcal{V}_s := \{tP_\nu^m(\cos\theta)\sin m\varphi \mid t \in \mathbf{R}\}$), then the dimension of the range of $P(\mu, v, l(\mu, v))$ ($v \in \mathcal{V}_c$) is 1. Hence, by similar arguments to Proposition 3.2, we can prove the existence of a non-trivial solution:

Proposition 3.4 *Assume the same assumptions as in Proposition* 3.3. *Then, for* $t_* = 0$ *or* $s_* = 0$, *there exists a nontrivial solution of* $K(\mu, v) = 0$.

Proof. First we assume $s_* = 0$, and then $t_* = 1$ ($t_*^2 + s_*^2 = 1$). We define

$$\mathcal{V}_c := \{tP_\nu^m(\cos\theta)\cos m\varphi \mid t \in \mathbf{R}\}.$$

For $\mathcal{V}_c$, we define

$$\mathcal{X}_c := \mathcal{V}_c \oplus \mathcal{W},$$
$$\mathcal{Y}_c := \mathcal{V}_c \oplus \mathcal{R}.$$

Let $v_c(t) := tP_\nu^m(\cos\theta)\cos m\varphi$. We restrict $f(\lambda, u) : \mathcal{X}_c \to \mathcal{Y}_c$. Then, by almost the same arguments as in the proof of Proposition 3.1, we can prove that there exists an implicit function $l(\mu, v)$ near $(\mu, v) = (0, P_\nu^m(\cos\theta)\cos m\varphi)$ such that

$$N_1(\mu, t) = 0 \qquad \text{for } |\mu| + |t-1| < \delta \tag{3.22}$$

with some $\delta > 0$. On the other hand, by Lemma 3.3, $N_2(\mu, t) = 0$ also holds. Since $N_1(\mu, t) = N_2(\mu, t) = 0$ is equivalent to $K(\mu, v_c(t)) = 0$, Proposition 3.4 with $s_* = 0$ is proved.

Next we assume $t_* = 0$, and then $s_* = 1$. Let $v_s(t) := sP_\nu^m(\cos\theta)\sin m\varphi$, and we define

$$\mathcal{V}_s := \{tP_\nu^m(\cos\theta)\sin m\varphi \mid t \in \mathbf{R}\},$$
$$\mathcal{X}_s := \mathcal{V}_s \oplus \mathcal{W},$$
$$\mathcal{Y}_s := \mathcal{V}_s \oplus \mathcal{R}.$$

Now we replace $\mathcal{V}_c$, $\mathcal{X}_c$ and $\mathcal{Y}_c$ with $\mathcal{V}_s$, $\mathcal{X}_s$ and $\mathcal{Y}_s$, respectively. Then, from almost the same arguments above, Proposition 3.4 with $t_* = 0$ is proved. ■

From Propositions 3.2–3.4, Theorem 1.2 follows.

Proof of Theorem 1.2. Recall (3.2) and Proposition 3.1. If $m = 0$, then, from arguments above and Proposition 3.2, we see that, for $\mu \le 0$ near $\mu = 0$,

$$u(\theta, \varphi; \mu + \lambda_{j,\epsilon}^0) = |\mu|^{\frac{1}{p-1}}\{t(\mu)v_* + l(\mu, t(\mu)v_*)\}$$

is a solution to (1.1). Here $t(0) = 1$, $l(0, v_*) = 0$ and

$$v_* = M_0 P_{\nu_{j,\epsilon}^0}^0(\cos\theta).$$

Similarly if $1 \le m \le j$, then, from arguments above and Propositions 3.3 and 3.4, we see that, for $\mu \le 0$ near $\mu = 0$,

$$u(\theta, \varphi; \mu + \lambda_{j,\epsilon}^m) = |\mu|^{\frac{1}{p-1}}\{v(t(\mu), s(\mu)) + l(\mu, v(t(\mu), s(\mu))\}$$

is a solution to (1.1). Here $t(0) = t_*$, $s(0) = s_*$, $l(0, t_*, s_*) = 0$, $t_*^2 + s_*^2 = 1$,

$$v(t, s) = tM_m P_\nu^m(\cos\theta)\cos m\varphi + sM_m P_\nu^m(\cos\theta)\sin m\varphi,$$

and $t(\mu) \equiv 0$ ($s(\mu) \equiv 0$) when $t_* = 0$ ($s_* = 0$). Theorem 1.2 is proved. ■

4 Appendix

4 Appendix In Appendix A we show Lemmas 2.2 and 2.3. Additionally, in Appendix B, we prove Lemma 3.1.

4.1 Proofs of Lemmas 2.2 and 2.3 In this appendix, we prove Lemmas 2.2 and 2.3. Arguments below follows results in [10].

Recall (1.8), and we define

$$F(a,b,c;x) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \tag{4.1}$$

where c is a non-positive integer. The function $F(a,b,c;x)$ is said to be the *Gauss hypergeometric function.* The radius of convergence of (4.1) is 1, and $F = F(a,b,c;x)$ satisfies the *Gauss hypergeometric equation*

$$x(1-x)\frac{d^2F}{dx^2} + \{c - (a+b+1)x\}\frac{dF}{dx} - abF = 0 \qquad (-1 < x < 1). \tag{4.2}$$

In Appendix we use some properties of $F(a,b,c;x)$, and those results follow from [10].

First, for $P_\nu^m(x)$ and $F(a,b,c;x)$, the following relation holds (p.319 in [10]):

$$\begin{aligned} P_\nu^m(x) = {} & \frac{(-1)^m \Gamma(\nu+m+1)(1-x^2)^{\frac{m}{2}}}{2^m \Gamma(\nu-m+1) m!} \\ & \times F\left(-\nu+m, \nu+m+1, m+1; \frac{1-x}{2}\right). \end{aligned} \tag{4.3}$$

By (4.3), we can prove Lemma 2.2.

Proof of Lemma 2.2. First we assume that ν is an integer. If $-\nu + m = -n$ (n be a non-negative integer), then

$$\begin{aligned} P_\nu^m(x) = {} & \frac{\Gamma(n+2\nu+1)(x^2-1)^{\frac{m}{2}}}{2^m \Gamma(-n+1) m!} \\ & \times F\left(-n, n+2\nu+1, m+1; \frac{1-x}{2}\right). \end{aligned}$$

If $n = 0$ ($\nu = m$), then, from (4.1), it follows that

$$\begin{aligned} F\left(0, 2\nu+1, m+1; \frac{1-x}{2}\right) &= \sum_{k=0}^{+\infty} \frac{(0)_k (2\nu+1)_k}{(m+1)_k k!} \left(\frac{1-x}{2}\right)^k \\ &\equiv 0 \end{aligned}$$

Hence $P_m^m(x) \equiv 0$ holds. On the other hand, if $n \geq 1$ ($\nu > m$), then, since $\Gamma(x)$ has singularity at $x = 0, -1, -2, ...$, the identity $P_{m+n}^m(x) \equiv 0$ holds (see (4.3)).

Next we assume that ν is not an integer. Then, from (4.1), it holds that

$$F\left(-\nu+m, \nu+m+1, m+1, \frac{1-x}{2}\right) > 0 \qquad \text{for } -1 < x < 1.$$

Therefore, from (4.3), $P_\nu^m(x)$ does not have a zero for $-1 < x < 1$, and Lemma 2.2 is proved. ■

Next we show Lemma 2.3 by using properties of $F(a,b,c;x)$. To investigate the behavior of $P_\nu^m(x)$ near $x=-1$, it suffices to consider that of $F(a,b,c;x)$ near $x=1$. For the purpose we use the following formula (see p.273 in [10])

$$\begin{aligned} F(a,b,a+b+1-c;1-x) = &\frac{\Gamma(a+b+1-c)\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}F(a,b,c;x) \\ &+\frac{\Gamma(a+b+1-c)\Gamma(c-1)}{\Gamma(a)\Gamma(b)}x^{1-c}F(a+1-c,b+1-c,2-c;x). \end{aligned} \tag{4.4}$$

In addition we define

$$\begin{aligned} U(a,b,c;x) :=&\frac{\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}F(a,b,c;x) \\ &+\frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)}x^{1-c}F(a+1-c,b+1-c,2-c;x). \end{aligned} \tag{4.5}$$

The function $U(a,b,c;x)$ is also a solution to (4.2), and moreover $F(a,b,c;x)$ and $U(a,b,c;x)$ are linearly independent (see p.274 in [10]). Thus, from (4.3)–(4.5), it follows that

$$\begin{aligned} P_\nu^m(-1+2x) = &\frac{(-1)^m\Gamma(\nu+m+1)\Gamma(m+1)x^{\frac{m}{2}}(1-x)^{\frac{m}{2}}}{\Gamma(\nu-m+1)m!} \\ &\times U\left(-\nu+m,\nu+m+1,m+1;x\right). \end{aligned} \tag{4.6}$$

Furthermore if c is an integer ($c=n$), then, for $U(a,b,n;x)$, the following formula holds (see p.275 in [10]):

$$\begin{aligned} U(a,b,n;x) = &\frac{(-1)^n}{\Gamma(a+1-n)\Gamma(b+1-n)(n-1)!}\Bigg[F(a,b,n;x)\log x \\ &+\sum_{k=0}^{+\infty}\frac{(a)_k(b)_k}{(n)_k k!}\{\psi(a+n)+\psi(b+n)-\psi(k+1)-\psi(n+k)\}x^k\Bigg] \\ &+\frac{(m-2)!}{\Gamma(a)\Gamma(b)}x^{1-n}\sum_{k=0}^{n-2}\frac{(a+1-n)_k(b+1-n)_k}{(2-n)_k k!}x^k. \end{aligned} \tag{4.7}$$

Proof of Lemma 2.3. Lemma 2.3 follows from (4.6) and (4.7). ■

4.2 Proof of Lemma 3.1 In this section, $H^k(\Omega|\mathbf{R}^2)$ denotes the usual Sobolev space on $\Omega\subset\mathbf{R}^2$. Before beginning to prove Lemma 3.1, we introduce the Sobolev spaces $H^k(B_{\pi-\epsilon})$ on $B_{\pi-\epsilon}\subset\mathbf{S}^2$ ($k=0,1,2$ and $H^0(B_{\pi-\epsilon})=L^2(B_{\pi-\epsilon})$). Now we introduce the stereographic projection from $\mathbf{S}^2$ to $\mathbf{R}^2$. Namely let

$$\left(\frac{2x_1}{1+|x|^2},\frac{2x_2}{1+|x|^2},\frac{1-|x|^2}{1+|x|^2}\right)\in\mathbf{S}^2,$$

where $(x_1,x_2)\in\Omega_{R_\epsilon}:=\{x\in\mathbf{R}^2\mid |x|<R_\epsilon\}$ ($R_\epsilon:=\tan[(\pi-\epsilon)/2]$). Then norms of $H^k(B_{\pi-\epsilon})$ are expressed as, respectively,

$$\|u\|^2_{H^k(B_{\pi-\epsilon})}=\sum_{s=0}^{k}\int_{\Omega_{R_\epsilon}}|D^s_{\mathbf{S}^2}u|^2q^{2-2s}dx_1dx_2. \tag{4.8}$$

Here

$$|D^0_{\mathbf{S}^2} u|^2 = |u|^2, \tag{4.9}$$

$$|D^1_{\mathbf{S}^2} u|^2 = \left|\frac{\partial u}{\partial x_1}\right|^2 + \left|\frac{\partial u}{\partial x_2}\right|^2, \tag{4.10}$$

and

$$|D^2_{\mathbf{S}^2} u|^2 = \sum_{i,j=1}^{2} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{n=1}^{2} \Gamma^n_{ij} \frac{\partial u}{\partial x_n} \right|^2, \tag{4.11}$$

where

$$q := \frac{2}{1+|x|^2},$$

$-\Gamma^1_{11} = -\Gamma^2_{12} = -\Gamma^1_{22} = 2x_1/(1+|x|^2)$ and $\Gamma^2_{11} = -\Gamma^1_{12} = -\Gamma^2_{22} = 2x_2/(1+|x|^2)$ (e.g., see Definitions 2.2 and 2.3 in [2] or Definition 2.1 in Hebey [12]). Moreover we define $H^k_0(B_{\pi-\epsilon})$ as the closure of $C^\infty_0(B_{\pi-\epsilon})$ in $H^k(B_{\pi-\epsilon})$.

From the definition above, it follows that

$$\Delta_{\mathbf{S}^2} u = q^{-2} \Delta u, \tag{4.12}$$

where Δ is the Laplace operator on $\mathbf{R}^2$.

The relation between the stereographic projection and the polar coordinates is as follows:

$$x_1 = \tan\left(\frac{\theta}{2}\right) \cos\varphi,$$
$$x_2 = \tan\left(\frac{\theta}{2}\right) \sin\varphi.$$

Thus, from

$$\begin{aligned} q^2 dx_1 dx_2 &= \frac{4\tan\left(\frac{\theta}{2}\right)}{\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)^2} \cdot \frac{1}{2\cos^2\left(\frac{\theta}{2}\right)} d\theta d\varphi \\ &= 2\tan\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) d\theta d\varphi \\ &= \sin\theta d\theta d\varphi, \end{aligned}$$

it follows that

$$\langle u, v\rangle = \int_{\Omega_{R_\epsilon}} u v q^2 dx_1 dx_2.$$

Therefore we prove Lemma 3.1 with the stereographic projection. Now recall definitions of the operator L and subspaces of $\mathcal{V}$, $\mathcal{W}$, $\mathcal{R}$ (see Section 3).

Proof of Lemma 3.1. We remark that, for any $g \in L^2(B_{\pi-\epsilon})$, there exists a unique solution $u \in H^1_0(B_{\pi-\epsilon}) \cap H^2(B_{\pi-\epsilon})$ to

$$\begin{cases} \Delta_{\mathbf{S}^2} u = g & \text{in } B_{\pi-\epsilon} \\ u = 0 & \text{on } \partial B_{\pi-\epsilon} \end{cases}$$

(e.g., see Theorem 4.8 in [2]). Hence there exists a unique inverse operator $\mathcal{G} : L^2(B_{\pi-\epsilon}) \to H^2(B_{\pi-\epsilon}) \subset L^2(B_{\pi-\epsilon})$ of $\Delta_{\mathbf{S}^2} : H^2(B_{\pi-\epsilon}) \to L^2(B_{\pi-\epsilon})$. Now we prove that $\mathcal{G}$ is compact and self-adjoint.

First we show that $\mathcal{G}$ is compact. From (4.11), the Minkowski inequality and $(a+b)^2 \leq 2(a^2+b^2)$, it holds that

$$
\begin{aligned}
|D^2_{\mathbf{S}^2} u|^2 &\leq \left[\left\{ \sum_{i,j=1}^{2} \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 \right\}^{\frac{1}{2}} + \left\{ \sum_{i,j=1}^{2} \left| \sum_{n=1}^{2} \Gamma^n_{ij} \frac{\partial u}{\partial x_n} \right|^2 \right\}^{\frac{1}{2}} \right]^2 \\
&\leq 2 \sum_{i,j=1}^{2} \left\{ \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|^2 + 2 \sum_{n=1}^{2} |\Gamma^n_{ij}| \left| \frac{\partial u}{\partial x_n} \right|^2 \right\}
\end{aligned} \tag{4.13}
$$

Hence, from (4.8)–(4.10) and (4.13), it follows that

$$\|u\|_{H^2(B_{\pi-\epsilon})} \leq K_1 \|u\|_{H^2(\Omega_{R_\epsilon}|\mathbf{R}^2)}, \tag{4.14}$$

where $K_1 > 0$ is some constant. In addition, from (4.8) and (4.9), it holds that

$$\|u\|_{L^2(\Omega_{R_\epsilon}|\mathbf{R}^2)} \leq K_2 \|u\|_{L^2(B_{\pi-\epsilon})}, \tag{4.15}$$

where some constant $K_2 > 0$. Moreover we apply the regularity theorem of elliptic equations for

$$
\begin{cases}
q^{-2}\Delta u = g & \text{in } \Omega_{R_\epsilon}, \\
u = 0 & \text{on } \partial\Omega_{R_\epsilon},
\end{cases}
$$

where $g \in L^2(\Omega_{R_\epsilon}|\mathbf{R}^2)$ (e.g., see Theorem 8.12 in Gilbarg and Trudinger [11]). Then we obtain

$$\|u\|_{H^2(\Omega_{R_\epsilon}|\mathbf{R}^2)} \leq K_3 \|g\|_{L^2(\Omega_{R_\epsilon}|\mathbf{R}^2)} = K_3 \|q^{-2}\Delta u\|_{L^2(\Omega_{R_\epsilon}|\mathbf{R}^2)} \tag{4.16}$$

with some constant $K_3 > 0$. Hence, from (4.8) and (4.14)–(4.16), we obtain

$$\|u\|_{H^2(B_{\pi-\epsilon})} \leq K_1K_2K_3 \|q^{-2}\Delta u\|_{L^2(\Omega_{R_\epsilon}|\mathbf{R}^2)} \leq 2K_1K_2K_3 \|\Delta_{\mathbf{S}^2} u\|_{L^2(B_{\pi-\epsilon})}.$$

for $u \in H^1_0(B_{\pi-\epsilon}) \cap H^2(B_{\pi-\epsilon})$ (see (4.12)). Thus the operator $\mathcal{G} : L^2(B_{\pi-\epsilon}) \to H^2(B_{\pi-\epsilon})$ is bounded. Furthermore the imbedding $H^2(B_{\pi-\epsilon}) \hookrightarrow L^2(B_{\pi-\epsilon})$ is compact by the *Rellich–Kondrachov theorem* (e.g., see Theorem 2.34 in [2]). Thus $\mathcal{G} : L^2(B_{\pi-\epsilon}) \to L^2(B_{\pi-\epsilon})$ is compact.

Second we show that $\mathcal{G}$ is self-adjoint. Let $\mathcal{G}^*$ be the adjoint operator of $\mathcal{G}$. Then, for any $u, v \in L^2(B_{\pi-\epsilon})$, it holds that

$$\langle u, \mathcal{G}^*(\Delta_{\mathbf{S}^2} v) \rangle = \langle \Delta_{\mathbf{S}^2}(\mathcal{G}u), v \rangle = \langle u, v \rangle.$$

Thus, by the uniqueness of the inverse operator $\mathcal{G}$, it holds that $\mathcal{G} = \mathcal{G}^*$ on $L^2(B_{\pi-\epsilon})$.

Thus we can apply the *Fredholm alternative theorem* (e.g., see Theorem 3 in p.284 of Yosida [18]) for $\mathcal{G}$. Namely, for any $a \in L^2(B_{\pi-\epsilon})$,

$$\lambda^{-1}u + \mathcal{G}(u) = a \qquad \text{for } u \in H^1_0(B_{\pi-\epsilon}) \cap H^2(B_{\pi-\epsilon})$$

has a solution u if and only if $\langle a, b \rangle = 0$ for any $b \in \mathrm{Ker}(\lambda^{-1} + \mathcal{G})$. Especially, for any $w \in \mathcal{R} \subset L^2(B_{\pi-\epsilon})$, $a := \lambda^{-1}\mathcal{G}(w) \in L^2(B_{\pi-\epsilon})$ holds. Hence it holds that

$$0 = \langle a, b \rangle = \langle \lambda^{-1}\mathcal{G}(w), b \rangle = \lambda^{-2} \langle w, \lambda\mathcal{G}(b) \rangle = -\lambda^{-2} \langle w, b \rangle.$$

Thus, for any $w \in \mathcal{R}$ and $b \in \mathrm{Ker}(\lambda^{-1} + \mathcal{G})$, we obtain $\langle w, b \rangle = 0$.

On the other hand, if $b \in \mathrm{Ker}(L)(= \mathrm{Ker}(\Delta_{\mathbf{S}^2} + \lambda))$, then, from $\lambda^{-1} b = \mathcal{G}(b)$, we obtain

$$\mathcal{G}(\Delta_{\mathbf{S}^2} b + \lambda b) = \lambda(\lambda^{-1} b + \mathcal{G}(b)) = 0.$$

Thus it holds that $b \in \mathrm{Ker}(\lambda^{-1} + \mathcal{G})$. Similarly if $b \in \mathrm{Ker}(\lambda^{-1} + \mathcal{G})$, then $b \in \mathrm{Ker}(L)$, and hence $\mathrm{Ker}(L) = \mathrm{Ker}(\lambda^{-1} + \mathcal{G})$. Therefore, since $\langle w, b \rangle = 0$ holds for any $w \in \mathcal{R}$ and $b \in \mathrm{Ker}(L)$. Lemma 3.1 is proved. ■

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