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ON RELATIVE UNIVERSALITY AND Q-UNIVERSALITY OF FINITELY GENERATED VARIETIES OF HEYTING ALGEBRAS

VÁCLAV KOUBEK AND JIŘÍ SICHLER

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ABSTRACT. Surprisingly small varieties of Heyting algebras have very complex categorical structure and contain the largest possible number of subquasivarieties.

1. INTRODUCTION

A quasivariety is a class of algebraic systems of a similarity type Δ closed under isomorphisms, products, subsystems and ultraproducts. For a family of algebraic systems $\mathcal{U}$ of a type Δ, let $\mathrm{Quasi}(\mathcal{U})$ denote the least quasivariety containing $\mathcal{U}$.

If $\mathbb{Q}$ is a quasivariety then subquasivarieties of $\mathbb{Q}$ form a complete lattice $\mathrm{Lat}(\mathbb{Q})$ with respect to the inclusion. One topic of universal algebra is an investigation of properties of $\mathrm{Lat}(\mathbb{Q})$ for some quasivarieties. M. V. Sapir [23] defined that a quasivariety $\mathbb{Q}$ of a finite type Δ is Q-universal if for every quasivariety $\mathbb{V}$ of a finite type the lattice $\mathrm{Lat}(\mathbb{V})$ is a quotient of a sublattice of $\mathrm{Lat}(\mathbb{Q})$. He proved that a variety of commutative 3-nilpotent semigroups is Q-universal and he asked which other quasivarieties are Q-universal.

A family $\{\mathbf{A}_U \mid U \in \mathcal{P}(\omega)\}$ of algebraic systems indexed by the set $\mathcal{P}(\omega)$ of all finite subsets of the set ω of all natural numbers is called an A-D family if

(p1) $\mathbf{A}_\emptyset$ is a terminal object of $\mathbb{Q}$;
(p2) if $U, V \in \mathcal{P}(\omega)$ then $\mathbf{A}_{U\cup V} \in \mathrm{Quasi}\{\mathbf{A}_U, \mathbf{A}_V\}$;
(p3) if $U, V \in \mathcal{P}(\omega)$ such that $U \neq \emptyset$ and $\mathbf{A}_U \in \mathrm{Quasi}\{\mathbf{A}_V\}$ then $U = V$;
(p4) if for $V \in \mathcal{P}(\omega)$ there exist a finite set $\mathcal{U} \subseteq \mathcal{P}(\omega)$, finite algebraic systems $\mathbf{B}, \mathbf{C} \in \mathrm{Quasi}\{\mathbf{A}_U \mid U \in \mathcal{U}\}$ and an injective homomorphism $f\mathbf{A}_V \to \mathbf{B} \times \mathbf{C}$ then there exists an injective homomorphism $g\mathbf{A}_V \to \mathbf{B}$ or there exists an injective homomorphism $g\mathbf{A}_V \to \mathbf{C}$ or there exist subsets $V_1, V_2 \subseteq V$ and injective homomorphisms $g_1\mathbf{A}_{V_1} \to \mathbf{B}$ and $g_2\mathbf{A}_{V_2} \to \mathbf{C}$ with $V = V_1 \cup V_2$.

M. E. Adams and W. Dziobiak [6, 7] proved that the existence of an A-D family in $\mathbb{Q}$ implies that $\mathrm{Lat}(\mathbb{Q})$ satisfies no nontrivial lattice identity. Then M. E. Adams and W. Dziobiak improved these results by proving in [2] that a quasivariety $\mathbb{Q}$ of algebras of a finite type is Q-universal whenever there exists a sublattice of $\mathrm{Lat}(\mathbb{Q})$ isomorphic to the lattice of all ideals of a free lattice over an infinite countable set, and that the existence of this sublattice in $\mathrm{Lat}(\mathbb{Q})$ follows from the existence of an A-D family in $\mathbb{Q}$. Many quasivarieties are Q-universal, see the excellent survey paper [1] by M. E. Adams et al.

Another important and interesting theme is the algebraic structure of morphisms of a category. One of the notions indicating its complexity is the alg-universality. A category

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$\mathbb{K}$ is alg-universal if every category of algebras and all their homomorphisms can be fully embedded into $\mathbb{K}$, or equivalently, if the category $\mathbb{GR}$ of all graphs and their homomorphisms can be fully embedded into $\mathbb{K}$. Many familiar concrete categories are alg-universal, see the monograph by A. Pultr and V. Trnková [22]. Experience shows that this notion is often too restrictive, and this is because there are many familiar categories rich in their structure which are not alg-universal, for example categories of topological spaces and varieties of lattices or those of monoids. Two obstacles to alg-universality are illustrated in the following theorem.

Theorem 1.1 ([12])**.** *The variety $\mathbb{V}$ of semigroups is alg-universal if and only if $\mathbb{V}$ contains all commutative semigroups and for every $n > 1$ the power law $(xy)^n = x^n y^n$ fails in $\mathbb{V}$.*

The condition that $\mathbb{V}$ contains all commutative semigroups is equivalent to the property that for every $n > 1$ the law $x^n = x$ fails in $\mathbb{V}$ and this holds exactly when there exists a semigroup $S \in \mathbb{V}$ without idempotents. We can say that for a rich category there are two main obstacles, the forced existence of endomorphisms whose images belongs to a poor subcategory (in our earlier examples the existence of endomorphisms whose images belong to the trivial variety) or the existence of canonical symmetries of objects (in our example the existence of endomorphisms $x \mapsto x^n$ for some $n > 1$). These facts motivate the following modifications. Let $\mathbb{K}$ be a category. Then a class I of $\mathbb{K}$-morphism is an ideal of $\mathbb{K}$ if $f \circ g \in I$ whenever $f \in I$ or $g \in I$. A faithful functor $F : \mathbb{L} \to \mathbb{K}$ is called an I-relatively full embedding if

(f1) $Fg \notin I$ for every $\mathbb{L}$-morphism g;
(f2) if $f : FA \to FB$ is a $\mathbb{K}$-morphism for some $\mathbb{L}$-objects A and B then either there exists an $\mathbb{L}$-morphism $g : A \to B$ with $Fg = f$ or $f \in I$.

A category $\mathbb{K}$ is I-relatively alg-universal if there exists an I-relatively full embedding $F : \mathbb{GR} \to \mathbb{K}$. If, moreover, $\mathbb{K}$ is concrete and the underlying set of $F\mathbf{G}$ is finite for every finite graph $\mathbf{G}$ then we say that $\mathbb{K}$ is I-relatively ff-alg-universal. If $\mathbb{K}$ is concrete and I consists of all $\mathbb{K}$-morphisms whose underlying mappings are constant, then $\mathbb{K}$ is almost alg-universal. Observe that if $I = \emptyset$ then $\mathbb{K}$ is I-relatively alg-universal exactly when it is alg-universal. Ideals determined by a subvariety play an important role. Let $\mathbb{Q}$ be a quasivariety and let $\mathbb{V}$ be a subvariety of $\mathbb{Q}$. Let us denote $I_{\mathbb{V}}$ the class of all homomorphisms $f \in \mathbb{Q}$ such that $\mathrm{Im}(f) \in \mathbb{V}$. It is easy to verify that $I_{\mathbb{V}}$ is an ideal. If $\mathbb{Q}$ is $I_{\mathbb{V}}$-relatively alg-universal for some a subvariety $\mathbb{V}$ of $\mathbb{Q}$ then we say that $\mathbb{Q}$ is var-relatively alg-universal. A var-relatively ff-alg-universal quasivariety is defined analogously. The idea of this notion is that many types of algebraic systems allow a retract to an object of a small proper subvariety while the overall morphism structure of $\mathbb{Q}$ remains rich.

Let $\mathbf{M}$ be a monoid viewed as a one-object category. Then $\mathbb{GR} \times \mathbf{M}$ is a category whose objects are all undirected graphs and morphisms from a graph G_1 to a graph G_2 are all pairs (f, m) where $f : G_1 \to G_2$ is a graph homomorphism and $m \in \mathbf{M}$, and if (f', m') is morphism from G_2 to G_3 then $(f, m) \circ (f', m') = (f \circ f', mm')$. We say that a category $\mathbb{K}$ is alg-universal modulo $\mathbf{M}$ if there exists a full embedding of $\mathbb{GR} \times \mathbf{M}$ into $\mathbb{K}$. This notion plays an important role for a characterization of richness of an algebraic structure of homomorphisms of a finitely generated variety of distributive dp-algebras, see [13] – in this case $\mathbf{M}$ is one of the powers $\mathbf{C}_2^0, \mathbf{C}_2$, $\mathbf{C}_2^2$, $\mathbf{C}_2^3$ of the two-element cyclic group $\mathbf{C}_2$. In the present paper we also combine these notions. We say that a category $\mathbb{K}$ is I-relatively alg-universal modulo $\mathbf{M}$ if there exists an I-relatively full embedding F from $\mathbb{GR} \times \mathbf{M}$ into $\mathbb{K}$ where I is an ideal of $\mathbb{K}$. If, moreover, $\mathbb{K}$ is concrete and FG is finite for every finite graph then $\mathbb{K}$ is I-relatively ff-alg-universal modulo $\mathbf{M}$. If $\mathbb{Q}$ is an $I_{\mathbb{V}}$-relatively alg-universal quasivariety modulo $\mathbf{M}$ where $\mathbb{V}$ is a proper subvariety of $\mathbb{Q}$ then $\mathbb{Q}$ is var-relatively alg-universal quasivariety modulo $\mathbf{M}$. A var-relatively ff-alg-universal quasivariety modulo

$\mathbf{M}$ is defined analogously. Observe that if $\mathbf{M}$ is the singleton monoid then a category $\mathbb{K}$ is I-relatively alg-universal modulo $\mathbf{M}$ exactly when $\mathbb{K}$ is I-relatively alg-universal. For algebraic systems, the elements of $\mathbf{M}$ often correspond to derived operations that are their homomorphisms.

M. E. Adams and W. Dziobiak proved the following remarkable result connecting Q-universality and ff-alg-universality.

Theorem 1.2 ([3]). *Every ff-alg-universal quasivariety of algebraic systems contains an A-D family and thus it is Q-universal.*

This result was strengthened to almost ff-alg-universal quasivarieties in [14]. An analogous result was proved for special var-relatively ff-alg-universal quasivarieties in [15], but in general this does not hold: there exists a var-relatively ff-alg-universal variety of distributive *dp*-algebras that is not Q-universal, see [16]. On the other hand, a construction of $I_{\mathbb{V}}$-relatively ff-full embedding can be often modified to a proof that the given quasivariety contains an A-D family. We will illustrate this observation on finitely generated varieties of Heyting algebras. We give a partial answer to the problem formulated in [1] asking to characterize Q-universal varieties of Heyting algebras. We also extend [9], where some finitely generated var-relatively ff-alg-universal varieties of Heyting algebras were exhibited.

For a family $\mathcal{U}$ of algebras of a similarity type Δ, we let $\mathrm{Var}(\mathcal{U})$ denote the least variety containing $\mathcal{U}$. Thus $\mathrm{Var}(\mathcal{U})$ is the variety generated by $\mathcal{U}$. We will work with the Priestley duals of Heyting algebras. It is well-known that duals of finite Heyting algebras are finite posets. Thus for a set $\mathcal{S}$ of finite posets we say that a variety $\mathbb{V}$ of Heyting algebras is generated by $\mathcal{S}$ if $\mathbb{V}$ is generated by Heyting algebras whose duals belong to $\mathcal{S}$. For a finite poset $\mathbf{P}$, we denote $\mathbf{DP}$ the Heyting algebra whose Priestley dual is the poset $\mathbf{P}$. The principal aim of this paper is to prove

Theorem 1.3. *For each $i = 0, 1, \ldots, 10$ the variety of Heyting algebras generated by $\mathbf{DQ}_i$ is var-relatively alg-universal modulo $\mathbf{C}_2$ or $\mathbf{C}_2^0$ and contains an A-D family, thus it is Q-universal. The posets $\mathbf{Q}_i$ are shown in Figure 1.*

For $i = 0, 1, 2$ the variety of Heyting algebras generated by $\{\mathbf{DF}_i, \mathbf{DG}_i\}$ is var-relatively ff-alg-universal and contains an A-D family, so that it is Q-universal. The posets $\mathbf{F}_i$ and $\mathbf{G}_i$ with $i = 0, 1, 2$ are shown in Figure 2.

It should be pointed out that it was already known to the authors of [1] that any variety $\mathbb{V}$ of Heyting algebras containing $\mathbf{DQ}_i$ with $i = 0, 1, 3, 6$ is Q-universal.

M. E. Adams and W. Dziobiak [4] proved that the variety of Heyting algebras generated by $\{\mathbf{DH}_1, \mathbf{DG}_1\}$ has an A-D family and thus it is Q-universal. This variety is a subvariety of the variety generated by $\{\mathbf{DF}_1, \mathbf{DG}_1\}$. This then generalizes our result. It is an open question whether the variety generated by $\{\mathbf{DH}_1, \mathbf{DG}_1\}$ is var-relatively alg-universal modulo $\mathbf{M}$ for some monoid $\mathbf{M}$ and it is also an open problem whether the varieties generated by $\{\mathbf{DF}_i, \mathbf{DG}_i\}$ for $i = 1, 2$ are minimal such that they contain an A-D family or whether they are var-relatively alg-universal modulo $\mathbf{M}$ for a monoid $\mathbf{M}$. The varieties generated by $\{\mathbf{DQ}_i\}$ for $i = 0, 1, \ldots, 10$ are minimal with respect to Q-universality and minimal var-relative alg-universal varieties modulo a monoid $\mathbf{M}$.

As a consequence we then obtain

Corollary 1.4. *The variety of Heyting algebras generated by $\mathbf{DS}_i$ with $i = 0, 1, \ldots, 11$ is var-relatively alg-universal and contains an A-D family; thus it is Q-universal. The posets $\mathbf{S}_i$ for $i = 0, 1, \ldots, 11$ are shown in Figure 3.*

Corollary 1.5. *Let $\mathbb{V}$ be a variety of Heyting algebras such that there exist a finite poset $\mathbf{P}$ whose dual belongs to $\mathbb{V}$ and there exist a three-element antichain $\{x_1, x_2, x_3\}$ in $\mathbf{P}$ and an*

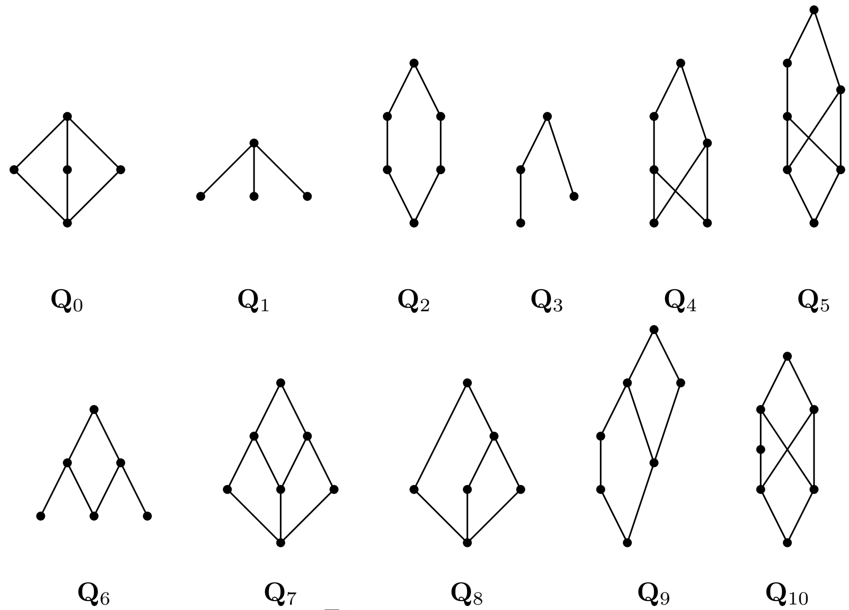


Figure 1

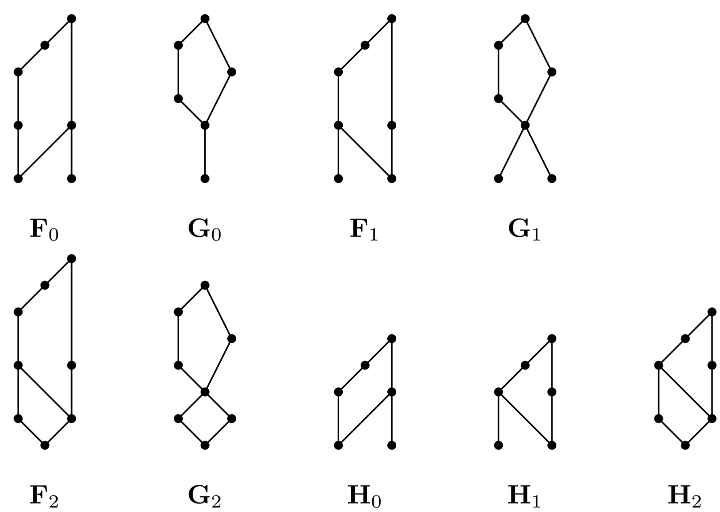


Figure 2

element y from $\mathbf{P}$ such that $x_i \leq y$ for $i = 1, 2, 3$. Then $\mathbb{V}$ is var-relatively ff-alg-universal and contains an A-D family, so that it is also Q-universal.

Our results use Priestley duality and the category of functors from a poset into compact totally disconnected spaces. We recall the Priestley duality for Heyting algebras in the second section. Let $\mathbb{C}$ be the category of compact totally disconnected spaces and continuous mappings between them, and for a poset $\mathbf{P}$, let $\mathbb{C}^{\mathbf{P}}$ be the category of all functors from $\mathbf{P}$ into $\mathbb{C}$ and all natural transformations between them. The third section is devoted to construction of relatively full embeddings from a suitable subcategory of $\mathbb{C}^{\mathbf{P}}$ to categorical duals of varieties of Heyting algebras. The subsequent sections apply these results to finitely generated varieties of Heyting algebras. The last section summarizes our results in detail.

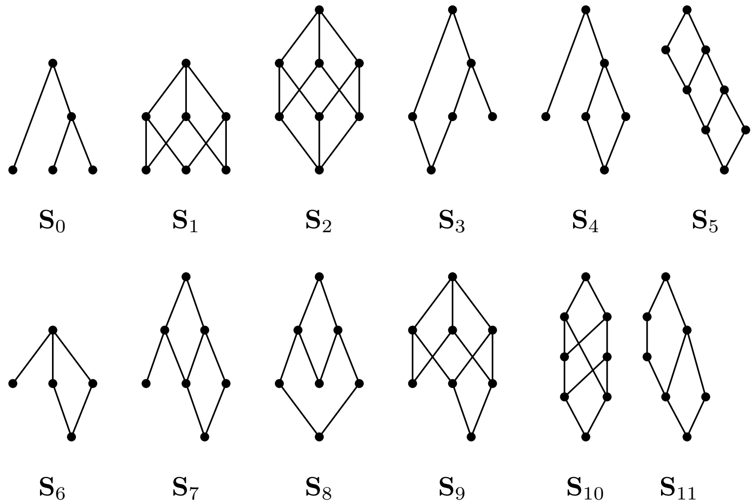


Figure 3

2. Priestley duality and some basic facts

Throughout the paper, we extensively use Priestley's duality for distributive (0,1)-lattices. We begin with a brief review of Priestley's duality and its application to Heyting algebras.

For a poset P and any $Q \subseteq P$ denote $[Q) = \{x \in P \mid \exists y \in Q, x \geq y\}$ and $(Q] = \{x \in P \mid \exists y \in Q, x \leq y\}$. A subset Q of P is decreasing if $(Q] = Q$, increasing if $[Q) = Q$, and convex if $[Q) \cap (Q] = Q$. Let $\mathrm{Min}(P)$ denote the set of all minimal elements of P and $\mathrm{Max}(P)$ denote the set of all maximal elements of P. For $Q \subseteq P$ denote $\mathrm{Min}(Q) = (Q] \cap \mathrm{Min}(P)$ and $\mathrm{Max}(Q) = [Q) \cap \mathrm{Max}(P)$. If $x \in P$ we shall write $(x]$ or $[x)$ instead of $(\{x\}]$ or $[\{x\})$ and, analogously, we shall write $\mathrm{Min}(x)$ and $\mathrm{Max}(x)$. A poset P is connected if for every pair $x, y \in P$ there exists a sequence $\{x_0, x_1, ..., x_n\}$ such that $x = x_0$, $x_n = y$, and x_i is comparable to x_{i+1} in P for every $i = 0, 1, ..., n-1$. If P is a poset then a maximal connected subposet (with respect to inclusion) is called its order component. We say that y covers x and x is covered by y if $\{y, x\} = (y] \cap [x)$; then (y, x) is called a covering pair. For $x \in P$, let $\mathrm{Cov}(x)$ denote the set consisting of all elements covered by x.

Let $\mathcal{X} = (X; \leq, \tau)$ be an ordered topological space; that is, $(X; \tau)$ is a topological space and $(X; \leq)$ is a poset. A subset $Z \subseteq X$ is clopen if it is both closed and open in τ. We say that $\mathcal{X}$ is a Priestley space if τ is compact and totally order disconnected; this means that for any $x, y \in X$ with $x \not\leq y$ there exists a clopen decreasing set $U \subseteq X$ with $y \in U$, $x \notin U$. Let $\mathbb{P}$ denote the category of all Priestley spaces and all their continuous order preserving mappings.

Clopen decreasing sets of any Priestley space form a distributive (0,1)-lattice, and the inverse image map f^{-1} of any $\mathbb{P}$-morphism f is a (0,1)-homomorphism of these lattices. This gives rise to a contravariant functor $\mathbf{D} : \mathbb{P} \to \mathbb{D}$ from the category of Priestley spaces into the category $\mathbb{D}$ of all distributive (0,1)-lattices and their (0,1)-homomorphisms. Conversely, for any distributive (0,1)-lattice L, let $\mathbf{P}(L) = (F(L); \leq, \tau)$ where $(F(L); \leq)$ is the poset of all prime filters of L ordered by the reversed inclusion, and an open subbasis of τ is formed by all sets $\{x \in F(L) \mid a \in x\}$, $\{x \in F(L) \mid a \notin x\}$ for $a \in L$. According to H. A. Priestley [19], $\mathbf{P}(L)$ is a Priestley space and if $f : L \to L'$ is a (0,1)-homomorphism then f^{-1} is continuous order preserving mapping from $\mathbf{P}(L')$ to $\mathbf{P}(L)$. This determines a contravariant functor $\mathbf{P} : \mathbb{D} \to \mathbb{P}$.

Theorem 2.1 ([19, 20])**.** *The two functors $\mathbf{P}\circ\mathbf{D} : \mathbb{P} \to \mathbb{P}$ and $\mathbf{D}\circ\mathbf{P} : \mathbb{D} \to \mathbb{D}$ are naturally equivalent to the identity functors. Therefore $\mathbb{D}$ is dually isomorphic to $\mathbb{P}$. Moreover, a Priestley space $\mathcal{X}$ is finite if and only if $\mathbf{P}\mathcal{X}$ is a finite distributive $(0,1)$-lattice.*

A Priestley space $\mathcal{X} = (X; \leq, \tau)$ is called an h-space if $[U)$ is clopen for every clopen set $U \subseteq X$. A mapping $h : P \to P'$ between two posets has an h-property if $(f(x)] = f((x])$ for all $x \in P$. An order preserving continuous mapping $f : (X; \leq, \tau) \to (Y; \leq, \sigma)$ with h-property is called an h-mapping. The subcategory of $\mathbb{P}$ formed by all h-spaces and h-mappings is denoted by $\mathbb{PH}$. The following claim is folklore.

Theorem 2.2 ([21])**.** *For a Priestley space $\mathcal{X}$, $\mathbf{D}\mathcal{X}$ is a Heyting algebra if and only if $\mathcal{X}$ is an h-space. For an order preserving continuous mapping $f : \mathcal{X} \to \mathcal{Y}$ between two h-spaces, $\mathbf{D}f$ is a homomorphism of Heyting algebras if and only if f is an h-mapping. Therefore $\mathbb{PH}$ is dually isomorphic to the variety $\mathbb{H}$ of all Heyting algebras and their homomorphisms.*

For an h-space $\mathcal{X} = (X, \leq, \tau)$, clearly $A \to B = X \setminus (A \setminus B)$ for any clopen decreasing sets $A, B \subseteq X$.

The claim below is folklore.

Statement 2.3. *A homomorphism $h : H \to H'$ between Heyting algebras is injective if and only if $\mathbf{P}h$ is surjective. A homomorphism $h : H \to H'$ between Heyting algebras is surjective if and only if $\mathbf{P}h$ is injective.*

The authors [17] have shown that for a family $\{f_i : A_i \to B \mid i \in I\}$ of order preserving continuous mappings the lattice $\mathbf{D}B$ is a subdirect product of $\{\mathbf{D}A_i \mid i \in I\}$ whenever $\cup\{\mathrm{Im}(f_i) \mid i \in I\}$ is dense in X. Since products of Heyting algebras and distributive (0,1)-lattices coincide, we immediately obtain the following folklore statement in which, for a variety $\mathbb{V}$ of Heyting algebras, $\mathbb{PHV}$ denotes the full subcategory of $\mathbb{PH}$ determined by all h-spaces $\mathbf{P}H$ with $H \in \mathbb{V}$.

Theorem 2.4. *An h-space $\mathcal{X} = (X; \leq, \tau)$ is a dual of subdirectly irreducible Heyting algebra if and only if $\mathrm{Max}(X)$ is a clopen singleton. For a variety $\mathbb{V}$ of Heyting algebras one has $\mathcal{X} = (X, \leq, \tau) \in \mathbb{PHV}$ if and only if $(x] \in \mathbb{PHV}$ for every $x \in X$.*

Note that for any h-map $f : (X; \leq, \tau) \to (Y; \leq, \sigma)$ of h-spaces, its image $\mathrm{Im}(f) \subseteq Y$ is closed decreasing and for all $y \in \mathrm{Im}(f)$ if $v \in f^{-1}(y)$ and $x \in (y]$ there is some $u \in (v]$ with $f(u) = x$. Thus f factorizes through $\mathbb{PHV}$ for a variety $\mathbb{V}$ of Heyting algebras if and only if the h-subspace of $(Y; \leq, \sigma)$ on $\mathrm{Im}(f)$ belongs to $\mathbb{PHV}$. Since the congruence lattice of any Heyting algebra is distributive, any finitely generated variety $\mathbb{V}$ of Heyting algebras has only finitely many subdirectly irreducible algebras and all of these are finite. Their duals are also finite, and we may use Theorem 2.4 to determine whether $\mathcal{X} = (X; \leq, \tau)$ belongs to $\mathbb{PHV}$. Hence for every finitely generated variety of Heyting algebras $\mathbb{V}$ there exists a natural number n such that for every $\mathcal{X} = (X; \leq, \tau) \in \mathbb{PHV}$ and for every $x \in X$ we have $|(x]| \leq n$.

If $\mathcal{S}$ is a class of h-spaces then $\mathrm{Var}(\mathcal{S})$ denotes the least variety $\mathbb{V}$ of Heyting algebras such that $\mathbf{D}\mathcal{X} \in \mathbb{V}$ for all $\mathcal{X} \in \mathcal{S}$. If $\mathcal{S} = \{\mathcal{X}\}$ then we shall write only $\mathrm{Var}(\mathcal{X})$ instead of $\mathrm{Var}(\{\mathcal{X}\})$.

Clearly, any finite poset $(X, \leq)$ with the discrete topology is an h-space. Thus

Theorem 2.5. *Let $\mathcal{S} = \{P_i \mid i \in I\}$ be a finite set of finite posets such that $|\mathrm{Max}(P_i)| = 1$ for all $i \in I$ and let $\mathbb{V} = \mathrm{Var}\{P_i \mid i \in I\}$ be a variety of Heyting algebras. Then an h-space $\mathcal{X} = (X; \leq, \tau)$ belongs to $\mathbb{PHV}$ if and only if for every $x \in \mathrm{Max}(X)$ there exist $i \in I$, a finite poset P, an injective order preserving mapping $f : (x] \to P$ with the h-property and a surjective order preserving mapping $g : P_i \to P$ with the h-property.*

Proof. Let $\mathbf{A} \in \mathbb{V}$ be a Heyting algebra and let $\mathcal{X} = (X; \leq, \tau)$ be its h-space. Since $\mathbf{A} \in \mathbb{V}$, there exists a family $\{\mathbf{A}_j \mid j \in J\}$ of Heyting algebras such that for every $j \in J$ there exists $i \in I$ such that $\mathbf{A}_j$ is isomorphic to $\mathbf{D}P_i$, an injective Heyting algebra homomorphism $f : \mathbf{B} \to \prod_{j \in J} \mathbf{A}_j$ and a surjective homomorphism $g : \mathbf{B} \to \mathbf{A}$. Let us denote $\mathbf{PB} = (Y; \leq, \tau)$ and $f_1 = \mathbf{P}f$, $g_1 = \mathbf{P}g$. Then, by Statement 2.3, g_1 is injective and f_1 is surjective. Since the product of Heyting algebras coincides with the product of underlying distributive $(0,1)$-lattices we conclude that the dual h-space of $\prod_{j \in J} \mathbf{A}_j$ coincides with the Priestley space of the corresponding underlying distributive $(0,1)$-lattice. Thus, by [17], every order component of the dual h-space $(Z; \leq, \tau)$ of $\prod_{j \in J} \mathbf{A}_j$ is isomorphic to P_i for some $i \in I$. Hence for every $y \in \mathrm{Max}(Y)$ there exists $z \in \mathrm{Max}(Z)$ such that $f_1(z) = y$. Choose $x \in \mathrm{Max}(X)$ and $y \in \mathrm{Max}(Y)$ with $g_1(x) \in (y]$. Let us denote $P = (y]$. Then the domain-range restriction g_2 of g_1 to $(x]$ and P is an injective order preserving mapping from $(x]$ into P with the h-property. Since for every $z \in \mathrm{Max}(Z)$ there exists $i \in I$ such that $(z]$ is isomorphic to P_i and since there exists $z \in \mathrm{Max}(Z)$ with $f_1(z) = y$, there exists an surjective order preserving mapping $f_2 : P_i \to P$ with the h-property. Thus for every $x \in \mathrm{Max}(X)$ there exist $i \in I$, a finite poset P, an injective order preserving mapping $f : (x] \to P$ with the h-property and a surjective order preserving mapping $g : P_i \to P$ with the h-property.

Conversely, assume that $\mathcal{X} = (X; \leq, \tau)$ is an h-space such that for every $x \in \mathrm{Max}(X)$ there exist $i_x \in I$, a finite poset P_x, an injective order preserving mapping $f : (x] \to P_x$ with the h-property and a surjective order preserving mapping $g : P_{i_x} \to P_x$ with the h-property. Since $\mathbb{V}$ has products, we infer that $\mathbb{PHV}$ has coproducts and, by the universal property of coproducts, there exists a surjective h-map $g : \coprod_{x \in \mathrm{Max}(X)} \to \mathcal{X}$. By Statement 2.3, $\mathbf{D}g\mathbf{D}(\coprod_{x \in \mathrm{Max}(X)} P_x) \to \mathbf{D}\mathcal{X}$ is injective, and, by Theorem 2.2, $\mathbf{D}\coprod_{x \in \mathrm{Max}(X)} P_x = \prod_{x \in \mathrm{Max}(X)} \mathbf{D}P_x \in \mathbb{V}$. Whence $\mathcal{X} \in \mathbb{PHV}$. ☐

Corollary 2.6. *Let $\mathcal{X} = (X; \leq, \tau)$ and $\mathcal{Y} = (Y; \leq, \tau)$ be h-spaces. Then* $\mathrm{Var}(\mathcal{X}) \subseteq \mathrm{Var}(\mathcal{Y})$ *whenever for every $x \in \mathrm{Max}(X)$ there exists some $y \in \mathcal{Y}$ such that the h-subspaces $(x]$ and $(y]$ are isomorphic.*

If there exists a natural number n such that $|(x]| \leq n$ for all $x \in \mathrm{Max}(X)$ and if $f : \mathcal{X} \to \mathcal{X}$ is an endomorphism of $\mathcal{X}$ such that there exists $x \in \mathrm{Max}(X)$ satisfying

- *if there exist $y \in \mathrm{Max}(X)$, a poset P, a surjective h-map $g : (y] \to P$ and an injective h-map $h : P \to (x]$, then $(x]$ and $(y]$ are isomorphic;*
- *there exists no $z \in \mathrm{Im}(f)$ such that $(z]$ and $(x]$ are isomorphic*

then $\mathrm{Var}((\mathrm{Im}(f))) \neq \mathrm{Var}(\mathcal{X})$.

3. General constructions

A poset $P = (P; \leq)$ will be considered as a category in which $\eta_{q,p} : q \to p$ denotes the P-morphism for each pair $q \geq p$ in P.

In what follows we assume that $\mathbb{V}$ is a finitely generated variety of Heyting algebras and $\mathcal{X} = (X; \leq, \tau) \in \mathbb{PHV}$ is an h-space. A finite convex open subset $U \subseteq X$ is called <u>functorial</u>, and $U(\mathcal{X})$ denotes the induced subposet of $(X; \leq)$ on the set U. Let us denote $\Gamma_{\mathcal{X},U}$ the constant functor from $\mathbb{C}^{U(\mathcal{X})}$ to $\mathbb{PHV}$ with the value $\mathcal{X}$. For a functorial set U of an h-space $\mathcal{X} = (X; \leq, \tau) \in \mathbb{PHV}$ we give a canonical construction of a functor $\Psi_{\mathcal{X},U} : \mathbb{C}^{U(\mathcal{X})} \to \mathbb{PHV}$ and a natural transformation $\psi : \Psi_{\mathcal{X},U} \to \Gamma_{\mathcal{X},U}$. For simplicity, let us denote $U = U(\mathcal{X}) = (U; \leq)$. For a functor $F : U \to \mathbb{C}$ define

- a set $Y_F = (X \setminus U) \cup (\bigcup_{u \in U} Fu)$ (we assume that $X \cap Fu = \emptyset$ for all $u \in U$ and that $Fu \cap Fv = \emptyset$ whenever $u, v \in U$ are distinct);

- a mapping $\psi^F : Y_F \to X$ given by
$$\psi^F(y) = \begin{cases} y & \text{if } y \in X \setminus U, \\ u & \text{if } y \in Fu \text{ for some } u \in U; \end{cases}$$
- a binary relation $\leq$ on Y_F such that $y \leq z$ for $y, z \in Y_F$ just when
  $y, z \in X \setminus U$ and $y \leq z$ in $\mathcal{X}$;
  $y \in X \setminus U$, $z \in Fu$ for some $u \in U$ with $y \leq u$ in $\mathcal{X}$;
  $y \in Fu$, $z \in X \setminus U$ for some $u \in U$ with $u \leq z$ in $\mathcal{X}$;
  $y \in Fu$, $z \in Fv$ for some $u, v \in U$ with $u \leq v$ in $\mathcal{X}$ and $F\eta_{v,u}(z) = y$;
- a topology τ as the union topology of the finitely many topologies of the Boolean spaces $Fu \in \mathbb{C}$ with $u \in U$ and the topology on $\mathcal{X} \setminus U$.

It is straightforward to verify that $\leq$ is a partial order on Y_F and that $Z \subseteq Y_F$ is τ-open exactly when $Z \cap Fu$ is an open set of Fu for all $u \in U$ and also $Z \setminus (\psi^F)^{-1}(U)$ is an open set of $\mathcal{X}$. Clearly, for any functor $F : U \to \mathbb{C}$ and for every $x \in \Psi_{\mathcal{X},U}F$ we have

$$\psi^F(\mathrm{Cov}(x)) = \mathrm{Cov}(\psi^F(x)). \tag{1}$$

For a natural transformation $\phi : F \to G$ between functors $F, G : U \to \mathbb{C}$ define a mapping $\Psi_{\mathcal{X},U}\phi : Y_F \to Y_G$ by
$$\Psi_{\mathcal{X},U}\phi(y) = \begin{cases} y & \text{if } y \in X \setminus U, \\ \phi^u(y) & \text{if } y \in Fu \text{ for some } u \in U. \end{cases}$$

We say that a family of mappings $\{f_i : X \to Y_i \mid i \in I\}$ with the same domain X is <u>separating</u> if for any two distinct $x, y \in X$ there is some $i \in I$ with $f_i(x) \neq f_i(y)$. A family of mappings $\{f_i : X_i \to Y \mid i \in I\}$ with the same codomain Y is <u>covering</u> if $Y = \bigcup_{i \in I} \mathrm{Im}(f_i)$. We reformulate this for transformations between functors from a poset P to $\mathbb{C}$ as follows. A family $\{\phi_i : F_i \to F \mid i \in I\}$ of transformations is a covering family in $\mathbb{C}^P$ if $Fp = \bigcup_{i \in I} \mathrm{Im}(\phi_i^p) = \bigcup_{i \in I} \phi_i^p(F_i p)$ for every $p \in P$. Analogously, a family $\{\phi_i : F \to F_i \mid i \in I\}$ of transformations is a separating family in $\mathbb{C}^P$ if for every $p \in P$ and every distinct $x, y \in Fp$ there exists $i \in I$ with $\phi_i^p(x) \neq \phi_i^p(y)$.

We prove

Theorem 3.1. *For every functorial set U of $\mathcal{X} \in \mathbb{PHV}$, $\Psi_{\mathcal{X},U} : \mathbb{C}^{U(\mathcal{X})} \to \mathbb{PHV}$ is a functor and $\psi = \{\psi^F \mid F \in \mathbb{C}^{U(\mathcal{X})}\} : \Psi_{\mathcal{X},U} \to \Gamma_{\mathcal{X},U}$ is a natural transformation such that ψ^F is surjective for every $F \in \mathbb{C}^{U(\mathcal{X})}$ and ψ^F is injective on $(y]$ for every $F \in \mathbb{C}^{U(\mathcal{X})}$ and every element y of $\Psi_{\mathcal{X},U}F$. Furthermore,*

(1) *a family $\{\phi_i : F_i \to F \mid i \in I\}$ is a covering family in $\mathbb{C}^{U(\mathcal{X})}$ if and only if $\{\Psi_{\mathcal{X},U}\phi_i \mid i \in I\}$ is a covering family in $\mathbb{PHV}$;*
(2) *a family $\{\phi_i : F \to F_i \mid i \in I\}$ is a separating family in $\mathbb{C}^{U(\mathcal{X})}$ if and only if $\{\Psi_{\mathcal{X},U}\phi_i \mid i \in I\}$ is a separating family in $\mathbb{PHV}$.*

Proof. For simplicity, let us denote $U = U(\mathcal{X}) = (U; \leq)$. For a functor $F : U \to \mathbb{C}$ let us denote $\Psi_{\mathcal{X},U}F = (Y_F; \leq, \tau)$. First we prove that $(Y_F; \leq, \tau)$ is an h-space from $\mathbb{PHV}$ and $\psi^F : \Psi_{\mathcal{X},U}F \to \mathcal{X}$ is a surjective h-map such that ψ^F is injective on $(y]$ for all $y \in Y_F$.

Since every $Fv \in \mathbb{C}$ is non-void and because ψ^F is the identity map on $X \setminus U$, the mapping ψ^F is surjective. For every $y \in Fv$, the subposet
$$(y] = ((X \setminus U) \cap (v]) \bigcup \{F\eta_{v,u}(y) \mid u \leq v\}$$
of Y_F is isomorphic to the poset $(v] \subseteq \mathcal{X}$, so that ψ^F preserves the order, is injective on $(y]$ and has the h-property. Since $(\psi^F)^{-1}\{v\} = Fv$ for every $v \in U$ and $(\psi^F)^{-1}\{x\} = \{x\}$ for

every $x \in X \setminus U$, the continuity of ψ^F follows. Once we show that $\Psi_{\mathcal{X},U}F$ is an h-space, it will follow that ψ^F is an h-map.

As the union of finitely many compact spaces, the space $(Y_F;\tau)$ is compact. Let $V \subseteq Y_F$ be a τ-clopen set we prove that $[V)$ is also τ-clopen. By the definition of topology, $\psi^F(V)$ is clopen in $\mathcal{X}$ and $V \cap Fu$ is clopen in Fu for every $u \in U$. The set $[\psi^F(V)) \setminus U$ is τ-clopen because $\psi^F(V)$ and U are clopen in $\mathcal{X}$. Since $[V \cap Fu) \cap Fv = (F\eta_{v,u})^{-1}(V \cap Fu)$ is non-void only when $u \leq v$ and $V \cap Fu \neq \emptyset$ because the $\mathbb{C}$-morphism $F\eta_{v,u}$ is continuous, the latter subset of Fv is clopen in Fv, and hence also in Y_F. Then

$$[V) = ([\psi^F(V)) \setminus U) \cup \bigcup \{[V \cap Fu) \cap Fv \mid u, v \in U,\, u \leq v,\, Fu \cap V \neq \emptyset\}$$

and therefore $[V)$ is τ-clopen for every τ-clopen $V \subseteq Y_F$. Thus if we prove that $(Y_F;\leq,\tau)$ is totally order disconnected then τ has the basis consisting of all clopen sets and hence $[V)$ is τ-open for every τ-open $V \subseteq Y_F$.

To show that $\Psi_{\mathcal{X},U}F$ is totally order disconnected, we suppose that $y \not\leq z$ in $(Y_F;\leq)$ and exhibit a τ-clopen decreasing set $Z \subseteq Y_F$ containing z but not y. Since ψ^F is continuous and preserves order, if $\psi^F(y) \not\leq \psi^F(z)$ then the required Z exists (indeed since $\mathcal{X}$ is an h-space there exists a clopen decreasing set $Z' \subseteq X$ with $\psi^F(z) \in Z'$ and $\psi^F(y) \notin Z'$ and it suffices to take $Z = (\psi^F)^{-1}(Z')$). Thus it is enough to consider $y \not\leq z$ such that $\psi^F(y) \leq \psi^F(z)$ in $\mathcal{X}$. If $\psi^F(y) \notin U$ then $y = \psi^F(y)$ and hence $y \leq v$ for all $v \in (\psi^F)^{-1}(\psi^F(z))$, contrary to this hypothesis. Analogously, if $\psi^F(z) \notin U$ then $z = \psi^F(z)$ and hence $v \leq z$ for all $v \in (\psi^F)^{-1}(\psi^F(y))$, contrary to this hypothesis. Thus $y \in Fu$ and $z \in Fv$ for some $u, v \in U$ with $u \leq v$ and $y \neq F\eta_{v,u}(z)$. There is a set $W \subseteq Fu$ clopen in Fu such that $F\eta_{v,u}(z) \notin W$ and $y \in W$. Since W is τ-clopen, the set $[W)$ is also τ-clopen. Clearly $z \in Z = Y_F \setminus [W)$ and $y \notin Z$. Thus $\Psi_{\mathcal{X},U}F$ is totally order disconnected, and this completes the proof that $\Psi_{\mathcal{X},U}F$ is an h-space.

The spaces $(y]$ and $(\psi_F(y)]$ are isomorphic for every $y \in Y_F$ and, by Corollary 2.6, we obtain that $\mathcal{X} \in \mathbb{PHV}$ implies that $\Psi_{\mathcal{X},U}F \in \mathbb{PHV}$.

To complete the proof that $\Psi_{\mathcal{X},U}$ is a functor, consider functors $F, G : U \to \mathbb{C}$ and a natural transformation $\phi : F \to G$. Let us denote $\Psi_{\mathcal{X},U}F = (Y_F;\leq,\tau)$ and $\Psi_{\mathcal{X},U}G = (Y_G;\leq,\tau)$. Since for any $u, v \in U$ with $u \leq v$ and for every $y \in Fv$ we have $\phi^u(F\eta_{v,u}(y)) = G\eta_{v,u}(\phi^v(y))$, we conclude that $\Psi_{\mathcal{X},U}\phi$ preserves order and has the h-property.

To see that $\Psi_{\mathcal{X},U}\phi$ is continuous, let $V \subseteq Y_G$ be clopen in $\Psi_{\mathcal{X},U}G$. Then $V \setminus (\psi^F)^{-1}(U)$ and the finitely many sets $V \cap Gv$ with $v \in U$ are clopen in $\Psi_{\mathcal{X},U}G$. Since every $\mathbb{C}$-morphism $\phi^v : Fv \to Gv$ is continuous, the subset $(\phi^v)^{-1}(V \cap Gv)$ of Fv is clopen in Fv. But then the finite union

$$(\Psi_{\mathcal{X},U}\phi)^{-1}(V) = (V \setminus (\psi^F)^{-1}(U)) \bigcup \{(\phi^v)^{-1}(V \cap Gv) \mid v \in U\}$$

is clopen in $\Psi_{\mathcal{X},U}F$, and the continuity of $\Psi_{\mathcal{X},U}F\phi$ follows because every h-space has an open basis formed by clopen sets. Altogether, $\Psi_{\mathcal{X},U}$ is a well-defined functor from $\mathbb{C}^U$ to $\mathbb{PHV}$, and it is also clear that $\Psi_{\mathcal{X},U}$ is faithful. It is routine to verify that $\psi^F = \psi^G \circ \Psi_{\mathcal{X},U}\phi$ for every transformation $\phi : F \to G$ where $F, G \in \mathbb{C}^{\mathbf{U}}$, thus $\psi = \{\psi^F \mid F\mathbf{U} \to \mathbb{C}$ is a functor $\}$ is a surjective transformation from $\Psi_{\mathcal{X},U}$ into the constant functor with the value $\mathcal{X}$. Hence $\Psi_{\mathcal{X},U} : \mathbb{C}^U \to \mathbb{PHV}$ is a functor and $\psi : \Psi_{\mathcal{X},U} \to \Gamma_{\mathcal{X},U}$ is a natural transformation. It is then routine to verify (1) and (2). $\square$

For an increasing open subset $A \subseteq X$, let $\mathcal{X} \setminus A$ denote the h-space $(X \setminus A;\leq,\tau)$ where both the partial order and the topology on $X \setminus A$ are inherited from $\mathcal{X}$.

Next we generalize the notion of a relatively full embedding for our purposes. Let $\mathbb{K}$ be a category and $\Phi : \mathbb{K} \to \mathbb{PHV}$ be a functor. Assume that $\mathcal{Z} = \{Z_K \mid K \in \mathbb{K}^o\}$ is a family of sets with $Z_K \subseteq \mathrm{Max}(\Phi K)$ for all $K \in \mathbb{K}^o$ and $\mathcal{G} = \{\mu_K \mathbf{G} \to \mathrm{Aut}(\Phi K) \mid K \in \mathbb{K}^o\}$ is a family of injective group homomorphisms from a fixed group $\mathbf{G}$. Then we say that Φ is a $(\mathcal{Z},\mathcal{G})$-relatively full embedding if

(e0) $\mu_K(a)(Z_K) = Z_K$ for every element a of the group $\mathbf{G}$ and for every $\mathbb{K}$-object K;
(e1) Φ is faithful;
(e2) $Z_{K'} \cap \mathrm{Im}(\Phi f) \neq \emptyset$ for every $\mathbb{K}$-morphism $f : K \to K'$;
(e3) if $f : \Phi K \to \Phi K'$ is an h-map for $\mathbb{K}$-objects K and K' then either $Z_{K'} \cap \mathrm{Im}(f) = \emptyset$ or there exist a $\mathbb{K}$-morphism $g : K \to K'$ and an element a of $\mathbf{G}$ with $f = \mu_{K'}(a) \circ \Phi g = \Phi g \circ \mu_K(a)$.

If $\mathbf{G}$ is a singleton group then we omit $\mathcal{G}$. Thus if we say that Φ is a $\mathcal{Z}$-relatively full embedding then Φ is a $(\mathcal{Z},\mathcal{G})$-relatively full embedding where $\mathbf{G}$ is a singleton group.

If $\mathbb{W}$ is a proper subvariety of $\mathbb{V}$ such that for every $\mathbb{K}$-object K we have $(x] \in \mathbb{PHW}$ for all elements x of $\Phi K \setminus Z_K$ and $(z] \notin \mathbb{PHW}$ for all $z \in Z_K$, then $\Phi : \mathbb{K} \to \mathbb{PHV}$ is a $\mathbb{W}$-relatively full embedding modulo $\mathbf{G}$ (and if $\mathbf{G}$ is a singleton group then Ψ is a $\mathbb{W}$-relatively full embedding).

We will develop specific proof techniques for the existence of an A-D family and for relative var-universality.

An h-space $\mathcal{X} = (X; \leq, \tau) \in \mathbb{V}$ is a (U, C)-representing object of $\mathbb{V}$ if

(r1) $U \subseteq X$ is functorial and $C \subseteq \mathrm{Max}(U)$;
(r2) $f(u) = u$ for every automorphism f of $\mathcal{X}$ and every $u \in U$;
(r3) $|\mathrm{Cov}(c)| \geq 2$ for all $c \in C$ and $\mathrm{Cov}(c) \neq \mathrm{Cov}(x)$ for all $c \in C$ and $x \in X \setminus \{c\}$;
(r4) if $f : \mathcal{X} \setminus C \to \mathcal{X}$ is an h-map then either $f = g \circ \iota$ where $\iota : X \setminus C \to X$ is the inclusion and g is an automorphism of $\mathcal{X}$ or else f is not injective on $X \setminus U$.

We say that an h-space $\mathcal{X} = (X; \leq, \tau) \in \mathbb{PHV}$ is a (U, C, Z)-testing object of $\mathbb{V}$ if

(t1) $\mathcal{X}$ is a (U, C)-representing object and $Z \subseteq \mathrm{Max}(X)$;
(t2) $f(Z) = Z$ for every automorphism f of $\mathcal{X}$;
(t3) if $z_1, z_2 \in Z$ then $(z_1]$ and $(z_2]$ are isomorphic;
(t4) either $C \cap Z = \emptyset$ and for every $z \in Z$ and $c \in C$ there exists no surjective h-map from $(c]$ onto $(z]$, or $C \subseteq Z$ and $(x]$ is isomorphic to $(z]$ for every $x \in \mathrm{Max}(X)$ and $z \in Z$ such that there exists a surjective h-map from $(x]$ onto $(z]$;
(t5) for every h-map $f : \mathcal{X} \setminus C \to \mathcal{X}$ such that f is not injective on $X \setminus U$ we have $\mathrm{Im}(f) \cap Z = \emptyset$ and either for every $z \in Z$ there exists $u_z \in \mathrm{Cov}(z)$ such that $u_z \notin \mathrm{Im}(g)$ for every h-map $g : \mathcal{X} \setminus C \to \mathcal{X}$ with $g \restriction X \setminus U = f \restriction X \setminus U$ or else for every $c \in C \cap Z$ there exists $u_c \in (c] \setminus C$ such that every h-map $g : \mathcal{X} \setminus C \to \mathcal{X}$ with $g \restriction X \setminus U = f \restriction X \setminus U$ is not injective on $(u_c]$.

Let us assume that $\mathcal{X}$ is a (U, C, Z)-testing object. Then for every $F \in \mathbb{C}^{U(\mathcal{X})}$ we set $Z_F = (\psi^F)^{-1}(Z)$. Let $\mu_F : \mathrm{Aut}(\mathcal{X}) \to \mathrm{Aut}(\Psi_{\mathcal{X},U} F)$ be a mapping such that for every automorphism $f \in \mathrm{Aut}(\mathcal{X})$

$$\mu_F f(u) = \begin{cases} u & \text{if } u \notin X \setminus U, \\ f(u) & \text{if } u \in X \setminus U, \end{cases}$$

for every element u of $\Psi_{\mathcal{X},U} F$. Since f is an automorphism of $\mathcal{X}$ then, by (r2) and the fact that ψ is a surjective transformation with $(\psi^F)^{-1}(x) = \{x\}$ for all $x \in X \setminus U$ we conclude that $\mu_F(f)$ preserves order and has the h-property. Since U is a clopen set we obtain the continuity of $\mu_F(f)$. Thus $\mu_F(f)$ is an h-map and the bijectivity of $\mu_F(f)$ immediately follows from the definition. Hence $\mu_F(f)$ is an automorphism of $\Psi_{\mathcal{X},U} F$. By a straightforward calculation, we find that μ_F maps the identity mapping to the identity mapping and preserves composition. Thus μ_F is a group homomorphism from $\mathrm{Aut}(\mathcal{X})$ to

$\mathrm{Aut}(\Psi_{\mathcal{X},U}F)$. Clearly, μ_F is injective. Set $\mathcal{Z} = \{Z_F \mid F \in \mathbb{C}^{U(\mathcal{X})}\}$ and $\mathcal{G} = \{\mu_F : \mathrm{Aut}(\mathcal{X}) \to \mathrm{Aut}(\Psi_{\mathcal{X},U}F) \mid F \in \mathbb{C}^{U(\mathcal{X})}\}$.

Let $U = (U; \leq)$ be a poset with an increasing set C and let $F : U \to \mathbb{C}$ be a functor. Then a family $\{x_u \mid u \in U \setminus C\}$ is called a C-coherent family if

(c1) $x_u \in Fu$ for all $u \in U \setminus C$;
(c2) if $u, u' \in U \setminus C$ with $u \leq u'$ then $F\eta_{u',u}(x_{u'}) = x_u$.

If for every $u_0 \in U \setminus C$ and every $x \in Fu_0$ there exists a C-coherent family $\{x_u \mid u \in U \setminus C\}$ such that $x = x_{u_0}$, then we say that the functor F is C-coherent. Let $C(\mathbb{C}^U)$ denote the full subcategory of $\mathbb{C}^U$ formed by C-coherent functors from U to $\mathbb{C}$.

Theorem 3.2. *If $\mathcal{X}$ is a (U, C)-representing object of $\mathbb{V}$ then*

$$\Psi_{\mathcal{X},U} : C(\mathbb{C}^{U(\mathcal{X})}) \to \mathbb{PHV}$$

is a faithful functor and if $f : \Psi_{\mathcal{X},U}F \to \Psi_{\mathcal{X},U}G$ is a h-map for $F, G \in C(\mathbb{C}^{U(\mathcal{X})})$ then either there exist a natural transformation $\phi : F \to G$ and an automorphism $g \in \mathrm{Aut}(\mathcal{X})$ with $f = \mu_G(g) \circ \Psi_{\mathcal{X},U}\phi = \Psi_{\mathcal{X},U}\phi \circ \mu_F(g)$ or else $\psi^G \circ f$ is not injective on $X \setminus U$. Moreover, if $\mathcal{X}$ is a (U, C, Z)-testing object then $\Psi_{\mathcal{X},U} : C(\mathbb{C}^{U(\mathcal{X})}) \to \mathbb{PHV}$ is a $(\mathcal{Z}, \mathcal{G})$-relatively full embedding.

Proof. By Theorem 3.1, $\Psi_{\mathcal{X},U} : C(\mathbb{C}^{U(\mathcal{X})}) \to \mathbb{PHV}$ is a faithful functor. Consider an h-map $f : \Psi_{\mathcal{X},U}F \to \Psi_{\mathcal{X},U}G$ for $F, G \in C(\mathbb{C}^{U(\mathcal{X})}$. For every C-coherent family $\mathcal{F} = \{x_u \mid u \in U \setminus C\}$ of F define $f_{\mathcal{F}} : \mathcal{X} \setminus C \to \Psi_{\mathcal{X},U}F$ by

$$f_{\mathcal{F}}(u) = \begin{cases} u & \text{if } u \in X \setminus U, \\ x_u & \text{if } u \in U \setminus C. \end{cases}$$

Since U is finite and clopen and because $V \cap (X \setminus U)$ is clopen for every clopen set $V \subseteq \Psi_{\mathcal{X},U}F$, we conclude that $f_{\mathcal{F}}$ is continuous. Since the partial orders of $\mathcal{X}$ and of $\Psi_{\mathcal{X},U}F$ coincide on $X \setminus U$ and because $x_u \leq x_v$ in $\Psi_{\mathcal{X},U}F$ for $u, v \in U \setminus C$ if and only if $u \leq v$ in $\mathcal{X}$, we conclude that $f_{\mathcal{F}}$ preserves order and has the h-property. Thus $f_{\mathcal{F}} : \mathcal{X} \setminus C \to \Psi_{\mathcal{X},U}F$ is an h-map for every C-coherent family $\mathcal{F}$ of F. Then $\psi^G \circ f \circ f_{\mathcal{F}} : \mathcal{X} \setminus C \to \mathcal{X}$ is an h-map for every C-coherent family $\mathcal{F}$ of F.

By (r4), either $\psi^G \circ f \circ f_{\mathcal{F}} = g \circ \iota$ for $g \in \mathrm{Aut}(\mathcal{X})$ and the inclusion ι or $\psi^G \circ f \circ f_{\mathcal{F}}$ is not injective on $X \setminus U$. From $f_{\mathcal{F}}(x) = x$ for all $x \in X \setminus U$ we infer that $\psi^G \circ f$ is injective in the first case and $\psi^G \circ f$ is not injective on $X \setminus U$ in the second case. Suppose that there exists a C-coherent family $\mathcal{F}'$ of F such that $\psi^G \circ f \circ f_{\mathcal{F}} = g \circ \iota$ for some $g \in \mathrm{Aut}(\mathcal{X})$. Then $\psi^G \circ f \circ f_{\mathcal{F}}(x) = \psi^G \circ f \circ f_{\mathcal{F}'}(x)$ for every C-coherent family $\mathcal{F}$ of F and every $x \in X \setminus U$. Thus $\psi^G \circ f \circ f_{\mathcal{F}}$ is injective on $X \setminus U$ and, by (r4), there exists $g_{\mathcal{F}} \in \mathrm{Aut}(\mathcal{X})$ with $\psi^G \circ f \circ f_{\mathcal{F}} = g_{\mathcal{F}} \circ \iota$. By (r2), $g(u) = u$ for every $u \in U$ and every $g \in \mathrm{Aut}(\mathcal{X})$, thus there exists $g \in \mathrm{Aut}(\mathcal{X})$ such that $g = g_{\mathcal{F}}$ and $\psi^G \circ f \circ f_{\mathcal{F}}(u) = u$ for all $u \in U \setminus C$ and any C-coherent family $\mathcal{F}$ of F. From this it follows that $f(Fu) \subseteq Gu$ for all $u \in U \setminus C$.

From (r3) it follows that $|\mathrm{Cov}(c)| \geq 2$ for all $c \in C$ and that $\mathrm{Cov}(c) \neq \mathrm{Cov}(x)$ for all $x \in X \setminus \{c\}$. Consider $c \in C$. By (1), $\psi^F(\mathrm{Cov}(x)) = \mathrm{Cov}(c)$ for every $x \in Fc$. Hence $\psi^G \circ f(x) = c$ and therefore $f(Fc) \subseteq Gc$. We conclude that $f(Fu) \subseteq Gu$ for all $u \in U$. Let ϕ^u be the domain-range restriction of f to Fu and Gu. Then ϕ^u is a continuous mapping from Fu to Gu and since f preserves order we infer that

$$\phi^v \circ F\eta_{u,v} = G\eta_{u,v} \circ \phi^u$$

for all $u, v \in U$ with $u \leq v$. Whence $\phi = \{\phi^u \mid u \in U\} : F \to G$ is a natural transformation such that $\Psi_{\mathcal{X},U}\phi(x) = f(x)$ for all $x \in \bigcup_{u \in U} Fu$ and $f(x) = g(x)$ for all $x \in X \setminus U$. Since $\mu_G(g)(y) = y$ for all $y \in \bigcup_{u \in U} Gu$ and $\mu_F(g)(x) = x$ for all $x \in \bigcup_{u \in U} Fu$ we conclude that

$f = \Psi_{\mathcal{X},U}\phi \circ \mu_F(g) = \mu_G(g) \circ \Psi_{\mathcal{X},U}\phi$ or $\psi^G \circ f$ is not injective on $X \setminus U$. Thus the first statement holds.

Let $\mathcal{X}$ be a (U, C, Z)-testing object. Then (e0) follows from (t2). By (t1), $\mathcal{X}$ is a (U, C)-representing object and, by the foregoing part of the proof, we obtain (e1). For every natural transformation $\phi : F \to G$ with $F, G \in \mathbb{C}^U$ we have $\Psi_{\mathcal{X},U}\phi(Fu) \subseteq Gu$ for every $u \in U$, and since $Fu \neq \emptyset$ for every $u \in U$ we conclude that $\mathrm{Im}(\Psi_{\mathcal{X},U}\phi) \cap Gz \neq \emptyset$ for every $z \in Z \cap U$. By (r1) and (r3), $\Psi_{\mathcal{X},U}\phi(X \setminus U) = X \setminus U$, thus $\mathrm{Im}\,\Psi_{\mathcal{X},U}\phi \cap (\psi^G)^{-1}(Z) \neq \emptyset$ and (e2) holds.

To prove (e3), consider an h-map $f : \Psi_{\mathcal{X},U}F \to \Psi_{\mathcal{X},U}G$ for $F, G \in C(\mathbb{C}^{U(\mathcal{X})})$ such that $f = \mu_G(g) \circ \Psi_{\mathcal{X},U}\phi$ for no pair (g, ϕ) where $g \in \mathrm{Aut}(\mathcal{X})$ and $\phi : F \to G$ is a natural transformation. Then, by the foregoing part of the proof, we conclude that $\psi^G \circ f$ is not injective on $X \setminus U$. Consider that $\mathrm{Im}(\psi^G \circ f) \cap Z \neq \emptyset$. Then there exist $z \in Z$ and $x \in \mathrm{Max}(\Psi_{\mathcal{X},U}F)$ with $\psi^G \circ f(x) = z$. Let $y = \psi^F(x)$. If $y \notin C$ then there exists a C-coherent family $\mathcal{F}$ with $x = f_{\mathcal{F}}(y)$. Hence $\psi^G \circ f \circ f_{\mathcal{F}}(y) = z$ and this is a contradiction because, by (t5), $\mathrm{Im}(g) \cap Z = \emptyset$ for every h-map $g : \mathcal{X} \setminus C \to \mathcal{X}$ such that g is not injective on $X \setminus U$. Thus we can assume that $y \in C$. Since ψ^F is injective on $(x]$ there exists a mapping $h : (y] \to (z]$ with $h \circ \psi^F \restriction (x] = \psi^G \circ f \restriction (x]$. Since $\psi^G \circ f$ and ψ^F are h-maps, we infer that h is also an h-map, and thus it is surjective. Hence, by (t4), $C \subseteq Z$. Choose a C-coherent family $\mathcal{F}$ then $\psi^G \circ f \circ f_{\mathcal{F}}$ is not injective on $X \setminus U$. By (t5), either there exists $u_z \in \mathrm{Cov}(z)$ such that $u_z \notin \mathrm{Im}(g)$ for every $g : \mathcal{X} \setminus C \to \mathcal{X}$ with $g \restriction X \setminus U = \psi^G \circ f \circ f_{\mathcal{F}} \restriction X \setminus U$ or there exists $u_y \in (y] \setminus C$ such that every $g : \mathcal{X} \setminus C \to \mathcal{X}$ with $g \restriction X \setminus U = \psi^G \circ f \circ f_{\mathcal{F}} \restriction X \setminus U$ is not injective on $(u_y]$. In the first case, $z = f(x)$ implies that $u_z \in \psi^G \circ f(x]$. Thus there exists $v_x \in (x] \setminus \{x\}$ with $f(v_x) = u_z$. Hence $v_x \notin \mathrm{Max}(X)$ and thus $\psi^F(v_x) = y_x \notin C$. Therefore there exists a C-coherent family $\mathcal{F}'$ containing v_x. Then $\psi^G \circ f \circ f_{\mathcal{F}'}(y_x) = \psi^G \circ f(v_x) = u_x$ but $\psi^G \circ f \circ f_{\mathcal{F}'} \restriction X \setminus U = \psi^G \circ f \circ f_{\mathcal{F}} \restriction X \setminus U$ and this is a contradiction. In the second case, there exists $w_x \in (x]$ with $\psi^F(w_x) = u_y$ because ψ^F is a h-map and $\psi^F(x) = y$. Let $\mathcal{F}'$ be a C-coherent family containing w_x. Since $\psi^G \circ f \circ f_{\mathcal{F}} \restriction X \setminus U = \psi^G \circ f \circ f_{\mathcal{F}'} \restriction X \setminus U$ we conclude that $\psi^G \circ f \circ f_{\mathcal{F}'}$ is not injective on $(u_y]$, but $f_{\mathcal{F}'}$ is injective and thus $\psi^G \circ f$ is not injective on $(w_x]$ and whence $\psi^G \circ f$ is not injective on $(x]$. Since $(x]$ is isomorphic to $(y]$ we conclude, by (t3), that $(x]$ is isomorphic to $(z]$ and this is a contradiction with $\psi^G \circ f(x) = z$ because $(x]$ is finite. Whence $\mathrm{Im}(f) \cap (\psi^G)^{-1}(Z) = \mathrm{Im}(f) \cap Z_G = \emptyset$ and (e3) is true. $\square$

To apply Theorem 3.2 in a construction of an A-D family we need the following concepts and technical statements.

Let $P = (P; \leq)$ be a poset. A functor $F : P \to \mathbb{C}$ is an isofunctor, if $F\eta_{p,q}$ is an homeomorphism for every $p, q \in P$ with $p \leq q$. We recall that every continuous bijection between compact Hausdorff spaces is a homeomorphism. Further a functor $F : P \to \mathbb{C}$ is finite if Fp is finite for every $p \in P$.

We claim that if $\{\phi_i : F_i \to F \mid i \in I\}$ is a family of natural transformations between functors from P into $\mathbb{C}$ such that F is finite then there exist a finite functor $G : P \to \mathbb{C}$, a covering family $\{\rho_i : F_i \to G \mid i \in I\}$ of natural transformations and an injective natural transformation $\nu : G \to F$ such that $\phi_i = \nu \circ \rho_i$ for all $i \in I$. Indeed, for every $p \in P$ let us define $Gp = \bigcup_{i \in I} \mathrm{Im}(\phi_i)^p \subseteq Fp$. From the finiteness of Fp it follows that $Gp \in \mathbb{C}$. Consider $q, p \in P$ with $q \leq p$. If $x \in Gp$ then there exist $i \in I$ and $y \in F_i p$ with $(\phi_i)^p(y) = x$. Hence $F\eta_{p,q}(x) = F\eta_{p,q} \circ (\phi_i)^p(y) = (\phi_i)^q \circ F_i\eta_{p,q}(y) \in Gq$. Thus we can define $G\eta_{p,q}$ as the domain-range restriction of $F\eta_{p,q}$ and we conclude that G is a finite functor from P into $\mathbb{C}$ and if ν^p is the inclusion from Gp into Fp then $\nu : G \to F$ is an injective natural transformation. For every $i \in I$ and $p \in P$, let ρ_i^p be the range restriction of ϕ_i^p. Then, by a direct calculation, we obtain that $\rho_i : F_i \to G$ is a natural transformation, that $\phi_i = \nu \circ \rho_i$ for all $i \in I$ and that $\{\rho_i \mid i \in I\}$ is a covering family in $\mathbb{C}^{\mathbf{P}}$. We shall write $G = \bigcup_{i \in I} \rho_i$.

The technical lemma below gives a stronger version of a diagonalization property of a factorization system for h-spaces which are $\Psi_{\mathcal{X},U}$-images of finite isofunctors.

Lemma 3.3. *Let $\mathcal{X} = (X; \leq, \tau)$ be a (U, C, Z)-testing object such that X is finite and $\Psi_{\mathcal{X},U}$ is a $(\mathcal{Z}, \mathcal{G})$-relatively full embedding. Let $\{F_i \mid i \in I\}$ and F be C-coherent finite isofunctors from $U = U(\mathcal{X})$ to $\mathbb{C}$ such that*

1. *for every $i \in I$, if $\alpha, \beta : F_i \to F$ are natural transformations, then there exists a natural equivalence $\nu : F_i \to F_i$ with $\alpha = \beta \circ \nu$;*
2. *$C \neq U$.*

Let $\mathcal{Y} = (Y; \leq, \tau)$ be a finite h-space such that there exist an injective h-map $g : \mathcal{Y} \to \Psi_{\mathcal{X},U} F$ and a covering family $\{f_{i,j} : \Psi_{\mathcal{X},U} F_i \to \mathcal{Y} \mid i \in I, j \in J_i\}$ of h-maps (the case $J_i = \emptyset$ is allowed) such that

$$I' = \{i \in I \mid \exists j \in J_i, \operatorname{Im}(g \circ f_{i,j}) \cap (\psi^F)^{-1} Z \neq \emptyset\}$$

is a non-empty set. Then there exist a family $\{\alpha_i : F_i \to F \mid i \in I'\}$ of natural transformations and h-maps $h' : \Psi_{\mathcal{X},U}(\bigcup_{i \in I'} \alpha_i) \to \mathcal{Y}$ and $h : \mathcal{Y} \to \Psi_{\mathcal{X},U}(\bigcup_{i \in I'} \alpha_i)$ such that $h \circ h'$ is the identity map of $\Psi_{\mathcal{X},U}(\bigcup_{i \in I'} \alpha_i)$.

Proof. Assume that $\mathcal{X} = (X; \leq, \tau)$ is a finite (U, C, Z)-testing object satisfying (2) and such that $\Psi_{\mathcal{X},U}$ is a relatively $(\mathcal{Z}, \mathcal{G})$-full embedding where $\mathcal{Z}$ and $\mathcal{G}$ are defined just before Theorem 3.2, and $\{F_i \mid i \in I\}$ and F are C-coherent finite isofunctors from U to $\mathbb{C}$ satisfying (1). Let $\mathcal{Y} = (Y; \leq, \tau)$ be an h-space such that there exists an injective h-map $g : \mathcal{Y} \to \Psi_{\mathcal{X},U} F$ and a covering family $\{f_{i,j} : \Psi_{\mathcal{X},U} F_i \to \mathcal{Y} \mid i \in I, j \in J_i\}$ of h-maps (the case $J_i = \emptyset$ is allowed) such that

$$I' = \{i \in I \mid \exists j \in J_i, \operatorname{Im}(g \circ f_{i,j}) \cap (\psi^F)^{-1}(Z) \neq \emptyset\}$$

is a non-empty set. For every $i \in I'$ set

$$J'_i = \{j \in J_i \mid \operatorname{Im}(g \circ f_{i,j}) \cap (\psi^F)^{-1} Z \neq \emptyset\}.$$

Then, by the hypothesis, for every $i \in I'$ and $j \in J'_i$ there exist a natural transformation $\alpha_{i,j} : F_i \to F$ and an h-automorphism $\sigma_{i,j}$ of $\mathcal{X}$ with $g \circ f_{i,j} = \Psi_{\mathcal{X},U}(\alpha_{i,j}) \circ \mu_{F_i}(\sigma_{i,j})$. For every $i \in I'$ choose $j(i) \in J'_i$. By (1), for every $i \in I'$ and $j \in J'_i$ there exists a natural equivalence $\beta_{i,j}$ of F_i with $\alpha_{i,j} = \alpha_{i,j(i)} \circ \beta_{i,j}$. Thus $\operatorname{Im}(\alpha_{i,j})^u = \operatorname{Im}(\alpha_{i,j(i)})^u$ for all $u \in U$. For simplicity we shall write α_i instead of $\alpha_{i,j(i)}$ and we set $G = \bigcup_{i \in I'} \alpha_i$. By the definition of union of functors, there exist a family $\{\rho_i : F_i \to G \mid i \in I'\}$ of natural transformations and a natural transformation $\nu : G \to F$ such that $\nu \circ \rho_i = \alpha_i$ for all $i \in I'$, the family $\{\rho_i \mid i \in I'\}$ is covering, and ν is an injective natural transformation.

By Theorem 3.1, $\{\Psi_{\mathcal{X},U}(\rho_i) \mid i \in I'\}$ is a covering family and $\Psi_{\mathcal{X},U}(\nu)$ is injective. Thus $\{\Psi_{\mathcal{X},U}(\rho_i \circ \beta_{i,j}) \circ \mu_{F_i}(\sigma_{i,j}) \mid i \in I', j \in J'_i\}$ is a covering family (because $\Psi_{\mathcal{X},U}(\beta_{i,j})$ and $\mu_{F_i}(\sigma_{i,j})$ are h-isomorphisms). Since g is also an injective h-map and

$$g \circ f_{i,j} = \Psi_{\mathcal{X},U}(\alpha_{i,j}) \circ \mu_{F_i}(\sigma_{i,j}) = \Psi_{\mathcal{X},U}(\nu) \circ \Psi_{\mathcal{X},U}(\rho_i \circ \beta_{i,j}) \circ \mu_{F_i}(\sigma_{i,j})$$

for all $i \in I'$ and $j \in J'_i$, by the diagonalization property of the factorization system, there exists an h-map $h' : \Psi_{\mathcal{X},U} G \to \mathcal{Y}$ with $f_{i,j} = h' \circ \Psi_{\mathcal{X},U}(\rho_i \circ \beta_{i,j}) \circ \mu_{F_i}(\sigma_{i,j})$ for all $i \in I'$ and $j \in J'_i$, and $g \circ h' = \Psi_{\mathcal{X},U}(\nu)$. By Theorem 3.1, $\Psi_{\mathcal{X},U}(\nu)$ is injective and hence the h-map h' is also injective.

It remains to produce an h-map $h : Y \to Y_G$ such that $h \circ h'$ is the identity mapping of $\Psi_{\mathcal{X},U} G$.

First, for any $y \in Y$ of the form $y = h'(x)$ for some $x \in Y_G$ we set $h(y) = x$. Then $h((y]) = (h(y)]$ for all $y \in \operatorname{Im}(h')$ because $(h'(y)] \subseteq \operatorname{Im}(h')$ for all $y \in \operatorname{Im}(h')$. Thus h has the h-property on $\operatorname{Im}(h') \subseteq Y$, and $h \circ h'$ will be the identity on $\Psi_{\mathcal{X},U} G$ regardless of how h will be defined on $Y \setminus \operatorname{Im}(h')$.

Let us denote $Y' = Y \setminus \mathrm{Im}(h')$ and $Y'' = (\psi^F \circ g)^{-1}(X \setminus U)$. Then $Y' \cap Y'' = \emptyset$ because $(\psi^F)^{-1}(X \setminus U) \subseteq \mathrm{Im}(\Psi_{\mathcal{X},U}\alpha_i)$ for all $i \in I'$ and thus $Y'' \subseteq \mathrm{Im}(h')$.

Consider $y \in Y \setminus (Y' \cup Y'')$. Then there exist $i \in I'$ and $a \in F_i u$ for $u = \psi^F \circ g(y) \in U$ with $f_{i,j(i)}(a) = y$ and $(y] \subseteq \mathrm{Im}(f_{i,j(i)}) \subseteq \mathrm{Im}(h')$. If $y' \in [y)$ then $g(y') \geq g(y)$ and thus $\psi^G \circ g(y') \geq \psi^G \circ g(y)$. From the injectivity of g and of ψ^G on $(u]$ for every $u \in \Psi_{\mathcal{X},U}F$ we infer that $\psi^G \circ g(y') = \psi^G \circ g(y)$ if and only if $y' = y$. Then either $\psi^G \circ g(y') \in U$ or $\psi^G \circ g(y') \in X \setminus U$.

First we assume that $\psi^G \circ g(y') = u' \in U$. Since F_i is an isofunctor, there exists $a' \in F_i u'$ with $F_i\eta_{u',u}(a') = a$. Then $g(y') \geq g(y) = g \circ f_{i,j}(a) \leq g \circ f_{i,j}(a')$ and $g(y'), g \circ f_{i,j}(a') \in Fu'$. Since F is an isofunctor we conclude that $g(y') = g \circ f_{i,j}(a')$ and the injectivity of g implies $f_{i,j}(a') = y'$. If $\psi^G \circ g(y') = w \notin X \setminus U$, then $g \circ f_{i,j} = \Psi_{\mathcal{X},U}(\alpha_{i,j}) \circ \mu_{F_i}(\sigma_{i,j})$ implies, by (r2), that

$$g \circ f_{i,j}(\mu_{F_i}(\sigma_{i,j})^{-1}(w)) = \Psi_{\mathcal{X},U}(\alpha_{i,j}) \circ \mu_{F_i}(\sigma_{i,j})(\mu_{F_i}(\sigma_{i,j})^{-1}(w)) = \Psi_{\mathcal{X},U}(\alpha_{i,j})(w) = w.$$

From the injectivity of g it then follows that $f_{i,j}(w) = y'$ and thus $y' \in \mathrm{Im}(h')$. Therefore we conclude that $[y) \subseteq \mathrm{Im}(f_{i,j(i)}) \subseteq \mathrm{Im}(h')$. Whence $(Y'] \subseteq Y' \cup Y''$ and $[Y') \subseteq Y'$.

Consider $y \in Y'$. Then there exist $i \in I$, $j \in J_i$ and $a \in F_i$ with $f_{i,j}(a) = y'$ and either $i \notin I'$ or $i \in I'$ and $j \notin J'_i$. Hence $g(y) \notin (\psi^F)^{-1}(Z)$.

Since $I' \neq \emptyset$ we can select $i \in I'$ and a C-coherent family $\{x_v \mid v \in U \setminus C\}$ of F_i because F_i is C-coherent, any space in $\mathbb{C}$ is non-empty and, by (2), $C \neq U$. Then $\{\Psi_{\mathcal{X},U}(\alpha_i)^v(x_v) \mid v \in U \setminus C\} \subseteq \Psi_{\mathcal{X},U}G$ is a C-coherent family of F. Thus we can fix a C-coherent family of $\{x_v \mid v \in U \setminus C\} \subseteq \Psi_{\mathcal{X},U}G$ of F.

Since $C \subseteq Z$ we infer $\psi^G \circ g(y) \notin Z$ for all $y \in Y'$ and thus there exist a C-coherent family $\{x'_v \mid v \in U \setminus C\}$ such that $g(y) = x'_w$ for some $w \in U \setminus C$ because F is C-coherent. Since F is isofunctor two distinct C-coherent families are disjoint. Thus for every $v \in U \setminus C$ we set $h(g^{-1}(x'_v)) = x_v$. Then h is defined on Y, preserves order and has the h-property.

Since X is finite we obtain that $\Psi_{\mathcal{X},U}F$ is a finite h-space and also $\Psi_{\mathcal{X},U}G$ is a finite h-space. The finiteness of $\mathcal{Y}$ implies the continuity of h. Therefore $h : \mathcal{Y} \to \Psi_{\mathcal{X},U}G$ is an h-map such that $h \circ h'$ is the identity of $\Psi_{\mathcal{X},U}G$. $\square$

Let $\mathcal{P}(\omega)$ be the set of all finite subsets of the set ω of all natural numbers and let $\mathcal{P}(\omega_0)$ be the subset of $\mathcal{P}(\omega)$ consisting of all non-empty finite subsets of ω. Denote $\mathbb{N}$ the poset of all members of $\mathcal{P}(\omega)$ ordered by the reversed inclusion.

Theorem 3.4. *Let $\mathcal{X}$ be a finite (U, C, Z)-testing object of $\mathbb{V}$ such that $U(\mathcal{X})$ is order connected, $Z \cap U \neq \emptyset$ and $C \neq U$. Let $\Phi : \mathbb{N} \to \mathbb{C}^U$ be a functor such that*

(1) *if $\alpha : \Phi A \to \Phi B$ is a natural transformation for $A, B \in \mathcal{P}(\omega_0)$, then $A \subseteq B$ and there exists a natural equivalence $\mu : \Phi A \to \Phi A$ with $\alpha = \Phi\eta_{A,B} \circ \mu$;*
(2) *ΦA is a finite C-coherent isofunctor for all $A \in \mathcal{P}(\omega_0)$;*
(3) *if $A, B \in \mathcal{P}(\omega_0)$ with $A \subseteq B$ then $\Phi\eta_{A,B}$ is an injective natural transformation;*
(4) *if $A, B, C \in \mathcal{P}(\omega_0)$ then $A = B \cup C$ if and only if $\{\Phi\eta_{C,A}, \Phi\eta_{B,A}\}$ is a covering family.*

Then $\mathbb{V}$ contains an A-D family and thus $\mathbb{V}$ is Q-universal.

Proof. Set $\Lambda = \Psi_{\mathcal{X},U} \circ \Phi$. Since $\mathcal{X}$ is finite we infer, by (2), that ΛA is finite for every $A \in \mathcal{P}(\omega_0)$. We prove that the family $\{\Lambda A \mid A \in \mathcal{P}(\omega_0)\}$ satisfies the following conditions:

(dp2) if $A, B_1, B_2 \in \mathcal{P}(\omega_0)$ with $A = B_1 \cup B_2$ then $\{\Lambda\eta_{B_1,A}, \Lambda\eta_{B_2,A}\}$ is a covering family;
(dp3) if $A, B \in \mathcal{P}(\omega_0)$ are such that the family of all h-maps from ΛA into ΛB is a covering family then $A = B$;
(dp4) if $\mathcal{Y}$ and $\mathcal{Z}$ are finite h-spaces such that for a finite subset $\mathcal{A} \subseteq \mathcal{P}(\omega_0)$ the families $\{f : \Lambda B \to \mathcal{Y} \mid B \in \mathcal{A}\}$ and $\{f : \Lambda B \to \mathcal{Z} \mid B \in \mathcal{A}\}$ of h-maps are covering and if

$A \in \mathcal{P}(\omega_0)$ is a set such that there exists a surjective h-map $g : \mathcal{Y} \vee \mathcal{Z} \to \Lambda A$ then one of the following conditions holds:

- there exists a surjective h-map $h : \mathcal{Y} \to \Lambda A$,
- there exists a surjective h-map $h : \mathcal{Z} \to \Lambda A$,
- there exist $A_1, A_2 \in \mathcal{P}(\omega)$ with $A = A_1 \cup A_2$ and surjective h-maps $h_1 : \mathcal{Y} \to \Lambda A_1$, $h_2 : \mathcal{Z} \to \Lambda A_2$.

Then, by Priestley duality, the singleton Heyting algebra and the family of algebras dual to the members of $\{\Lambda A \mid A \in \mathcal{P}(\omega_0)\}$ constitute an A-D family and by Adams-Dziobiak theorem [3], $\mathbb{V}$ is Q-universal.

The condition (dp2) follows from (4) and Theorem 3.1.

Now we prove for a finite $\mathcal{A} \subseteq \mathcal{P}(\omega_0)$ and $A \in \mathcal{P}(\omega_0)$ that the family of all h-maps from ΛB, $B \in \mathcal{A}$ to ΛA is covering if and only if there exists $\mathcal{A}' \subseteq \mathcal{A}$ with $A = \bigcup_{B \in \mathcal{A}'} B$. If there exists $\mathcal{A}' \subseteq \mathcal{A}$ with $A = \bigcup_{B \in \mathcal{A}'} B$ then, by (4), we obtain that $\{\Lambda\eta_{B,A} \mid B \in \mathcal{A}'\}$ is a covering family, by an easy induction over $|\mathcal{A}'|$. Thus we assume that the family of all h-maps from ΛB, $B \in \mathcal{A}$ into ΛA is covering. Let us define

$$\mathcal{A}' = \{B \in \mathcal{A} \mid \exists \text{ an } h\text{-map } f : \Lambda B \to \Lambda A, \text{ with } \operatorname{Im}(f) \cap (\psi^{\Phi A})^{-1} Z \neq \emptyset\}$$

and $A' = \bigcup_{B \in \mathcal{A}'} B$. Since $\mathcal{X}$ is a (U, C, Z)-testing object of $\mathbb{V}$ we conclude, by Theorem 3.2, that $\Psi_{\mathcal{X},U} : C(\mathbb{C}^U) \to \mathbb{PHV}$ is a $(\mathcal{G}, \mathcal{Z})$-relatively full embedding. Thus for an h-map $f : \Lambda B \to \Lambda A$ and for $B \in \mathcal{A}$ such that $\operatorname{Im}(\psi^{\Phi A}) \circ f \cap Z \neq \emptyset$ there exist an automorphism $\alpha \in \operatorname{Aut}(\mathcal{X})$ and a natural transformation $\beta : \Phi B \to \Phi A$ with $f = \Psi_{\mathcal{X},U}\beta \circ \mu_B(\alpha) = \mu_A(\alpha) \circ \Psi_{\mathcal{X},U}\beta$ (μ_A is defined just before Theorem 3.2). Whence, by (1), $B \subseteq A$ and there exists a natural equivalence ν of ΦA such that $\beta = \Phi\eta_{B,A} \circ \nu$ and thus $f = \Psi_{\mathcal{X},U}(\Phi\eta_{B,A} \circ \nu) \circ \mu_B(\alpha) = \Lambda\eta_{B,A} \circ \Psi_{\mathcal{X},U}\nu \circ \mu_B(\alpha)$. As a consequence we obtain that $A' \subseteq A$ and $f = \Lambda\eta_{A',A} \circ \Lambda_{B,A'} \circ \Psi_{\mathcal{X},U}\nu \circ \mu_B(\alpha)$. Hence $\operatorname{Im}(f) \subseteq \operatorname{Im}(\Lambda\eta_{A',A})$. We recall that ΛC is finite for all $C \in \mathcal{P}(\omega)$. Since $\mathcal{A}$ is finite there are only finitely many h-maps from ΛB, $B \in \mathcal{A}$ to ΛA. Since the family of h-maps from ΛB, $B \in \mathcal{A}$ to ΛA is covering we conclude that $(\psi^{\Phi A})^{-1}(Z) \subseteq \bigcup \operatorname{Im}(f)$ where the union is taken over all h-maps $f : \Lambda B \to \Lambda A$ with $\operatorname{Im}(f) \cap (\psi^{\Phi A})^{-1}(Z) \neq \emptyset$ and hence we infer that $(\psi^{\Phi A})^{-1}(Z) \subseteq \operatorname{Im}\Lambda\eta_{A',A}$. Thus for $z \in U \cap Z$ we deduce that $(\Lambda\eta_{A',A})^z$ is surjective and, by (3) and Theorem 3.1, it is also injective. Hence $(\Lambda\eta_{A',A})^z$ is a homeomorphism, and, by Theorem 3.1, $(\Phi\eta_{A',A})^z$ is a homeomorphism. By (2), $\Phi A'$ and ΦA are finite isofunctors and, thus for $u, v \in U$ with $u \leq v$ we have that $(\Phi A)\eta_{v,u}$ and $(\Phi A')\eta_{v,u}$ are homeomorphisms. Hence $(\Phi\eta_{A',A})^u$ is a homeomorphism if and only if $(\Phi\eta_{A',A})^v$ is a homeomorphism and since $U(\mathcal{X})$ is order connected and $z \in U$ we infer that $(\Phi\eta_{A',A})^u$ is a homeomorphism for all $u \in U$. Thus, $\Phi\eta_{A',A}$ is a natural equivalence and, by (1), we obtain that $A = A'$ and the proof of (dp2) is complete.

To prove (dp3) consider $A, B \in \mathcal{P}(\omega_0)$ such that the family of h-maps from ΛB in ΛA is covering. Then by the foregoing part of the proof, for $\mathcal{A} = \{B\}$ we conclude that $A = B$ and (dp3) follows.

To prove (dp4), assume that there exist finite h-spaces $\mathcal{Y}$ and $\mathcal{Z}$, and $A \in \mathcal{P}(\omega_0)$ such that there exist a surjective h-map $g : \mathcal{Y} \vee \mathcal{Z} \to \Lambda A$ and a finite subset $\mathcal{A} \subseteq \mathcal{P}(\omega_0)$ and covering families $\{f : \Lambda B \to \mathcal{Y} \mid B \in \mathcal{A}\}$ and $\{f : \Lambda B \to \mathcal{Z} \mid B \in \mathcal{A}\}$ of h-maps. Let $\iota : \mathcal{Y} \to \mathcal{Y} \vee \mathcal{Z}$ be the sum inclusion. Then there exist a surjective h-map $h_1 : \mathcal{Y} \to \mathcal{Y}'$ and an injective h-map $h_2 : \mathcal{Y}' \to \Lambda A$ with $h_2 \circ h_1 = g \circ \iota$. Since the assumption (1) of Lemma 3.3 follows from (1) and the assumptions (2) and (3) of Lemma 3.3 are our assumptions on $\mathcal{X}$, U, C and Z, we conclude that the hypotheses of Lemma 3.3 are satisfied for the family of all h-maps from ΛB with $B \in \mathcal{A}$ to $\mathcal{Y}'$ and $h_2 : \mathcal{Y}' \to \Lambda A$. Let us denote

$$\mathcal{A}' = \{B \in \mathcal{A} \mid \exists \text{ an } h\text{-map } f : \Lambda B \to \mathcal{Y}' \text{ with } \operatorname{Im} h_2 \circ f \cap (\psi^{\Phi A})^{-1}(Z) \neq \emptyset\}.$$

If $\mathcal{A}' \neq \emptyset$ then, by Lemma 3.3, there exists a surjective h-map $g_1 : \mathcal{Y}' \to \bigcup_{B\in\mathcal{A}'} \Lambda\eta_{B,A}$. If we set $A_1 = \bigcup_{B\in\mathcal{A}'} B$ then from (1), (2) and (3) we infer that $\bigcup_{B\in\mathcal{A}'} \Lambda\eta_{B,A} = \Lambda A_1$ and thus $g_1 \circ h_1 : \mathcal{Y} \to \Lambda A_1$ is a surjective h-map. If $\mathcal{A}' = \emptyset$ then we set $A_1 = \emptyset$. If $\upsilon : \mathcal{Z} \to \mathcal{Y} \vee \mathcal{Z}$ is the sum inclusion then, analogously, there exist a surjective h-map $k_1 : \mathcal{Z} \to \mathcal{Z}'$ and an injective h-map $k_2 : \mathcal{Z}' \to \Lambda A$ with $k_2 \circ k_1 = g \circ \upsilon$. Let us denote

$$\mathcal{A}'' = \{B \in \mathcal{A} \mid \exists f : \Lambda B \to \mathcal{Z}' \text{ with } \operatorname{Im}(k_2 \circ f) \cap (\psi^{\Phi A})^{-1}(Z) \neq \emptyset\}.$$

If $\mathcal{A}'' \neq \emptyset$ then, by Lemma 3.3, there exists a surjective h-map $g_2 : \mathcal{Z}' \to \bigcup_{B\in\mathcal{A}''} \Lambda\eta_{B,A}$. Analogously to the above, we set $A_2 = \bigcup_{B\in\mathcal{A}''} B$ and then $\bigcup_{B\in\mathcal{A}''} \Lambda\eta_{B,A} = \Lambda A_2$, thus $g_2 \circ k_1 : \mathcal{Z} \to \Lambda A_2$ is a surjective h-map. If $\mathcal{A}'' = \emptyset$ then we set $A_2 = \emptyset$. Hence to complete the proof of (dp4) it suffices to show that $A = A_1 \cup A_2$. Since $\{f : \Lambda B \to \mathcal{Y} \mid B \in \mathcal{A}\}$ and $\{f : \Lambda B \to \mathcal{Z} \mid B \in A\}$ are covering families of h-maps and since $g : \mathcal{Y} \vee \mathcal{Z} \to \Lambda A$ is a surjective h-map we infer that the family $\{f : \Lambda B \to \Lambda A \mid B \in \mathcal{A}\}$ of all h-maps from ΛG, $B \in \mathcal{A}$ into ΛA is a covering family of h-maps. Hence, by the foregoing part of the proof, $A = A_1 \cup A_2$. $\square$

Let $P = (P; \leq)$ be a poset, then a covering pair (a, b) is called f-covering pair if $b \in \operatorname{Max}(P)$ and $(a] \cap (a'] = \emptyset$ for every $a' \in \operatorname{Cov}(b)$ with $a \neq a'$.

Let $\{p_i\}_{i=0}^{\infty}$ be an increasing sequence of prime numbers with $p_0 \geq 11$.

Let $P = (P; \leq)$ be a poset and let (a, b) be a f-covering pair of P such that the poset $(P; \preceq)$ is order connected where $\preceq$ is the least partial order such that $u \preceq v$ for all covering pairs $u \leq v$ of P other than (a, b). Let us denote $P \setminus \{a, b\} = (P; \preceq)$. Define a functor $\Phi : \mathbb{N} \to C(\mathbb{C}^P)$ as follows:

- for an $A \subseteq \mathcal{P}(\omega)$ define $(\Phi A)p = \{(p, i, j) \mid j \in A, i \in p_j\}$ for all $p \in P$ with the discrete topology and for $p, q \in P$ with $q \leq p$ define

$$(\Phi A)\eta_{p,q}(p, i, j) = \begin{cases} (q, i + 1 \bmod p_j, j) & \text{if } q \leq a < b = p \\ (q, i, j) & \text{else.} \end{cases}$$

  Since $(\Phi A)p$ is finite and non-empty for all $p \in P$ and (a, b) is a f-covering pair of P we claim that ΦA is a functor from P into $\mathbb{C}$ because, by the definition of f-covering pair, for $u, v \in P$ with $u \leq v < b$ we have $u \leq a$ if and only if $v \leq a$.
- if $A, B \in \mathcal{P}(\omega)$ and $A \subseteq B$ then $(\Phi\eta_{A,B})^p(p, i, j) = (p, i, j)$ for all $p \in P$, $j \in A$ and $i \in p_j$. Since $A \subseteq B$, the definition of $(\Phi\eta_{A,B})^p$ is correct for all $p \in P$. From the finiteness of $(\Phi A)p$ follows the continuity of $(\Phi\eta_{A,B})^p$ for all $p \in P$. It is straightforward to verify that $(\Phi B)\eta_{p,q} \circ (\Phi\eta_{A,B})^p = (\Phi\eta_{A,B})^q \circ (\Phi A)\eta_{p,q}$ for all $p, q \in P$ with $q \leq p$. Hence $\Phi\eta_{A,B} : \Phi A \to \Phi B$ is a natural transformation and therefore $\Phi : \mathbb{N} \to \mathbb{C}^{\mathbf{P}}$ is a functor.

Finally, set $C = \{b\}$. We prove

Theorem 3.5. *If P is a poset with a f-covering pair $a < b$ such that $P \setminus \{a, b\}$ is order connected then $\Phi : \mathbb{N} \to \mathbb{C}^P$ is a faithful functor such that*

1. *$(\Phi A)p$ is finite and non-empty for all $A \in \mathcal{P}(\omega)$ and all $p \in P$;*
2. *$(\Phi\eta_{A,B})^p$ is injective for all $A, B \in \mathcal{P}(\omega)$ with $A \subseteq B$ and all $p \in P$;*
3. *if $A, B_1, B_2 \in \mathcal{P}(\omega)$ then $A = B_1 \cup B_2$ iff $\{(\Phi\eta_{B_1,A})^p, (\Phi\eta_{B_2,A})^p\}$ is a covering family of ΦAp for all $p \in P$, hence $\{\Phi\eta_{B_1,A}, \Phi\eta_{B_2,A}\}$ is a covering family in $\mathbb{C}^P$;*
4. *if $\phi : \Phi A \to \Phi B$ is a natural transformation then $A \subseteq B$ and there exists a natural equivalence $\mu : \Phi A \to \Phi A$ with $\phi = \Phi\eta_{A,B} \circ \mu$;*
5. *for all $A \in \mathcal{P}(\omega)$, ΦA is a C-coherent finite isofunctor;*
6. *$(\Psi A)\eta_{p,q}$ is the identity mapping for all $A \in \mathcal{P}(\omega)$ and all $p, q \in P$ such that $q \leq p$ and $q \notin (a]$ or $b \neq p$.*

Proof. By the foregoing note, $\Phi : \mathbb{N} \to C(\mathbb{C}^{\mathbf{P}})$ is a functor such that ΦA is a finite isofunctor for all $A \in \mathcal{P}(\omega)$. A verification of faithfulness of Φ and of statements (1), (2), (3), (5), and (6) is straightforward. It remains to prove the statement (4).

Let $\phi : \Phi A \to \Phi B$ be a natural transformation. If $p, q \in P$ with $q \leq p$ and $q \notin (a]$ or $b \neq p$ then

$$\phi^q(q,i,j) = \phi^q \circ (\Phi A)\eta_{p,q}(p,i,j) = (\Phi B)\eta_{p,q} \circ \phi^p(p,i,j)$$

and hence there exists a mapping

$$f : \{(i,j) \mid j \in A,\ i \in p_j\} \to \{(i,j) \mid j \in B,\ i \in p_j\}$$

with $\phi^p(p,i,j) = (p, f(i,j))$ for all $p \in P$ because $(P; \preceq)$ is order connected. From

$$\phi^a \circ (\Phi A)\eta_{b,a}(b,i,j) = (\Phi B)\eta_{b,a} \circ \phi^b(b,i,j)$$

it follows that if $f(i,j) = (k,l)$ then $f(i+1 \bmod p_j, j) = (k+1 \bmod p_l, l)$ because

$$(\Phi C)\eta_{b,a}(b,i,j) = (b, i+1 \bmod p_j, j)$$

for all $C \in \mathcal{P}(\omega)$, $j \in C$ and $i \in p_j$. Since p_j and p_l are primes we conclude that $j = l$ and whence $A \subseteq B$ and $f(i,j) = (k,j)$ for all $j \in A$ and $i, k \in p_j$. Further if $f(i,j) = (k,j)$ then $f(i+l \bmod p_j, j) = (k+l \bmod p_j, j)$ for all l. Conversely, if $\mathfrak{k} = \{k_j \mid j \in A\}$ is a family of natural numbers $k_j \in p_j$ then we define $(\mu_{\mathfrak{k}})^p(p,i,j) = (p, i+k_j \bmod p_j, j)$ for all $j \in A$, $i \in p_j$ and $p \in P$. By a direct calculation, we obtain that $\mu_{\mathfrak{k}} = \{(\mu_{\mathfrak{k}})^p \mid p \in P\}$ is a natural equivalence of ΦA (we exploit the finiteness of $(\Phi A)p$ for all $p \in P$). If $f(0,j) = (k_j, j)$ then $\phi = \Phi\eta_{A,B} \circ \mu_{\mathfrak{k}}$ for $\mathfrak{k} = \{k_j \mid j \in A\}$ and the proof is complete. □

We say that a finite h-space $\mathcal{X} = (X; \leq, \tau) \in \mathbb{PHV}$ is a standard Q-testing object of $\mathbb{V}$ if there exist a functorial set $U \subseteq X$ of $(X; \leq)$ and a f-covering pair $a < b$ of U such that $U(\mathcal{X}) \setminus \{a, b\}$ is order connected and a set $Z \subseteq \mathrm{Max}(X)$ such that $b \in Z$ and $\mathcal{X}$ is a (U, C, Z)-testing object for $C = \{b\}$. Observe that $C \neq U$.

Corollary 3.6. *A finitely generated variety $\mathbb{V}$ of Heyting algebras contains an A-D family and thus it is Q-universal whenever there exists a standard Q-testing object $\mathcal{X}$ of $\mathbb{V}$.*

Proof. If $\mathcal{X}$ is a standard Q-testing object of $\mathbb{V}$, then U and $a < b$ satisfy the hypotheses of Theorem 3.5. The functor Φ from Theorem 3.5 satisfies the hypotheses of Theorem 3.4 and $\mathcal{X}$ satisfies the hypotheses of Theorem 3.4 on testing object (for $C = \{b\}$). An application of Theorem 3.4 then completes the proof. □

Next we prepare for 'testing' relative universality.

Let $\mathbb{A}(1,1)$ be the category of all unary algebras with two unary operations and all their homomorphisms, let $\mathbb{D}$ be the variety of all distributive $(0,1)$-lattices and all their $(0,1)$-homomorphisms. Let us define a category $\mathbb{L}$ whose objects are $(L; \tau, \vee, \wedge, 0, 1, \alpha, \beta)$ where $(L; \tau)$ is an object of $\mathbb{C}$, $(L; \vee, \wedge, 0, 1)$ is an object of $\mathbb{D}$, $(L; \alpha, \beta)$ is an object of $\mathbb{A}(1,1)$ such that $\vee$ and $\wedge$ are continuous mappings from $(L; \tau)^2$ to $(L; \tau)$ and α and β are continuous mappings from $(L; \tau)$ to itself, and $\mathbb{L}$-morphisms from $(L; \tau, \vee, \wedge, 0, 1, \alpha, \beta)$ into $(L'; \tau, \vee, \wedge, 0, 1, \alpha, \beta)$ are all mappings $f : L \to L'$ such that $f : (L; \tau) \to (L'; \tau)$ is a morphism of $\mathbb{C}$, $f : (L; \vee, \wedge, 0, 1) \to (L'; \vee, \wedge, 0, 1)$ is a morphism of $\mathbb{D}$ and $f : (L; \alpha, \beta) \to (L'; \alpha, \beta)$ is a morphism of $\mathbb{A}(1,1)$. We recall

Theorem 3.7 ([5])**.** *There is a contravariant faithful full functor* \$$\Phi : \mathbb{A}(1,1) \to \mathbb{L}$ *such that*

1. *$(X; \alpha, \beta)$ is a finite algebra if and only if $\Phi(X; \alpha, \beta)$ is a finite object of $\mathbb{L}$;*
2. *if $(X; \alpha, \beta)$ and $(X; \alpha', \beta')$ are unary algebras with the same underlying set X and if $\Phi(X, \alpha, \beta) = (L; \tau, \vee, \wedge, 0, 1, \alpha, \beta)$ and $\Phi(X, \alpha', \beta') = (L'; \tau, \vee, \wedge, 0, 1, \alpha, \beta)$ then we have $(L; \vee, \wedge, 0, 1) = (L'; \vee, \wedge, 0, 1)$ and $(L; \tau) = (L'; \tau)$.*

Let $U = (U; \leq)$ be a poset and $V \subseteq U$ its increasing subset, and let $E = \{(u_i, v_i) \mid i \in 7\}$ be a family of seven distinct f-covering pairs such that

(u1) V is order connected;
(u2) the poset $(U \setminus V; \preceq)$ is order connected; here $\preceq$ is the least order such that $u \preceq v$ for $u, v \in U \setminus V$ if $u \in \operatorname{Cov}(v)$ in U and $(u, v) \neq (u_i, v_i)$ for $i \in 7$;
(u3) $u_i \in U \setminus V$ for all $i \in 7$, $v_i \in V$ for $i \in 3$ and $v_i \in U \setminus V$ for $i = 3, 4, 5, 6$;
(u4) there exist $v \in V$, $u \in U \setminus V$ such that $u \in \operatorname{Cov}(v)$ and $(u, v) \neq (u_i, v_i)$ for all $i \in 3$.

Then we say that $\mathfrak{U} = (U, V, E = \{(u_i, v_i) \mid i \in 7\})$ is a u-triple. For a u-triple $\mathfrak{U} = (U, V, E)$ we shall define a functor $\Omega_{\mathfrak{U}} : \mathbb{L} \to \mathbb{C}^U$. For an $\mathbb{L}$-object $\mathcal{L} = (L; \tau, \vee, \wedge, 0, 1, \alpha, \beta)$, let $\Omega_{\mathfrak{U}}(\mathcal{L}) = F$ be a functor from U to $\mathbb{C}$ given by

- $Fu = (L; \tau)^2$ for $u \in V$ and $Fu = (L; \tau)$ for $u \in U \setminus V$;
- if $u, v \in V$ with $u \leq v$ then $F\eta_{v,u}$ is the identity of L^2;
- if $u, v \in U \setminus V$ with $u \leq v$ then

$$F\eta_{v,u} = \begin{cases} \alpha & \text{if } u \leq u_3 < v_3 = v, \\ \beta & \text{if } u \leq u_4 < v_4 = v, \\ c_0 & \text{if } u \leq u_5 < v_5 = v, \\ c_1 & \text{if } u \leq u_6 < v_6 = v, \\ 1_L & \text{if either } v \neq v_j \text{ for all } j = 3,4,5,6 \\ & \quad \text{or } v = v_j \text{ and } u \notin (u_j] \text{ for some } j = 3,4,5,6, \end{cases}$$

  where c_x is the constant mapping with the value x and 1_X is the identity mapping of X;
- if $v \in V$ and $u \in U \setminus V$ are such that $u \leq v$ then for all $x, y \in L$

$$F\eta_{v,u}(x,y) = \begin{cases} x \vee y & \text{if } u \leq u_0 < v_0 = v, \\ x \wedge y & \text{if } u \leq u_1 < v_1 = v, \\ y & \text{if } u \leq u_2 < v_2 = v, \\ x & \text{if either } v \neq v_j \text{ for } j \in 3 \\ & \text{or } v = v_j \text{ and } u \notin (u_j] \text{ for some } j \in 3. \end{cases}$$

For any $\mathbb{L}$-morphism $f : (L; \tau, \vee, \wedge, 0, 1, \alpha, \beta) \to (L'; \tau, \vee, \wedge, 0, 1, \alpha, \beta)$, let $\Omega_{\mathfrak{U}}(f)^u = f$ for $u \in U \setminus V$ and $\Omega_{\mathfrak{U}}(f)^u = f \times f$ for $u \in V$.

Set $C = [\{v_i \mid i \in 7\})$.

Proposition 3.8. *Let $\mathfrak{U} = (U, V, E)$ be a u-triple. Then $\Omega_{\mathfrak{U}} : \mathbb{L} \to \mathbb{C}^U$ is a full embedding such that $\Omega_{\mathfrak{U}}(\mathcal{L})$ is C-coherent for all $\mathbb{L}$-objects $\mathcal{L}$. Moreover, $\Omega_{\mathfrak{U}}\mathcal{L}u$ is finite for all $u \in U$ whenever $\mathcal{L}$ is finite.*

Proof. First we prove that $\Omega_{\mathfrak{U}}(\mathcal{L}) = F$ is a correctly defined functor from U to $\mathbb{C}$. Let $\mathcal{L} = (L; \tau, \vee, \wedge, 0, 1, \alpha, \beta)$ be an $\mathbb{L}$-object. Clearly, $Fu \in \mathbb{C}$ for every $u \in U$ and $F\eta_{u,v}$ is a continuous mapping for every $u, v \in U$ with $v \leq u$. By the definition of f-covering pair, for every $u, v \in U$ with $u \leq v$ there exists at most one $i \in 7$ with $u \leq u_i < v_i = v$ and hence the definition of $F\eta_{u,v}$ is correct. From the definition of a f-covering pair it follows that if $u, v \in U$ with $u \leq v < v_j$ for some $j \in 7$ then $u \in (u_j]$ if and only if $v \in (u_j]$. Hence $F\eta_{w,u} = F\eta_{v,u} \circ F\eta_{w,v}$ and F is a functor from U to $\mathbb{C}$. Verification that F is C-coherent is straightforward.

Let $f : \mathcal{L} = (L; \tau, \vee, \wedge, 0, 1, \alpha, \beta) \to \mathcal{L}' = (L'; \tau, \vee\wedge, 0, 1, \alpha, \beta)$ be an $\mathbb{L}$-morphism. From the continuity of f it follows that $(\Omega_{\mathfrak{U}}f)^u$ is continuous for all $u \in U$. It is clear that $\Omega_{\mathfrak{U}}f$ commutes with both projections and the identity mappings. Since $f : (L; \vee, \wedge, 0, 1) \to (L'; \vee, \wedge, 0, 1)$ and $f : (L; \alpha, \beta) \to (L'; \alpha, \beta)$ are homomorphisms, $\Omega_{\mathfrak{U}}f$ commutes with $\vee$,

$\wedge$, α, β and the two constant mappings with values 0 or 1. Whence $\Omega_{\mathfrak{U}} f$ is a natural transformation from $\Omega_{\mathfrak{U}}(\mathcal{L})$ to $\Omega_{\mathfrak{U}}(\mathcal{L}')$.

Since $(f \times f) \circ (g \times g) = (f \circ g) \times (f \circ g)$, we have $\Omega_{\mathfrak{U}} f \circ \Omega_{\mathfrak{U}} g = \Omega_{\mathfrak{U}}(f \circ g)$. Hence $\Omega_{\mathfrak{U}}$ is a functor from $\mathbb{L}$ into $C(\mathbb{C}^{\mathbf{U}})$. It is clear that $\Omega_{\mathfrak{U}}$ is faithful. If $\mathcal{L}$ is finite it is easy to see that $\Omega_{\mathfrak{U}} \mathcal{L} u$ is finite for all $u \in U$.

It remains to prove that $\Omega_{\mathfrak{U}}$ is full. Let $g : \Omega_{\mathfrak{U}}(\mathcal{L}) \to \Omega_{\mathfrak{U}}(\mathcal{L}')$ be a transformation. By the definition of $\mathfrak{U}$, there exists $u \in U \setminus V$. Set $f = g^u$. Then the mapping f from $(L; \tau)$ to $(L'; \tau)$ is continuous because g^u from $\Omega_{\mathfrak{U}}(\mathcal{L})u = (L; \tau)$ to $\Omega_{\mathfrak{U}}(\mathcal{L}')u = (L'; \tau)$ is continuous. By (u1) and (u2), $f = g^u$ for all $u \in U \setminus V$ and $g^v = g^{v'}$ for all $v, v' \in V$ because for $u, v \in V$ or $u, v \in U \setminus V$ with $u \leq v$, $\Omega_{\mathcal{U}}(\mathcal{L})(\eta_{v,u})$ is the identity map whenever $u \leq u_i < v_i \leq v$ for no $i \in 7$. By (u4), there exist $v \in V$ and $u \in U$ with $u \leq v$ and $u \leq u_i < v_i \leq v$ for no $i \in 7$. If π_j (or ρ_j) is the j-th projection from L^2 to L (or $(L')^2$ to L', respectively), then, by the definition of $\Omega_{\mathfrak{U}}$, for $v \in V$ we have $f \circ \pi_j = \rho_j \circ g^v$ for both $j = 1, 2$. Whence, by (u1), $g^v = f \times f$ and thus $f \times f = g^{v'}$ for all $v' \in V$. From the definition of $(\Omega_{\mathfrak{U}} \mathcal{L})\eta_{u_i,v_i}$ and $(\Omega_{\mathfrak{U}} \mathcal{L}')\eta_{u_i,v_i}$ for $i \in 7$ it follows that f preserves $\vee$, $\wedge$, commutes with α and β and $f(0) = 0$ and $f(1) = 1$. Thus $f : \mathcal{L} \to \mathcal{L}'$ is an $\mathbb{L}$-morphism with $\Omega_{\mathfrak{U}} f = g$. Hence $\Omega_{\mathfrak{U}} : \mathbb{L} \to \mathbb{C}^U$ is a full embedding. $\square$

Let $\mathbb{W}$ be a proper subvariety of $\mathbb{V}$. We say that an h-space $\mathcal{X} = (X; \leq, \tau) \in \mathbb{PHV}$ is a <u>universal</u> <u>testing</u> <u>object</u> of $\mathbb{V}$ with respect to $\mathbb{W}$ if there exist a functorial set $U \subseteq X$ of $\mathcal{X}$, an increasing subset $V \subseteq U$, f-covering pairs (u_i, v_i) of $U(\mathcal{X})$ for $i \in 7$ and a set $Z \subseteq \operatorname{Max}(X)$ such that

(v1) $(U(\mathcal{X}), V, \{(u_i, v_i\} \mid i \in 7\})$ is a u-triple;
(v2) $\mathcal{X}$ is a (U, C, Z)-testing object of $\mathbb{V}$ for $C = [\{v_i \mid i \in 7\}) \subseteq Z$;
(v3) for $x \in X$, $(x]$ belongs to $\mathbb{PHW}$ if and only if $x \notin Z$.

Combining Theorems 3.2 and 3.7 with Proposition 3.8 we obtain

Theorem 3.9. *If there exists a universal testing object $\mathcal{X}$ of $\mathbb{V}$ with respect to $\mathbb{W}$ then $\mathbb{V}$ is $\mathbb{W}$-relatively alg-universal modulo* $\operatorname{Aut}(\mathcal{X})$. *If $\mathcal{X}$ is automorphism free then $\mathbb{V}$ is $\mathbb{W}$-relatively alg-universal, if $\mathcal{X}$ is finite then $\mathbb{V}$ is $\mathbb{W}$-relatively ff-alg-universal modulo* $\operatorname{Aut}(\mathcal{X})$, *and if $\mathcal{X}$ is finite and automorphism free then $\mathbb{V}$ is $\mathbb{W}$-relatively ff-alg universal.*

Proof. By Theorem 3.2, the functor $\Psi_{\mathcal{X},U} : C(\mathbb{C}^U) \to \mathbb{PHV}$ is a $(\mathcal{Z}, \mathcal{G})$-relatively full embedding for $\mathcal{Z} = \{Z_F \mid F \in C(\mathbb{C}^{U(\mathcal{X})})\}$ and $\mathcal{G} = \{\mu_F \mid F \in C(\mathbb{C}^{U(\mathcal{X})})\}$ defined just before Theorem 3.2. By Proposition 3.8, $\Psi_{\mathcal{X},U} \circ \Omega_{\mathfrak{U}}$ is also a $(\mathcal{Z}, \mathcal{G})$-relatively full embedding. From (v3) and from Theorem 3.1 it follows for $(x] \in \Psi_{\mathcal{X},U} F$ with $F \in C(\mathbb{C}^U)$ that $x \in \mathbb{PHW}$ if and only if $x \notin Z_F$. Thus if $\mathcal{L}_1$ and $\mathcal{L}_2$ are $\mathbb{L}$-objects and $f : \Psi_{\mathcal{X},U} \circ \Omega_{\mathfrak{U}}(\mathcal{L}_1) \to \Psi_{\mathcal{X},U} \circ \Omega_{\mathfrak{U}}(\mathcal{L}_2)$ is an h-map then $\operatorname{Im}(f) \notin \mathbb{PHW}$ if and only if there exist an $\mathbb{L}$-morphism $g : \mathcal{L}_1 \to \mathcal{L}_2$ and an automorphism $\alpha \in \operatorname{Aut}(\mathcal{X})$ such that $f = \Psi_{\mathcal{X},U} \circ \Omega_{\mathfrak{U}}(g) \circ \mu_{\mathcal{L}_1}(\alpha) = \mu_{\mathcal{L}_2}(\alpha) \circ \Psi_{\mathcal{X},U} \circ \Omega_{\mathfrak{U}}(g)$. Whence, by Theorem 3.7, $\mathbb{V}$ is $\mathbb{W}$-relatively alg-universal modulo the group $\operatorname{Aut}(\mathcal{X})$. If, moreover, $\mathcal{X}$ is finite then $\Psi_{\mathcal{X},U}(\Omega_{\mathfrak{U}} \mathcal{L}))$ is finite if and only if $\mathcal{L}$ is finite and hence $\mathbb{V}$ is $\mathbb{W}$-relatively ff-alg-universal modulo group $\operatorname{Aut}(\mathcal{X})$. The rest is straightforward. $\square$

Corollary 3.10. *If $\mathbb{W}$ is a proper subvariety of $\mathbb{V}$ and $\mathcal{X}$ is a finite universal testing object of $\mathbb{V}$ with respect to $\mathbb{W}$ then $\mathcal{X}$ is also a standard Q-testing object. Thus $\mathbb{V}$ contains an A-D family and it is Q-universal.*

4. Directed graphs

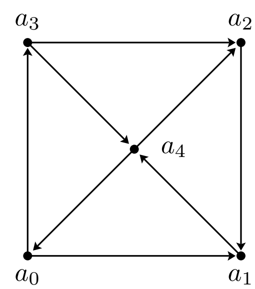
In this section we use special finite graphs in a construction of testing objects in several finitely generated varieties of Heyting algebras. A few notions concerning digraphs are needed.

We recall that a digraph is a pair (X, R) in which X is a set and $R \subseteq X \times X$, and that a digraph homomorphism from (X, R) into (X', R') is a mapping $f : X \to X'$ such that $(f(x), f(y)) \in R'$ for all $(x, y) \in R$. A digraph (X, R) is

- reflexive if $(x, x) \in R$ for all $x \in X$;
- antireflexive if $(x, x) \notin R$ for all $x \in X$;
- antisymmetric if $(x, y), (y, x) \in R$ implies $x = y$.

For any digraph $G = (X, R)$, respectively denote $G^r = (X, R^r)$ and $G^a = (X, R^a)$ its reflexive and antireflexive modifications, that is, let $R^r = R \cup \{(x, x) \mid x \in X\}$ and $R^a = R \setminus \{(x, x) \mid x \in X\}$.

Consider the finite reflexive antisymmetric graph $\mathbf{G}_0 = (A, B)$ shown in Figure 4 (where all the loops (a_i, a_i) are omitted); denote $A = \{a_i \mid i \in 5\}$ the set of all its vertices and B the set of all its edges.



The digraph $\mathbf{G}_0$

Figure 4

Z. Hedrlín described digraph homomorphisms from $\mathbf{G}_0$.

Lemma 4.1 ([8] or [22])**.** *Any digraph homomorphism from* $\mathbf{G}_0$ *to an antisymmetric graph* $\mathbf{G}$ *is either constant or injective. Moreover, the identity mapping is the only injective digraph endomorphism of* $\mathbf{G}_0$*.*

We use the digraph $\mathbf{G}_0$ in our construction.

Let $W = A \times \{0, 1, \dots 14\}/\sim$ where $\sim$ is the least equivalence on $A \times \{0, 1, \dots, 14\}$ such that $(a_3, j) \sim (a_0, j+1)$ and $(a_2, j) \sim (a_1, j+1)$ for all $j \in \{0, 1, \dots, 13\}$. If $[x]$ denotes the equivalence class of $\sim$ containing $x \in A \times \{0, 1, \dots, 14\}$ then set

$$S_0 = \{([x], [y]) \mid \exists (u, v) \in B,\ \exists j \in \{0, 1, \dots 14\},\ x = (u, j),\ y = (v, j)\},$$

$$S_1 = S_0 \cup \{([(a_4, 2k)], [(a_4, 2j)]) \mid j \in \{0, 1, 2, 3, 4\},\ k \in \{5, 6, 7\}\}.$$

Thus (W, S_0) is a 'chain' of fifteen successsive copies of G_0 in which any two successive copies of G_0 are amalgamated at their opposing vertical edges, and (W, S_1) is obtained from (W, S_0) by adding fifteen edges connecting the 'center' points of certain even-numbered copies of G_0 in (W, S_0).

Next we prove a technical lemma about these two graphs.

Lemma 4.2. *A mapping* $f : W \to W$ *is a digraph homomorphism from* (W, S_0) *into* (W, S_1) *if and only if it is a constant or the identity.*

Proof. Since $S_0 \subseteq S_1$ and (W, S_1) is a reflexive graph it is clear that the identity mapping of W and any constant mapping from W into itself is a digraph homomorphism from (W, S_0) to (W, S_1). Conversely, let $f : (W, S_0) \to (W, S_1)$ be a digraph homomorphism. For $j \in \{0, 1, \dots, 14\}$ let $g_j : A \to W$ be a mapping such that $g_j(a_k) = [(a_k, j)]$ for all $k \in \{0,1,\dots,14\}$. Then $g_j : \mathbf{G}_0 \to (W,S_0)$ is a digraph homomorphism for each $j \in \{0,1,\dots 14\}$. Since $|\operatorname{Im}(g_j) \cap \operatorname{Im}(g_{j+1})| = 2$ for all $j \in \{0,1,\dots,13\}$ we conclude, by Lemma 4.1, that $f \circ g_j$ is a constant mapping if and only if $f \circ g_{j+1}$ is a constant mapping. Moreover, if $f \circ g_j$ and $f \circ g_{j+1}$ are constant mappings then $\operatorname{Im}(f \circ g_j) = \operatorname{Im}(f \circ g_{j+1})$. Thus either $f \circ g_j$ are injective mappings for all $j \in \{0,1,\dots,14\}$ or there exists $x \in W$ such that $f \circ g_j$ is a constant mapping with value x for all $j \in \{0,1,\dots,14\}$. From $W = \bigcup_{j\in 15} \operatorname{Im}(g_j)$ it follows that in the second case, f is a constant mapping with value x. Thus it suffices to consider the first case. Observe that the sets $T_1 = \{(a_3,a_0),(a_0,a_4),(a_4,a_3)\}$, $T_2 = \{(a_1,a_0),(a_0,a_4),(a_4,a_1)\}$, $T_3 = \{(a_2,a_1),(a_2,a_4),(a_4,a_1)\}$ of B are three pairwise distinct cycles in (A,B) containing a_4 such that T_i and T_{i+1} have a common edge for $i = 1,2$. Then $f \circ g_j(a_4)$ must have the same property because $f \circ g_j$ is injective. But an element x of W is contained in three distinct cycles T_1', T_2' and T_3' such that T_i' and T_{i+1}' have a common edge for $i = 1,2$ if and only if $x = g_j(a_4)$ for some $j \in \{0,1,\dots,14\}$. Therefore, by Lemma 4.1, for every $j \in \{0,1,\dots,14\}$ there exists $k(j) \in \{0,1,\dots,12\}$ with $f \circ g_j = g_{k(j)}$. From $|\operatorname{Im}(g_j) \cap \operatorname{Im}(g_{j+1})| = 2$ it follows that $k(j+1) = k(j)+1$ and thus $k(j) = j$ for all $j \in \{0,1,\dots,14\}$ and $W = \bigcup_{j\in 15} \operatorname{Im}(g_j)$ implies that f is the identity mapping. □

We say that a digraph (X,R) is <u>strongly</u> <u>connected</u> if for every ordered pair (x,y) of distinct vertices there exists a sequence $x_0, x_1, \dots, x_l$ of vertices such that $x = x_0$, $y = x_l$ and $(x_i,x_{i+1}) \in R$ for all $i = 0,1,\dots,l-1$. Let $\mathbb{DG}_0$ denote the category of all finite, reflexive, antisymmetric and strongly connected digraphs with at least two vertices and all their digraph homomorphisms. Clearly, (W,S_0) and (W,S_1) belong to $\mathbb{DG}_0$.

Let $\mathcal{S} = (S;\le,\tau)$ be a finite h-space such that $\operatorname{Max}(S) = \{s_0\}$, $\operatorname{Cov}(s_0) = \{s_1,s_2\}$, $\operatorname{Cov}(s_1) = \{s_3\}$, $|\operatorname{Cov}(s_2)| \ne 1$, and $(s_2] \setminus \{s_2\} = (s_3] \setminus \{s_3\} = S \setminus \{s_i \mid i = 0,1,2,3\} = T$. The h-spaces Q_3, Q_4 and Q_5 of Figure 5 below are instances of such a space $\mathcal{S}$. Observe that $\operatorname{Cov}(s_2) = \operatorname{Cov}(s_3)$ and that any automorphism f satisfies $f(s_i) = s_i$ for all $i = 0,1,2,3$ for each of these three spaces.

Let $\mathcal{S}_0$ be the h-space $\mathcal{S}/\theta$ where θ is the least equivalence on S with $s_0 \theta s_1$ and $s_2 \theta s_3$. For the h-spaces Q_3, Q_4 and Q_5 of Figure 5, their respective quotients R_3, R_4 and R_5 are also shown in Figure 5. Let $\mathcal{S}_1$ be the h-space $\mathcal{S}/\theta_1$ where θ_1 is the equivalence on $\mathcal{S}$ whose only non-singleton class is T. The properties of $\mathcal{S}$ immediately imply that both $\mathcal{S}_0$ and $\mathcal{S}_1$ are quotient h-spaces of $\mathcal{S}$ via the maps whose respective kernels are defined above, and that their associated mappings are h-maps from $\mathcal{S}$ onto $\mathcal{S}_0$ and $\mathcal{S}_1$, respectively

For the variety $\mathbb{V} = \operatorname{Var}(\mathcal{S})$, we now construct an embedding of $\mathbb{DG}_0$ into $\mathbb{PHV}$.

Recalling that $T = S \setminus \{s_0,\dots,s_3\}$, for any finite digraph $\mathbf{G} = (X,R) \in \mathbb{DG}_0$ define $\Lambda\mathbf{G} = (Y_{\mathbf{G}};\le,\tau)$ such that $Y_{\mathbf{G}} = T \cup (X \times \{0,1\}) \cup R^a$ (we assume that T, $X \times \{0,1\}$ and R^a are pairwise disjoint), $\le$ is the least partial order such that

- (•) if $u,v \in T$ then $u \le v$ if and only if $u \le v$ in $(S;\le,\tau)$;
- (•) for all $u \in T$ and $x \in X$ we have $u \le (x,0) \le (x,1)$;
- (•) if $(x,y) \in R^a$ then $(x,1),(y,0) \le (x,y)$.

Let τ be the discrete topology on $Y_{\mathbf{G}}$. For a digraph homomorphism $f : (X,R) \to (X',R') \in \mathbb{DG}_0$ define Λf by

$$\Lambda f(u) = \begin{cases} u & \text{if } u \in T, \\ (f(x),i) & \text{if } u = (x,i) \in X \times \{0,1\}, \\ (f(x),f(y)) & \text{if } u = (x,y) \in R^a \text{ and } f(x) \ne f(y), \\ (f(x),0) & \text{if } u = (x,y) \in R^a \text{ and } f(x) = f(y). \end{cases}$$

The lemma below is easily verified.

Lemma 4.3. *For every digraph* $\mathbf{G} = (X,R) \in \mathbb{DG}_0$,

(1) $Y_{\mathbf{G}}$ *is finite and* $\leq$ *is a partial order;*
(2) $\mathrm{Max}(Y_{\mathbf{G}}) = R^a$ *and* $((x, y)]$ *is isomorphic to* $\mathcal{S}$ *for every* $(x, y) \in R^a$*;*
(3) $(Y_{\mathbf{G}}; \leq, \tau)$ *is an h-space from* $\mathbb{PHV}$*.*

If $f : (X, R) \to (X', R') \in \mathbb{DG}_0$ *is a digraph homomorphism then* $\Lambda f : \Lambda(X, R) \to \Lambda(X', R')$ *is an h-map.*

$\Lambda : \mathbb{DG}_0 \to \mathbb{PH}(\mathrm{Var}(\mathcal{S}))$ *is a faithful functor.*

Lemma 4.4. *Let* $\mathbf{G} = (X, R)$ *and* $\mathbf{G}' = (X', R')$ *be digraphs from* $\mathbb{DG}_0$ *and let* $f : \Lambda\mathbf{G} \to \Lambda\mathbf{G}'$ *be an h-map, then exactly one of the following possibilities occurs:*

(1) f *is not injective on* T, $f(T) \subset T$, $f(X \times \{0\}) \subseteq T$, $\mathrm{Im}(f) \cap \mathrm{Max}(Y_{\mathbf{G}'}) = \emptyset$ *and* $f^{-1}(X' \times \{1\}) \subseteq \mathrm{Max}(Y_{\mathbf{G}})$*;*
(2) $\mathrm{Im}(f) \cap (R')^a \neq \emptyset$, $f(X \times \{0\}) \subseteq X' \times \{0\}$, *the mapping* $f' : X \to X'$ *given by* $f(x, 0) = (f'(x), 0)$ *for all* $x \in X$ *is a non-constant digraph homomorphism from* $\mathbf{G}$ *to* $\mathbf{G}'$, *and there exists an automorphism* g *of* $\mathcal{S}$ *such that* $f(t) = g(t)$ *for* $t \in T$ *and* $f(u) = \Lambda f'(u)$ *for* $u \in Y_{\mathbf{G}} \setminus T$*;*
(3) $\mathrm{Im}(f) \cap (R')^a = \emptyset$, $f(X \times \{0\})$ *is a singleton in* $X' \times \{0\}$, $|\mathrm{Im}(f) \cap (X' \times \{1\})| \leq 1$, *and there exists an automorphism* g *of* $\mathcal{S}_1$ *such that* $f(t) = g(t)$ *for all* $t \in T$*.*

Proof. Since f has the h-property, by Lemma 4.3(2) and from the fact that $T = (s_2] \setminus \{s_2\} = (s_3] \setminus \{s_3\}$ we obtain that $f(T) \subseteq T$.

If f is not injective on T then $f(T) \subset T$ because T is finite and, by the h-property, $f(X \times \{0\}) \subseteq T$. But then $f(X \times \{1\}) \subseteq T \cup (X' \times \{0\})$ and $f(\mathrm{Max}(Y_{\mathbf{G}})) \cap \mathrm{Max}(Y_{\mathbf{G}'}) = \emptyset$. Thus $\mathrm{Im}(f) \cap \mathrm{Max}(Y_{\mathbf{G}'}) = \emptyset$ and $f^{-1}(X' \times \{1\}) \subseteq \mathrm{Max}(Y_{\mathbf{G}})$ and this fully describes the case in (1).

Suppose that (1) fails. Accordingly, we assume that $f(T) = T$. Then f is a permutation of T because of the finiteness of T and $f(X \times \{0\}) \subseteq X' \times \{0\}$ because $|\mathrm{Cov}(x, 0)| \neq 1$ for all $x \in X$.

Now also suppose that $f(X \times \{0\})$ is a singleton. Then $f(X \times \{0\}) = \{(x_1, 0)\}$. Now $((x_0, 0)]$ and $((x_1, 0)]$ are isomorphic to $\mathcal{S}_1$ for any $x_0 \in X$ and, since f is an h-map, we see that the permutation g of $\mathcal{S}_1$ given by $g(t) = f(t)$ for all $t \in T$ and $g(s) = s$ for all $s \notin T$ is an automorphism of $\mathcal{S}_1$. From the h-property of f it follows that $f(X \times \{0, 1\}) \subseteq \{(x_1, 0), (x_1, 1)\}$ and since $|((x, y)] \cap (X' \times \{0\})| = 2$ for all $(x, y) \in (R')^a$ we conclude that $\mathrm{Im}(f) \cap (R')^a = \emptyset$. Thus (3) is fully established.

Suppose that $|f(X \times \{0\})| \geq 2$. Since (X, R) is strongly connected, for every $x' \in X'$ there exists $(x, y) \in R^a$ such that $f(x, 0) = (x', 0)$ and $f(y, 0) = (y', 0) \neq (x', 0)$. Then $(x', 0), (y', 0) \in (f(x, y)]$ and from $x' \neq y'$ it follows that $f(x, y) = (x', y') \in (R')^a$. Whence $\mathrm{Im}(f) \cap (R')^a \neq \emptyset$ and a mapping $f' : X \to X'$ such that $f(x, 0) = (f'(x), 0)$ for all $x \in X$ is a nonconstant digraph homomorphism from (X, R) into (X', R'). Let $g : S \to S$ be the mapping given by

$$g(x) = \begin{cases} x & \text{if } x \in \{s_i \mid i \in 4\}, \\ f(x) & \text{if } x \in T. \end{cases}$$

Since f is an h-map, the mapping g preserves order and has the h-property on T. Since f is a permutation of T, it maps the set of all maximal elements of T onto itself, so that g preserves order and has the h-property on S. Since S is finite, g is an h-map and thus it is an automorphism of $\mathcal{S}$ because f is a permutation of T. This proves (2). $\square$

For a digraph $\mathbf{G} = (X, R) \in \mathbb{DG}_0$ denote $Z_{\mathbf{G}} = R^a = \mathrm{Max}(\Lambda\mathbf{G})$ and define a mapping $\mu_{\mathbf{G}} : \mathrm{Aut}(\mathcal{S}) \to \mathrm{Aut}(\Lambda\mathbf{G})$ by

$$\mu_{\mathbf{G}}(f)(u) = \begin{cases} f(u) & \text{if } u \in T, \\ u & \text{if } u \notin T \end{cases}$$

for all $f \in \mathrm{Aut}(\mathcal{S})$ and all elements of $\Lambda\mathbf{G}$. Since $f \in \mathrm{Aut}(\mathcal{S})$ and $\Lambda\mathbf{G}$ is finite, by a direct verification we obtain that $\mu_{\mathbf{G}}(f) \in \mathrm{Aut}(\Lambda\mathbf{G})$ and hence $\mu_{\mathbf{G}} : \mathrm{Aut}(\mathcal{S}) \to \mathrm{Aut}(\Lambda\mathbf{G})$ is an injective group homomorphism. Set $\mathcal{Z} = \{Z_{\mathbf{G}} \mid \mathbf{G} \in (\mathbb{DG}_0)^o\}$ and $\mathcal{G} = \{\mu_{\mathbf{G}} \mid \mathbf{G} \in (\mathbb{DG}_0)^o\}$.

Corollary 4.5. $\Lambda : \mathbb{DG}_0 \to \mathbb{PHV}$ *is a faithful functor preserving finiteness such that*

(1) *if* $f : \mathbf{G}_1 \to \mathbf{G}_2$ *is a non-constant* $\mathbb{DG}_0$*-homomorphism then* $\Lambda f(Z_{\mathbf{G}_1})] \cap Z_{\mathbf{G}_2} \neq \emptyset$;
(2) *if* $f : \mathbf{G}_1 \to \mathbf{G}_2$ *is a constant* $\mathbb{DG}_0$*-homomorphism then* $\mathrm{Im}(\Lambda f) \cap Z_{\mathbf{G}_2} = \emptyset$;
(3) *if* $f : \Lambda\mathbf{G}_1 \to \Lambda\mathbf{G}_2$ *is an h-map then either* $\mathrm{Im}(f) \cap Z_{\mathbf{G}_2} \neq \emptyset$ *and there exist a non-constant homomorphism* $g : \mathbf{G}_1 \to \mathbf{G}_2$ *of* $\mathbb{DG}_0$ *and* $\phi \in \mathrm{Aut}(\mathcal{S})$ *such that* $f = \mu_{\mathbf{G}_2}(\phi) \circ \Lambda g = \Lambda g \circ \mu_{\mathbf{G}_1}(\phi)$, *or else* $\mathrm{Im}(f) \cap Z_{\mathbf{G}_2} = \emptyset$ *and f is non-injective on T or f is non-injective on* $\{(x,0),(y,0)\}$ *for every edge* (x,y) *of* $\mathbf{G}_1$ *with* $y \neq x$.

Consider $\mathcal{X} = \Lambda(W, S_1)$, $U = [\{([(a_4, 2j)], 0) \mid j = 0,1,\dots,7\})$, $V = [\{([(a_4,10)],0)\}) = [g_{10}(a_4))$, $C = \{([(a_4,10)],[(a_4,2j)]) \mid j = 0,1,2\} \cup \{([(a_4,12)],[(a_4,2j)]) \mid j = 0,1,2,3\} \subseteq S_1^a$,

$$\begin{aligned}(u_i, v_i) &= (([(a_4, 2i)], 0), ([(a_4, 10)], [(a_4, 2i)])) \text{ for } i = 0,1,2, \text{ and}\\ (u_i, v_i) &= (([(a_4, 2(i-3))], 0), ([(a_4, 12)], [(a_4, 2(i-3))])) \text{ for } i = 3,4,5,6,\end{aligned}$$

and $Z = (S_1)^a$. Observe that U and V are increasing sets (thus U is convex), $(U; \preceq)$ and $(V; \leq)$ are order connected, $Z = \mathrm{Max}\,\Lambda\mathcal{X}$ and $C \subseteq Z$. Clearly, (u_i, v_i) for $i = 0,1,\dots,6$ are f-covering pairs satisfying (u3) and (u4), and thus $(U, V, \{(u_i, v_i) \mid i = 0,1,\dots 6\})$ is a u-triple. Clearly, $(x] \in \mathbb{PH}(\mathrm{Var}(\mathcal{S}_0))$ for all $x \in W \setminus Z$. Thus to obtain that $\mathcal{X}$ is a universal testing object of $\mathbb{V}$ with respect to $\mathrm{Var}(\mathcal{S}_0)$ it remains to prove that $\mathcal{X}$ is (U, C, Z)-testing object. For this it suffices to prove (t5); the other conditions are clearly fulfilled. Since $\Lambda(W, S_0)$ is a h-subspace of $\mathcal{X} \setminus C$, consider an h-map $f : \Lambda(W, S_0) \to \mathcal{X}$ that is not injective on $X \setminus U$. By Lemmas 4.2 and 4.4, either there exists $\phi \in \mathrm{Aut}(\mathcal{S})$ such that $f = \mu_{(W,S_1)}(\phi) \circ \Lambda\iota$ where ι is the identity mapping of W, or $\mathrm{Im}(f) \cap Z_{\mathbf{G}_2} = \emptyset$ and f is non-injective on T or on every set $\{(x,0),(y,0)\}$ with $(x,y) \in S_1^a$. Since in the first case $\mu_{(W,S_1)}(\phi) \circ \Lambda\iota$ is injective we restrict ourselves to the second case. Then for every $z \in Z$, $|\mathrm{Cov}(z)| = 2$ and $\mathrm{Cov}(z) \subseteq W \times \{0,1\}$. Hence for every $z \in Z$ there exists $u_z \in \mathrm{Cov}(z)$ such that $u_z \in \mathrm{Im}(g)$ for no h-map $g : \Lambda(W, S_0) \to \mathcal{X}$ with $g \restriction X \setminus U = f \restriction X \setminus U$. Thus $\mathcal{X}$ is a (U, C, Z)-testing object of $\mathbb{V}$. If T is finite then $\mathcal{X}$ is finite and we conclude, by Corollary 3.10 that in this case $\mathcal{X}$ is also a standard Q-testing object. By Theorem 3.9 and Corollary 3.6 we obtain

Corollary 4.6. *If T is finite then the variety* $\mathbb{V}$ *is* $\mathrm{Var}(\mathcal{S}_0)$*-relatively ff-alg-universal modulo the group* $\mathrm{Aut}(\mathcal{S})$. *The variety* $\mathbb{V}$ *contains an A-D family and thus it is Q-universal.*

We apply Corollary 4.6 to the varieties of Heyting algebras determined by h-spaces $\mathbf{Q}_3$, $\mathbf{Q}_4$ and $\mathbf{Q}_5$ given on Figure 5. Direct inspection shows that $\mathrm{Aut}(\mathbf{Q}_3)$ is a singleton group and $\mathrm{Aut}(\mathbf{Q}_4)$ and $\mathrm{Aut}(\mathbf{Q}_5)$ are isomorphic to the cyclic group $\mathbf{C}_2$ of order 2. Observe that if $\mathcal{S} = \mathbf{Q}_i$ then $\mathcal{S}_0 = \mathbf{R}_i$ for $i = 3,4,5$. Altogether, we have

Theorem 4.7. *The variety* $\mathrm{Var}(\mathbf{DQ}_3)$ *is* $\mathrm{Var}(\mathbf{DR}_3)$*-relatively ff-alg-universal.*

The variety $\mathrm{Var}(\mathbf{DQ}_4)$ *is* $\mathrm{Var}(\mathbf{DR}_4)$*-relatively ff-alg-universal modulo the group* $\mathbf{C}_2$.

The variety $\mathrm{Var}(\mathbf{DQ}_5)$ *is* $\mathrm{Var}(\mathbf{DR}_5)$*-relatively ff-alg-universal modulo the group* $\mathbf{C}_2$.

Each of the varieties $\mathrm{Var}(\mathbf{DQ}_3)$, $\mathrm{Var}(\mathbf{DQ}_4)$ *and* $\mathrm{Var}(\mathbf{DQ}_5)$ *contains an A-D family and is therefore Q-universal.*

We now turn our attention to three varieties generated by a pair of (finite) subdirectly irreducible algebras.

To this end, we employ another category $\mathbb{DG}_p$ of 'pointed' digraphs. Its objects are quadruples (X, R, x_0, x_1) such that

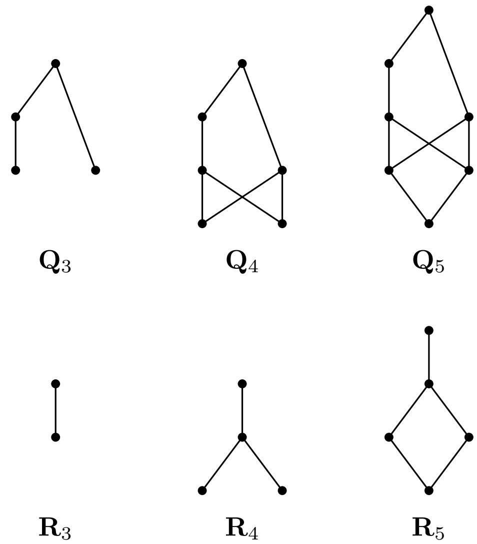


Figure 5

- (•) (X, R) is a finite connected reflexive antisymmetric digraph,
- (•) $x_0, x_1 \in X$,
- (•) the induced subdigraph of (X, R) on $X \setminus \{x_0, x_1\}$ belongs to $\mathbb{DG}_0$, that is, it is a strongly connected digraph and $|X \setminus \{x_0, x_1\}| \geq 2$,
- (•) $(x_0, x) \in R$ only when $x = x_0$,
- (•) $(x, x_1) \in R$ only when $x = x_1$,
- (•) $(x_0, x_1), (x_1, x_0) \notin R$,
- (•) there exist $x, x' \in X \setminus \{x_0, x_1\}$ with $(x_1, x), (x', x_0) \in R$.

And the $\mathbb{DG}_p$-morphisms from (X, R, x_0, x_1) to (Y, S, y_0, y_1) are all digraph homomorphisms $f : (X, R) \to (Y, S)$ with $f(x_0) = y_0$ and $f(x_1) = y_1$.

Recall the digraphs (W, S_0) and (W, S_1) defined just above Lemma 4.2. Choose distinct $x_0, x_1 \notin W$ and set $V = W \cup \{x_0, x_1\}$,

$$T_0 = S_0 \cup \{([a_0, 0], x_0), ([a_3, 14], x_0), (x_1, [a_1, 0]), (x_1, [a_2, 14])\}$$

and $T_1 = T_0 \cup S_1$. Then $(V, T_0, x_0, x_1), (V, T_1, x_0, x_1) \in \mathbb{DG}_p$ and, by Lemma 4.2 we obtain

Corollary 4.8. *If $f : (V, T_0, x_0, x_1) \to (V, T_1, x_0, x_1)$ is a $\mathbb{DG}_p$-morphism then f is the inclusion.*

Proof. Let $f : (V, T_0) \to (V, T_1)$ be a $\mathbb{DG}_p$-morphism. By the definition of $\mathbb{DG}_p$-morphisms we have $f(x_0) = x_0$, $f(x_1) = x_1$ and $f(W) \subseteq W$, so that f is not constant. But then f is the identity mapping by Lemma 4.2. □

Let $\mathbb{POS}$ denote the category of all finite posets and all their order preserving maps having the h-property.

For $\mathcal{X} = (X, R, x_0, x_1) \in \mathbb{DG}_p$ define a finite poset $\Pi\mathcal{X} = (A_X, \leq)$ such that $A_X = \{a, b\} \cup \big((X \times \{0, 1, 2\}) \setminus \{(x_0, 1)\}\big) \cup R^a$ where a and b are new distinct elements (i.e. $a, b \notin (X \times \{0, 1, 2\}) \cup R^a$) where $R^a = R \setminus \{(x, x) \mid x \in X\}$ and

- (•) $a \leq (x, 0) \leq (x, 1) \leq (x, 2) \geq b$ for all $x \in X \setminus \{x_0, x_1\}$;
- (•) $a \leq (x_i, 0) \leq (x_i, 2) \geq b$ for $i = 0, 1$, and $(x_1, 0) \leq (x_1, 1)$;
- (•) $(x, 1) \leq (x, y) \geq (y, 0)$ for all $(x, y) \in R^a$ (thus $x \neq y$).

Clearly,

Lemma 4.9. *For every $\mathcal{X} = (X, R, x_0, x_1)$ we have*

(1) $\mathrm{Max}(A_X) = X \times \{2\} \cup R^a$, $\mathrm{Min}(A_X) = \{a, b\}$;
(2) *if* $(u]$ *and* $(v]$ *are isomorphic for* $u, v \in \mathrm{Max}(A_X)$ *then either* $u, v \in R^a$ *or* $u, v \in (X \setminus \{x_0, x_1\}) \times \{2\}$ *or* $u, v \in \{x_0, x_1\} \times \{2\}$;
(3) *for every* $u \in \mathrm{Max}(A_X)$, *the only order preserving bijection from* $(u]$ *onto itself is the identity map;*
(4) $|((x, y)]| = 5$ *for all* $(x, y) \in R^a$, $|((x, 2)]| = 5$ *for all* $x \in X \setminus \{x_0, x_1\}$ *and* $|((x, 2)]| = 4$ *for* $x \in \{x_0, x_1\}$;
(5) $[b) = (X \times \{2\}) \cup \{b\}$, $[a) \cap [b) = X \times \{2\}$;
(6) *if* $(x_0, 0) < u$ *for some* $u \in A_X$ *then* u *covers* $(x_0, 0)$;
(7) *if* $u \in X \setminus \{x_0\}$ *and* $(u, 1) < v$ *for a* $v \in A_X$ *then* $v \in \mathrm{Max}(A_X)$.

For any $\mathbb{DG}_p$-morphism $f : (X, R, x_0, x_1) \to (Y, S, y_0, y_1)$ define $\Pi f : A_X \to A_Y$ by

$$\Pi f(u) = \begin{cases} u & \text{if } u \in \{a, b\}, \\ (f(x), i) & \text{if } u = (x, i) \in \big(X \times \{0, 1, 2\}\big) \setminus \{(x_0, 1)\}, \\ (f(x), f(y)) & \text{if } u = (x, y) \in R^a \text{ and } f(x) \neq f(y), \\ (f(x), 1) & \text{if } u = (x, y) \in R^a \text{ and } f(x) = f(y). \end{cases}$$

Since $(f(x), f(y)) \in S^a$ if $(x, y) \in R^a$, the definition of Πf is correct.

Direct calculations yield

Lemma 4.10. *Let* $f : \mathcal{X} = (X, R, x_0, x_1) \to \mathcal{Y} = (Y, S, y_0, y_1)$ *be a* $\mathbb{DG}_p$*-morphism. Then*

(1) *if* $u, v \in A_X$ *with* $u \leq v$ *in* $\Pi\mathcal{X}$ *then* $\Pi f(u) \leq \Pi f(v)$ *in* $\Pi\mathcal{Y}$;
(2) $(\Pi f(u)] = \Pi f((u])$ *for all* $u \in A_X$;
(3) $\Pi f(u) = u$ *for* $u = a, b$ *and* $\Pi f(x_i, j) = (y_i, j)$ *for all* $(i, j) \in \big(\{0, 1\} \times \{0, 1, 2\}\big) \setminus \{(0, 1)\}$;
(4) $\Pi f((X \setminus \{x_0, x_1\}) \times \{i\}) \subseteq (Y \setminus \{y_0, y_1\}) \times \{i\}$ *for* $i = 0, 1, 2$.

Thus Π *is a faithful functor from* $\mathbb{DG}_p$ *to* $\mathbb{POS}$.

Next we prove the basic lemma about Π.

Lemma 4.11. *Let* $\mathcal{X} = (X, R, x_0, x_1)$ *and* $\mathcal{Y} = (Y, S, y_0, y_1)$ *be objects of* $\mathbb{DG}_p$ *and let* $f : \Pi\mathcal{X} \to \Pi\mathcal{Y}$ *be a* $\mathbb{POS}$*-morphism such that* $f\big((X \setminus \{x_0, x_1\}) \times \{2\}\big) \cap \big((Y \setminus \{y_0, y_1\}) \times \{2\}\big) \neq \emptyset$. *Then there exists a* $\mathbb{DG}_p$*-morphism* $g : \mathcal{X} \to \mathcal{Y}$ *with* $\Pi g = f$.

Proof. By the assumption, there exist $x \in X \setminus \{x_0, x_1\}$ and $y \in Y \setminus \{y_0, y_1\}$ such that $f(x, 2) = (y, 2)$. Then, by Lemma 4.9(2), and (3), $f(u) = u$ for $u = a, b$ and $f(x, i) = (y, i)$ for $i = 0, 1, 2$. Thus, by Lemma 4.9(5), $f(X \times \{2\}) \subseteq Y \times \{2\}$. From $(f(u)] = f((u])$ it follows that $f(X \times \{0\}) \subseteq (Y \times \{0\}) \cup \{a\}$. Since a and b are incomparable, we infer that $f(u) \neq b$ for all $u \in A_X$ with $a \leq u$. Hence, by Lemma 4.9(1) and (5),

$$f(A_X \setminus \big((X \times \{2\}) \cup \{b\}\big)) \subseteq A_Y \setminus \big((Y \times \{2\}) \cup \{b\}\big).$$

By Lemma 4.9(4), we conclude that for $i = 0, 1$ there exists $k_i \in \{0, 1\}$ such that $f(x_i, j) = (y_{k_i}, j)$ for $j = 0, 2$. In particular, $f(\{(x_i, 0) \mid i = 0, 1\}) \subseteq \{(y_i, 0) \mid i = 0, 1\}$.

Observe that if $f(u, 0) = a$ for some $u \in X$ then, by the above, $u \neq x_0, x_1$ and $f(u, 1) \in \{(y_0, 0), (y_1, 0)\}$. Indeed, $f(u, 2) \in Y \times \{2\}$ and $|f((u, 2)])| \leq 4$ because $f(u, 0) = f(a) = a$. By 4.9(5), then $f(u, 2) \in \{(y_0, 2), (y_1, 2)\}$ and hence $f(u, 1) \in \{(y_0, 0), (y_1, 0)\}$.

Next we show that $f(v, 1) \in \{(y_0, 0), (y_1, 0)\}$ whenever $(u, v) \in R^a$ and $f(u, 1) \in \{(y_0, 0), (y_1, 0)\}$. Indeed, $f(u, 1) \in \{(y_0, 0), (y_1, 0)\}$ implies that $|(f(u, 1)]| = 2$ and hence, by Lemma 4.9(5), $f(u, v) \notin S^a$. Thus $f(u, v) \in \{(y_1, 1), (y_1, 0), (y_0, 0)\}$ and hence $f(v, 0) \in$

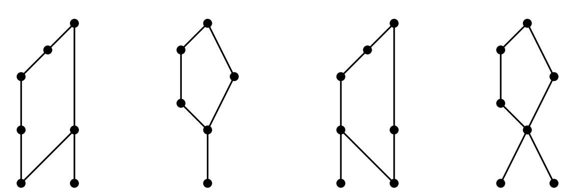
$\{(y_0,0),(y_1,0),a\}$. If $f(v,0)=a$ then, by the foregoing part of the proof, $f(v,1)\in\{(y_0,0),(y_1,0)\}$. If $f(v,0)=(y_j,0)$ for some $j=0,1$ then $f(v,2)\in Y\times\{2\}$ implies $f(v,2)=(y_j,2)$ because $[(y_j,0)\cap(Y\times\{2\})=\{(y_j,2)\}$. Since $(y_j,0)=f(v,0)\le f(v,1)\le f(v,2)=(y_j,2)$, we infer that $f(v,1)=(y_j,0)$ because $f(v,1)\notin Y\times\{2\}$. Since the subdigraph of (X,R) on $X\setminus\{x_0,x_1\}$ is strongly connected we deduce that from $f(u,1)\in\{(y_0,0),(y_1,0)\}$ if follows that $f((X\setminus\{x_0,x_1\})\times\{1\})\subseteq\{(y_0,0),(y_1,0)\}$, and this is a contradiction because $f(x,1)=(y,1)$ by the assumption.

We claim that $f((X\setminus\{x_0,x_1\})\times\{2\})\subseteq(Y\setminus\{y_0,y_1\})\times\{2\}$. Indeed, if there exists $u\in X$ with $f(u,2)\notin(Y\setminus\{y_0,y_1\})\times\{2\}$ then from $f(X\times\{2\})\subseteq Y\times\{2\}$ it follows that $f(u,2)\in\{(y_0,2),(y_1,2)\}$. Then $f(u,0)\in\{(y_0,0),(y_1,0),a\}$ and thus $f(u,1)\in\{(y_0,0),(y_1,0)\}$ and this is a contradiction. Therefore there exists a mapping $g':(X\setminus\{x_0,x_1\})\to Y\setminus\{y_0,y_1\}$ with $f(u,i)=(g'(u),i)$ for all $u\in X\setminus\{x_0,x_1\}$ and $i=0,1,2$. Let g be an extension of g' such that $g(x_0)=y_0$ and $g(x_1)=y_1$.

Next we prove that $f(x_i,j)=(y_i,j)$ for $i=0,1$ and $j=0,2$. By the assumption, there exists $u\in X\setminus\{x_0,x_1\}$ with $(u,x_0)\in R^a$. Then $(u,1)<(u,x_0)>(x_0,0)$. Hence $(g(u),1)=f(u,1)\le f(u,x_0)\ge f(x_0,0)$. We know that $f(x_0,0)\in\{(y_0,0),(y_1,0)\}$ and $[(Y\setminus\{y_0,y_1\})\times\{1\})\cap[y_1,0)=\emptyset$. Whence $f(x_0,0)=(y_0,0)$ and then $f(x_0,2)=(y_0,2)$. Further, by the assumption, there exists $v\in X\setminus\{x_0,x_1\}$ with $(x_1,v)\in R^a$. Then $(x_1,0)<(x_1,1)<(x_1,v)>(v,0)$. If $f(x_1,0)=(y_0,0)$ then, by Lemma 4.9(6), $|f\{(x_1,0),(x_1,1),(x_1,v)\}|\le 2$. Thus $|f(((x_1,v)])|\le 4$ and, by Lemma 4.9(5), $f(x_1,v)\notin S^a$. Since $f(v,0)=(g(v),0)\in(Y\setminus\{y_0,y_1\})\times\{0\}$ and since $[(Y\setminus\{y_0,y_1\})\times\{0\})\cap[(y_0,0))\subseteq S^a$ we obtain a contradiction. Whence $f(x_1,0)=(y_1,0)$ and also $f(x_1,2)=(y_1,2)$. Moreover, $f(x_1,v)\in S^a$ implies that $(y_1,0)<f(x_1,1)<f(x_1,v)$ and thus $f(x_1,1)=(y_1,1)$.

Finally, we prove that $g:\mathcal{X}\to\mathcal{Y}$ is a $\mathbb{DG}_p$-morphism and $\Pi g=f$. To do this, consider any $(u,v)\in R^a$. Then $(u,1)<(u,v)>(v,0)$ and $f(u,1)=(g(u),1)$, $f(v,0)=(g(v),0)$. If $g(u)\neq g(v)$ then $(g(u),1)<f(u,v)>(g(v),0)$ and thus $(g(u),g(v))\in S^a$ and $f(u,v)=(g(u),g(v))$. If $g(u)=g(v)$ then $f(u,v)\ge(g(u),1)$ and $|(f(u,v)]|\le 4$ because $f(u,0)=(g(u),0)=(g(v),0)=f(v,0)$. Hence $f(u,v)\notin S^a$ and, by Lemma 4.9(1), $f(u,v)\notin\mathrm{Max}(A_Y)$. Hence, by Lemma 4.9(7), $f(u,v)=(g(u),1)$ and the proof is complete. ☐

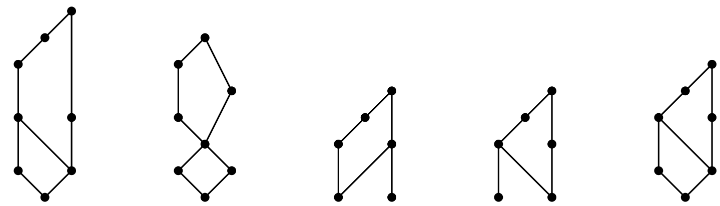
We extend the faithful functor Π to embeddings into the categories of h-spaces dual to some finitely generated varieties of Heyting algebras. Let $\mathbb{V}_i$ denote the variety of Heyting algebras generated by algebras dual to the h-spaces $\mathbf{F}_i$ and $\mathbf{G}_i$ and let $\mathbb{W}_i$ be a variety of Heyting algebras generated by algebras dual to h-spaces $\mathbf{G}_i$ and $\mathbf{H}_i$ for $i=0,1,2$ where $\mathbf{F}_i$, $\mathbf{G}_i$ and $\mathbf{H}_i$ for $i=0,1,2$ are shown in Figure 6 and Figure 7.



$\mathbf{F}_0$ $\mathbf{G}_0$ $\mathbf{F}_1$ $\mathbf{G}_1$

Figure 6

For $\mathcal{X}=(X,R,x_0,x_1)$, let $\Pi_0\mathcal{X}=(A_X\cup\{c,d\};\le,\tau)$ be an extension of $\Pi\mathcal{X}$ by new elements c and d such that $c\le a,b$ and $d\le b$, and τ is a discrete topology. Then $\Pi_0\mathcal{X}\in\mathbb{PHV}_0$. For a $\mathbb{DG}_p$-morphism $f:\mathcal{X}\to\mathcal{Y}$ let $\Pi_0 f$ be an extension of Πf such that $\Pi_0 f(c)=c$ and $\Pi_0 f(d)=d$.



$\mathbf{F}_2$ $\mathbf{G}_2$ $\mathbf{H}_0$ $\mathbf{H}_1$ $\mathbf{H}_2$

Figure 7

For $\mathcal{X} = (X, R, x_0, x_1)$, let $\Pi_1\mathcal{X} = (A_X \cup \{c, d\}; \leq, \tau)$ be an extension of $\Pi\mathcal{X}$ by new elements c and d such that $c \leq a$ and $d \leq a, b$, and τ is the discrete topology. Then $\Pi_1\mathcal{X} \in \mathbb{PHV}_1$. For a $\mathbb{DG}_p$-morphism $f : \mathcal{X} \to \mathcal{Y}$ let $\Pi_1 f$ be an extension of Πf such that $\Pi_1 f(c) = c$ and $\Pi_1 f(d) = d$.

For $\mathcal{X} = (X, R, x_0, x_1)$, let $\Pi_2\mathcal{X} = (A_X \cup \{c, d, e\}; \leq, \tau)$ be an extension of $\Pi\mathcal{X}$ by new elements c, d and e such that $c \leq a$, $d \leq a, b$, $e \leq c, d$ and τ is a discrete topology. Then $\Pi_2\mathcal{X} \in \mathbb{PHV}_2$. For a $\mathbb{DG}_p$-morphism $f : \mathcal{X} \to \mathcal{Y}$ let $\Pi_2 f$ be an extension of Πf such that $\Pi_2 f(c) = c$, $\Pi_2 f(d) = d$ and $\Pi_2 f(e) = e$.

Corollary 4.12. *For $i = 0, 1, 2$, Π_i is a $\mathcal{Z}$-relatively full embedding of $\mathbb{DG}_p$ into $\mathbb{PHV}_i$ where $\mathcal{Z} = \{Z_{(X,R,x_0,x_1)} = (X \setminus \{x_0, x_1\}) \times \{2\} \mid (X, R, x_0, x_1) \in \mathbb{DG}_p\}$.*

Proof. It is clear that for every $i = 0, 1, 2$ and every $\mathcal{X} = (X, R, x_0, x_1) \in \mathbb{DG}_p$ we have $\Pi_i\mathcal{X} \in \mathbb{PHV}_i$. Let us assume that $\mathcal{X} = (X, R, x_0, x_1)$ and $\mathcal{Y} = (Y, S, y_0, y_1)$ are objects of $\mathbb{DG}_p$ and let $f : \mathcal{X} \to \mathcal{Y}$ be a $\mathbb{DG}_p$-morphism. Then $f(X \setminus \{x_0, x_1\}) \subseteq Y \setminus \{y_0, y_1\}$ and thus there exists $y \in \mathrm{Im}(f) \setminus \{y_0, y_1\}$. Then $(y, 2) \in \mathrm{Im}(\Pi f)$ and $((y, 2)]$ in $\mathrm{Im}(\Pi_i f)$ is isomorphic to $\mathbf{F}_i$ for $i = 0, 1, 2$. Since there exists $x \in X \setminus \{x_0, x_1\}$ with $(x, x_0) \in R$ we deduce that $(f(x), y_0) \in S^a$ and $f(x, x_0) = (f(x), y_0) \in \mathrm{Im}(\Pi f)$. Then $((f(x), y_0)] \in \mathrm{Im}(\Pi_i f)$ is isomorphic to $\mathbf{G}_i$ for $i = 0, 1, 2$ and the variety generated by the Heyting algebra corresponding to the h-subspace of $\Pi_i\mathcal{Y}$ on $\mathrm{Im}(\Pi_i f)$ is $\mathbb{V}_i$ for $i = 0, 1, 2$.

Conversely, assume that $f : \Pi_i\mathcal{X} \to \Pi_i\mathcal{Y}$ is a h-map. If there exists $y \in \mathrm{Im}(f)$ such that $(y]$ is isomorphic to $\mathbf{F}_i$ then there exists $x \in \Pi_i\mathcal{X}$ such that $(x]$ is isomorphic to $(y]$ and $f(x) = y$. Since $\mathbf{F}_i$ is automorphism free we infer that $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = d$ (and if $i = 2$ then also $f(e) = e$) and $x \geq a, b$. Hence $f(A_X) \subseteq A_Y$ and we apply Lemma 4.10 to complete the proof. □

Set $\mathcal{X}_i = \Pi_i(V, T_1, x_0, x_1) = (X_i; \leq, \tau)$ for $i = 0, 1, 2$, so that $\mathcal{X}_i$ is an h-space from $\mathbb{PHV}_i$. Set $Z = S_1^a$, $U = [\{([(a_4, 2j)], 0) \mid j = 0, 1, \ldots, 7\})$, and

$$\emptyset \neq C \subseteq \{([(a_4, 10)], [(a_4, 2j)]) \mid j = 0, 1, 2\} \cup \{([(a_4, 12)], [(a_4, 2j)]) \mid j = 0, 1, 2, 3\}.$$

Then U is an increasing subset of X (thus it is convex), $\emptyset \neq Z \subseteq \mathrm{Max}(U)$ and $C \subseteq U \cap \mathrm{Max}(X)$. Observe that there exists no surjective h-map from $\mathbf{G}_i$ onto $\mathbf{F}_i$ for $i = 0, 1, 2$. For every $i = 0, 1, 2$ we have $C \cap Z = \emptyset$ in $\mathcal{X}_i$ and $(u]$ in $\mathcal{X}_i$ is isomorphic to $\mathbf{G}_i$ for every $u \in C$ and $(z]$ in $\mathcal{X}_i$ is isomorphic to $\mathbf{F}_i$ for every $z \in Z$ we deduce that (t4) holds. The condition (t5) follows from Corollary 4.8 and Lemma 4.11 or Corollary 4.12; we conclude that $\mathcal{X}_i$ is a (U, C, Z)-testing object of $\mathbb{V}_i$ for $i = 0, 1, 2$ because $\Pi_i(V, T_0, x_0, x_1)$ is a h-subspace of $\mathcal{X}_i \setminus C$. Set $V = [\{([(a_4, 10)], 0)\})$ and

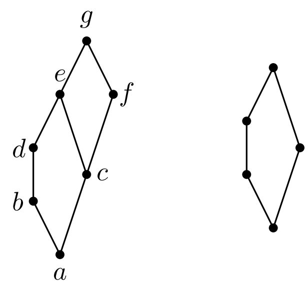
$$\begin{aligned}
(u_i, v_i) &= (([(a_4, 2i)], 0), ([(a_4, 10)], [(a_4, 2i)])) \text{ for } i = 0, 1, 2, \text{ and} \\
(u_i, v_i) &= (([(a_4, 2(i-3))], 0), ([(a_4, 12)], [(a_4, 2(i-3))])) \text{ for } i = 3, 4, 5, 6.
\end{aligned}$$

Observe that V is an increasing set of U, $(U;\preceq)$ and $(V;\leq)$ are order connected. Clearly, (u_i,v_i) for $i=0,1,\dots,6$ are f-covering pairs satisfying (u3) and (u4), thus $(U,V,\{(u_i,v_i)\mid i=0,1,\dots 6\})$ is a u-triple. Whence $(\mathcal{X}_i,U,C)$ is a universal testing object of $\mathbb{V}_i$ with respect to $\mathbb{W}_i$. Since $\mathcal{X}_i$ is finite, by Corollary 3.10 it is also a standard Q-testing object. By Theorem 3.9 and Corollary 3.6 we then obtain

Theorem 4.13. *For $i=0,1,2$, the variety $\mathbb{V}_i$ is $\mathbb{W}_i$-relatively ff-alg-universal and contains an A-D family, so that $\mathbb{V}_i$ is Q-universal for $i=0,1,2$.*

Finally let $\mathbb{DG}_1$ be the category of all finite antireflexive, antisymmetric, strongly connected digraphs (X,R) such that there exist $(x,y),(x,z),(y,z)\in R$ for pairwise distinct vertices $x,y,z\in X$, and all their digraph homomorphisms. Then the digraphs (W,S_0^a) and (W,S_1^a) defined just before Lemma 4.2 belong to $\mathbb{DG}_1$.

Consider a variety of Heyting algebras determined by the h-space $\mathbf{Q}_9$ given in Figure 8.



$\mathbf{Q}_9$ $\mathbf{R}_9$

Figure 8

We shall construct an embedding of $\mathbb{DG}_1$ into Var($\mathbf{DQ}_9$)

For a digraph $\mathbf{G}=(X,R)\in\mathbb{DG}_1$ let us define the h-space $\Theta\mathbf{G}=(Y_{\mathbf{G}};\leq,\tau)$ so that

- (•) $Y_{\mathbf{G}}=\{a,b,c\}\cup(X\times 2)\cup(R\times 4)$ (we assume that a, b and c are pairwise distinct elements with $a,b,c\notin(X\times 2)\cup(R\times 4)$ and that $(X\times 2)\cap(R\times 4)=\emptyset$);
- (•) $\leq$ is the least partial order such that
  $a<b,c$,
  $b<(x,0)<(x,1)>c$ for all $x\in X$,
  $c<(r,0)<(r,1)>b$ for all $r\in R$,
  $((x,y),0)<((x,y),2)>(x,1)$ and $((x,y),1)<((x,y),3)>(y,0)$ for all $(x,y)\in R$;
- (•) τ is the discrete topology on $Y_{\mathbf{G}}$.

For a digraph homomorphism $f:(X,R)\to(X',R')\in\mathbb{DG}_1$ define

$$\Theta f(u)=\begin{cases} u & \text{if } u\in\{a,b,c\},\\ (f(x),i) & \text{if } u=(x,i) \text{ for } x\in X \text{ and } i=0,1,\\ (f(x),f(y),i) & \text{if } u=((x,y),i) \text{ for } (x,y)\in R\\ & \text{and } i=0,1,2,3.\end{cases}$$

Since f is a digraph homomorphism, we conclude that $(f(x),f(y))\in R'$ for all $(x,y)\in R$ and hence Θf is correctly defined.

Direct calculations give

Lemma 4.14. *For every digraph $\mathbf{G}=(X,R)\in\mathbb{DG}_1$ we have*

1. $\mathrm{Max}(Y_{\mathbf{G}})=R\times\{2,3\}$ *and* $\mathrm{Min}(Y_{\mathbf{G}})=\{a\}$*;*
2. $((r,i)]$ *is isomorphic to* $\mathbf{Q}_9$ *for all* $r\in R$ *and* $i=2,3$*.*

(3) $\Theta\mathbf{G}$ *is a finite h-space from* $\mathbb{PH}(\mathrm{Var}(\mathbf{Q}_9))$*;*

For every digraph homomorphism $f : \mathbf{G} \to \mathbf{G}' \in \mathbb{DG}_1$*,* $\Theta f : \Theta\mathbf{G} \to \Theta\mathbf{G}'$ *is an h-map. Thus* $\Theta : \mathbb{DG}_1 \to \mathbb{PH}(\mathrm{Var}(\mathbf{Q}_9))$ *is a faithful functor.*

Lemma 4.15. *Let* $h : (Y_{\mathbf{G}}; \leq, \tau) \to (Y_{\mathbf{G}'}; \leq, \tau)$ *be an h-map where* $(Y_{\mathbf{G}}; \leq, \tau) = \Theta\mathbf{G}$*,* $(Y_{\mathbf{G}'}; \leq, \tau) = \Theta\mathbf{G}'$ *for digraphs* $\mathbf{G} = (X, R), \mathbf{G}' = (X', R') \in \mathbb{DG}_1$*. Then either there exists a digraph homomorphism* $g : \mathbf{G} \to \mathbf{G}'$ *with* $h = \Theta g$ *or else* $|h(\{a, b, c\})| \leq 2$*.*

Proof. By Lemma 4.14(1), $h(a) = a$. Clearly $a \in \mathrm{Cov}(x)$ for $x \in Y_{\mathbf{G}}$ (or $x \in Y_{\mathbf{G}'}$) if and only if $x \in \{b, c\}$ and thus $h(\{b, c\}) \subseteq \{a, b, c\}$. Hence $h(\{b, c\}) = \{b, c\}$ follows from $|h(\{a, b, c\})| = 3$. Thus it suffices to investigate the case of $h(\{b, c\}) = \{b, c\}$. Observe that any h-endomorphism of $\mathbf{Q}_9$ fixing elements b and c has to fix also d and e, and any endomorphism of $\mathbf{Q}_9$ fixing b, c and f is the identity mapping, see Figure 8. We exploit these facts in what follows.

Observe that $(X \times \{1\}) \cup (R \times \{1\})$ (or $(X' \times \{1\} \cup (R' \times \{1\}))$ is the set of all least elements above both b and c. Hence $h((X \times \{1\}) \cup (R \times \{1\})) \subseteq (X' \times \{1\}) \cup (R' \times \{1\})$. Since $\mathrm{Cov}(x, 1) = \{(x, 0), c\}$ and $\mathrm{Cov}(r, 1) = \{(r, 0), b\}$ for $x \in X$ and $r \in R$ (or $x \in X'$ and $r \in R'$) we infer that one of the following two possibilities occurs:

(a) $h(b) = b$, $h(c) = c$, there exist mappings $g : X \to X'$ and $f : R \to R'$ with $h(x, i) = (g(x), i)$ for all $x \in X$, $i \in 2$ and $h(r, i) = (f(r), i)$ for all $r \in R$, $i \in 2$;

(b) or $h(b) = c$, $h(c) = b$, there exist mappings $g : X \to R'$ and $f : R \to X'$ with $h(x, i) = (g(x), i)$ for all $x \in X$, $i \in 2$ and $h(r, i) = (f(r), i)$ for all $r \in R$, $i \in 2$.

If $(x, y) \in R$ then $h((x, y), 3) > (g(y), 0), (f(x, y), 1)$ and $h((x, y), 2) > (g(x), 1), (f(x, y), 0)$. If $h(b) = b$ and $h(c) = c$ then we infer that $f(x, y) = (g(x), g(y)) \in R'$ and $h((x, y), i) = (f(x, y), i)$ for $i = 2, 3$. If $h(b) = c$ and $h(c) = b$ then $h((x, y), 3) = (g(y), 2)$, $h((x, y), 2) = (g(x), 3)$ and $g(x) = (u, f(x, y)), g(y) = (f(x, y), v) \in R'$ for some $u, v \in X'$.

First consider the case $h(b) = c$ and $h(c) = b$. Then there exist pairwise distinct $x, y, z \in X$ with $(x, y), (x, z), (y, z) \in R$. Thus $(x, 1), ((x, y), 0) < ((x, y), 2)$ and $(x, 1), ((x, z), 0) < ((x, z), 2)$. Then $h((x, y), 2) = (r, 3) \geq (g(x), 1), (f(x, y), 0)$ and $h((x, z), 2) = (r', 3) \geq (g(x), 1), (f(x, z), 0)$ for some $r, r' \in R'$. Hence $r = g(x) = r'$, thus $h((x, y), 2) = (g(x), 3) = h((x, z), 2)$. Hence $f(x, y) = f(x, z)$. From $(y, 0), ((x, y), 1) \leq ((x, y), 3)$ it follows that $h((x, y), 3) = (r'', 2) \geq (g(y), 0), (f(x, y), 1)$ for some $r'' \in R'$ and hence $r'' = g(y) = (f(x, y), 1), v) \in R'$ for some $v \in X'$. Analogously, from $(z, 0), ((x, z), 1) < ((x, z), 3)$ and $(z, 0), ((y, z), 1) < ((y, z), 3)$ it follows that $h((x, z), 3) = (g(z), 2) = h((y, z), 3)$ and $f(x, z) = f(y, z)$. And then from $(y, 1), ((y, z), 0) \leq ((y, z), 2)$ it follows that

$$h((y, z), 2) = (r''', 3) \geq (g(y), 1), (f(y, z), 0)$$

for some $r''' \in R$ and hence $r''' = g(y) = (u, f(y, z)) \in R'$ for $u \in X'$. If we combine these facts we obtain that $g(y) = (f(x, y), f(y, z)) \in R'$ and $f(x, y) = f(x, z) = f(y, z)$ and this is a contradiction because (X', R') is antireflexive digraph.

Therefore $h(b) = b$ and $h(c) = c$. Then $g : (X, R) \to (X', R')$ is a digraph homomorphism and clearly $h = \Theta g$. □

For $\mathbf{G} = (X, R) \in \mathbb{DG}_1$ set $Z_{\mathbf{G}} = R \times \{2, 3\} = \mathrm{Max}(\Theta\mathbf{G})$. Since $\mathrm{Aut}(\mathbf{Q}_9)$ is a singleton group, we can omit $\mathcal{G}$ and set $\mathcal{Z} = \{Z_{\mathbf{G}} \mid \mathbf{G} \in (\mathbb{DG}_1)^o\}$. Thus

Corollary 4.16. *The functor* $\Theta : \mathbb{DG}_1 \to \mathbb{PH}(\mathrm{Var}(\mathbf{Q}_9))$ *is a* $\mathcal{Z}$*-relatively full embedding.*

Now we set

$$\mathcal{X} = \Theta(W, (S_1)^a), \quad U = [\{([(a_4, 2j)], 0) \mid j = 0, 1, \ldots, 6\}), \quad V = [\{([(a_4, 10)], 0)\}),$$

$$C = \{(([(a_4, 10)], [(a_4, 2j)]), 3) \mid j = 0, 1, 2\} \cup \{(([(a_4, 12)], [(a_4, 2j)]), 3) \mid j = 0, 1, 2, 3\},$$

$$Z = (S_1)^a \times \{2,3\}, \quad \text{and}$$

$$(u_i, v_i) = \begin{cases} (([(a_4, 2i)], 0), (([({}_4, 10)], [(a_4, 2i)]), 3)) & \text{if } i = 0,1,2, \\ (([(a_4, 2(i-3))], 0), (([(a_4, 12)], [(a_4, 2(i-3))]), 3)) & \text{if } i = 3, \dots, 6. \end{cases}$$

Thus $(U, V, \{(u_i, v_i) \mid i = 0, 1, \dots 6\})$ is a u-triple and $(x] \in \mathbb{PH}(\mathrm{Var}(\mathbf{R}_5)$ for all $x \in W \setminus Z$. Combine Lemma 4.2 and the fact that $\Theta(W, S_0^a)$ is an h-subspace of $\mathcal{X} \setminus C$ and Lemmas 4.14 and 4.15 we obtain that $\mathcal{X}$ is a (U, C, Z)-testing object of $\mathrm{Var}(\mathbf{Q}_9)$ because $\{a, b, c\} \subseteq (x]$ for every $x \in \mathrm{Max}(\mathcal{X})$. Hence $\mathcal{X}$ is a universal testing object of $\mathrm{Var}(\mathbf{Q}_9)$ with respect to $\mathrm{Var}(\mathbf{R}_9)$. Since $\mathcal{X}$ is finite, by Corollary 3.10, it is also standard Q-universal testing object. By Theorem 3.9 and Corollary 3.6, we obtain

Corollary 4.17. *The variety* $\mathrm{Var}(\mathbf{DQ}_9)$ *is* $\mathrm{Var}(\mathbf{DR}_9)$*-relatively* ff*-alg-universal. It also contains an A-D family and thus it is* Q*-universal.*

5. Undirected graphs

In this section we shall construct testing objects using undirected graphs. First we recall several notions for undirected graphs.

An undirected graph (or a graph) is a pair $\mathbf{G} = (V, E)$ where V is a set (of the vertices of $\mathbf{G}$) and E is a set of two-element subsets of V (the edges of $\mathbf{G}$). A graph homomorphism from (V, E) to (V', E') is a mapping $f : V \to V'$ such that $\{f(v), f(w)\} \in E'$ for every $\{v, w\} \in E$. A path between $u, v \in V$ of length k in a graph $\mathbf{G} = (V, E)$ is a sequence $P = \{u = x_0, x_1, \dots, x_k = v\}$ of vertices of $\mathbf{G}$ such that $\{x_i, x_{i+1}\} \in E$ for $i = 0, 1, \dots, k-1$ and these edges are pairwise distinct. If, moreover, $\{x_k, x_0\} \in E$ then it is a cycle of length $k+1$. A graph (V, E) is connected if for every pair of vertices $u, v \in V$ there exists a path between u and v. In [22] it is shown that the graph $\mathbf{F} = (T, F)$ shown in Fig 9 is rigid (that is, only the identity mapping is a graph homomorphism from $\mathbf{F}$ to itself).

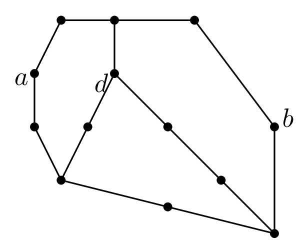


Figure 9

Let us define $\mathbf{F}_0 = (T_0, F_0)$, where

- ($\bullet$) $T_0 = (T \times 22)/\theta \cup D$ where $D = \{c_i \mid i \in 11\}$ and θ is the least equivalence on $T \times 22$ such that $(b, i)\theta(a, i+1)$ for all $i \in 21$ (we assume that c_i for $i \in 11$ are pairwise distinct new vertices, that is, $c_i \notin (T \times 22)/\theta$ for every $i \in 11$);
- ($\bullet$) if $[x]$ is the class of θ containing a vertex $x \in T \times 22$ then

$$\begin{aligned} F_0 &= \{\{[x], [y]\} \mid \exists i \in 22,\ \{u, v\} \in F,\ x = (u, i),\ y = (v, i)\} \\ &\cup \{\{[(d, i)], c_i\}, \{[(d, i+11)], c_i\} \mid i \in 11\}. \end{aligned}$$

A graph (V, E) is bipartite if there exist disjoint sets $V_1, V_2 \subseteq V$ such that $V_1 \cup V_2 = V$ and $|e \cap V_1| = |e \cap V_2| = 1$ for every $e \in E$. Then $\{V_1, V_2\}$ is a bipartite decomposition of (V, E).

For a bipartite graph $\mathbf{B} = (D, B)$, we define $\mathbf{F_B} = (T_0, F_0 \cup B)$. Then

Lemma 5.1. *For every bipartite graph* $\mathbf{B} = (D, B)$ *the identity mapping* $\iota : T_0 \to T_0$ *is the only graph homomorphism from* $\mathbf{F}_0$ *to* $\mathbf{F_B}$.

Proof. Since $F_0 \subseteq F_0 \cup B$, the identity mapping $\iota : T_0 \to T_0$ is obviously a graph homomorphism from $\mathbf{F}_0$ to $\mathbf{F_B}$. Conversely assume that $f : \mathbf{F}_0 \to \mathbf{F_B}$ is a graph homomorphism. Since the shortest odd cycle of $\mathbf{F}$ has length 7 and because every cycle of $\mathbf{F_B}$ containing c_i for some $i \in 11$ contains a path between (d, k) and (d, l) in $\mathbf{F_B}$ for distinct $k, l \in 22$ and this path has length at least 6), the shortest odd cycle of $\mathbf{F_B}$ has length 7. If shortest odd cycles in two graphs have the same length, then graph homomorphisms between these graphs preserve shortest odd cycles. For every $s, t \in T$, either there exists a cycle of length 7 containing s and t or there exist two cycles C_1, C_2 of length 7 such that C_1 contains s, C_2 contains t and C_1 and C_2 have a common edge. Hence the image of a graph homomorphism from $\mathbf{F}$ to $\mathbf{F_B}$ has also this property. By a routine inspection, we obtain that $\{[(t, i)] \mid t \in T\}$ for $i \in 22$ are all the subsets of $\mathbf{F_B}$ with this property. Moreover, the subgraph of $\mathbf{F_B}$ induced on the set $\{[(t, i)] \mid t \in T\}$ is isomorphic to $\mathbf{F}$ for all $i \in 22$. For every $i \in 22$ the mapping $g_i : T \to T_0$ such that $g_i(t) = [(t, i)]$ for all $t \in T$ is a graph homomorphism from $\mathbf{F}$ to $\mathbf{F}_0$. Since $\mathbf{F}$ is a rigid graph we deduce that for every $i \in 22$ there exists $j(i) \in 22$ with $f \circ g_i = g_{j(i)}$, hence $f([(t, i)]) = [(t, j(i))]$ for all $t \in T$ and $i \in 22$. From the definition of θ it follows that $j(i + 1) = j(i) + 1$ for all $i \in 22$ (with the usual addition) and thus $j(i) = i$. Thus $f([(t, i)]) = [(t, i)]$ for all $t \in T$ and $i \in 22$. Since for every $i \in 11$ there exist exactly two edges of F_0 or $F_0 \cup B$ between c_i and $(T \times 22)/\theta$ – the edges $\{[(d, i)], c_i\}$ and $\{[(d, i + 9)], c_i\}$, we conclude that $f(c_i) = c_i$ for all $i \in 11$ and f is the identity mapping of T_0. $\square$

A graph (V, E) has chromatic number 3 if it is not bipartite and there exists a graph homomorphism $f : (V, E) \to (\{0, 1, 2\}, \{\{0, 1\}, \{0, 2\}, \{1, 2\}\}) = \mathbf{K}_3$, the complete graph on three vertices. By a direct verification, there exists a graph homomorphism $f : \mathbf{F} \to \mathbf{K}_3$ such that $f(a) = f(b)$. Assume that $f(d) = l \in \{0, 1, 2\}$. Let $\mathbf{B} = (D, B)$ be a bipartite graph with a bipartite decomposition $\{D_1, D_2\}$ and define a mapping $g : T_0 \to \{0, 1, 2\}$ by $g([(u, i)]) = f(u)$ for all $u \in T$ and $g(d) = l'$ for $d \in D_1$ and $g(d) = l''$ for $d \in D_2$ where $\{l, l', l''\} = \{0, 1, 2\}$. Since $f(a) = f(b)$, the definition of g is correct and, by a routine calculation, we obtain that $g : \mathbf{F_B} \to \mathbf{K}_3$ is a graph homomorphism. Thus $\mathbf{F}$, $\mathbf{F}_0$ and $\mathbf{F_B}$ for any bipartite graph $\mathbf{B} = (D, B)$ have a chromatic number 3. We exploit these graphs to construct special embeddings into duals of some finitely generated varieties of Heyting algebras.

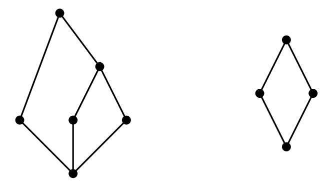
Let $\mathbb{GR}$ be the category of all finite connected graphs with at least two vertices having the chromatic number 3. Let $\mathbb{TGR}$ be a category whose object are triples (V, E, f) where (V, E) is a finite connected graph with chromatic number 3 and $f : (V, E) \to \mathbf{K}_3$ is a graph homomorphism, and whose morphisms from (V, E, f) into (V', E', f') are all graph homomorphisms $g : (V, E) \to (V', E')$ with $f = f' \circ g$.

We consider the h-space $\mathbf{Q}_8$ given in Figure 10.

We shall construct a $\mathcal{Z}$-relatively full embedding of $\mathbb{GR}$ into $\mathbb{PH}(\mathrm{Var}\,\mathbf{Q}_8)$. For a graph $\mathbf{G} = (V, E) \in \mathbb{GR}$, let us define $\Gamma\mathbf{G} = (X_{\mathbf{G}}; \leq, \tau)$ where $X_{\mathbf{G}} = \{a, b\} \cup V \cup (E \times 2)$ (we assume that the latter union is disjoint), and $\leq$ be the least partial order such that

- (•) $a < b < (e, 1)$ for all $e \in E$;
- (•) $a < v$ for all $v \in V$;
- (•) $v < (e, 0) < (e, 1)$ for all $v \in e \in E$;

and τ is the discrete topology on $X_{\mathbf{G}}$. Denote $Z_{\mathbf{G}} = E \times \{1\}$.



$\mathbf{Q}_8$ $\mathbf{R}_8$

Figure 10

For a graph homomorphism $f : (V, E) \to (V', E')$, let us define

$$\Gamma f(u) = \begin{cases} u & \text{if } u \in \{a, b\}, \\ f(v) & \text{if } u = v \in V, \\ (\{f(v), f(w)\}, i) & \text{if } u = (\{v, w\}, i) \text{ for } \{v, w\} \in E \text{ and } i = 0, 1. \end{cases}$$

By a direct verification, we obtain

Lemma 5.2. *For every graph $\mathbf{G} = (V, E) \in \mathbb{GR}$,*

(1) *$\leq$ is a partial order, $X_{\mathbf{G}}$ is finite and $\Gamma\mathbf{G}$ is an h-space;*
(2) *$\mathrm{Max}(\Gamma\mathbf{G}) = E \times \{1\} = Z_{\mathbf{G}}$;*
(3) *$(x]$ is isomorphic $\mathbf{Q}_8$ for $x \in X_{\mathbf{G}}$ if and only if $x = (e, 1)$ for $e \in E$.*

If $f : \mathbf{G} \to \mathbf{G}'$ is a graph homomorphism then Γf is an h-map. Thus $\Gamma : \mathbb{GR} \to \mathbb{PH}(\mathrm{Var}(\mathbf{Q}_8))$ is a faithful functor.

Lemma 5.3. *Let $\mathbf{G}_0 = (V_0, E_0)$ and $\mathbf{G}_1 = (V_1, E_1)$ belong to $\mathbb{GR}$ and let $f : \Gamma\mathbf{G}_0 \to \Gamma\mathbf{G}_1$ be an h-map. Then one of the following cases occurs:*

(1) *there exists a graph homomorphism $g : \mathbf{G}_0 \to \mathbf{G}_1$ with $\Gamma g = f$;*
(2) *$\mathrm{Im}(f) \subseteq \{a, b\}$;*
(3) *$f(b) = a$, $\mathrm{Im}(f) \cap (E_1 \times \{1\}) = \emptyset$, and $f(e, 0) \neq f(e, 1)$ implies that $f(e, 0) = a$ and $f(e, 1) \in V \cup \{b\}$ for all $e \in E_0$;*
(4) *$f(b) \in V_1$ and $\mathrm{Im}(f) \subseteq (\{(e, 0) \mid e \in E_1,\ f(b) \in e\}]$.*

Proof. Since f has the h-property we conclude that $f(a) = a$, $f(\{b\} \cup V_0) \subseteq \{a, b\} \cup V_1$ and $f(E_0 \times \{0\}) \cap (E_1 \times \{1\}) = \emptyset$.

First assume that $f(b) = a$. Then $|f(((e, 1)])| < 6 = |((e', 1)]|$ for all $e \in E_0$ and all $e' \in E_1$ thus $\mathrm{Im}(f)$ is disjoint with $E_1 \times \{1\}$. Since $a \in \mathrm{Min}(X_{\mathbf{G}_1})$ and $((e, 1)] = ((e, 0)] \cup \{b, (e, 1)\}$ we conclude that $f(((e, 1)]) = f(((e, 0)]) \cup \{f(e, 1)\}$ because f has the h-property. Thus if $|\operatorname{Cov}(f(e, 1))| = 2$ then $f(e, 0) = f(e, 1)$. Since for $y \in X_{\mathbf{G}_1}$ we have $|\operatorname{Cov}(y)| \leq 1$ if and only if $y \notin E_1 \times \{0, 1\}$ and since $E_1 \times \{1\} \cap \mathrm{Im}(f) = \emptyset$ we infer that if $f(e, 1) \neq f(e, 0)$ then $f(e, 1) \in \{b\} \cup V_1$ and hence $f(e, 0) = a$, and (3) holds.

Assume that $f(b) \in V_1$. Then for all $e \in E_0$ we have $f(e, 1) \in [f(b))$ because $b \leq (e, 1)$ for all $e \in E_0$ and $\mathrm{Max}([v)) = \{(e', 1) \mid e' \in E_1, v \in e'\}$ for all $v \in V_1$. Since $|V_0| \geq 2$ and $\mathbf{G}_0$ is connected, for every $v \in V_0$ there exists an edge $e \in E_0$ with $v \in e$. If $f(u) = b$ for some $u \in V_0$ then for an edge $e = \{u, v\} \in E_0$ we infer that $(f(e, 0)] = (b]$ because $|((e, 0)]| = 4$, $|((e', 1)]| = 6$ for all $e' \in E_1$ and $[b) = \{b\} \cup (E_1 \times \{1\})$ in $\Gamma\mathbf{G}_1$. Then $f((e, 0)), f(b) < f((e, 1))$ and hence $f((e, 1)) \in E_1 \times \{1\}$ but $|f(((e, 1)])| \leq 4$ – this is a contradiction. Thus $b \notin f(V_0)$ and hence $f((e, 1)) \notin E_1 \times \{1\}$. Therefore $\mathrm{Im}(f) \subseteq (\{(e, 0) \mid e \in E_1,\ f(b) \in e\}]$, and (4) holds.

Assume that $f(b) = b$. If $f(v) \in \{a, b\}$ for some $v \in V_0$ then for an edge $e \in E_0$ with $v \in e$ we have $|f(((e, 1)])| < 6$ and hence $f(e, 1) = b$. Then $f([(e, 1))) = \{a, b\}$ and, from the connectedness of $\mathbf{G}_0$, we conclude that $\mathrm{Im}(f) = \{a, b\}$ and (2) holds. Thus we can assume that $f(b) = b$ and $f(V_0) \subseteq V_1$. Let $\{v, w\} \in E_0$. Then $b, v, w < (\{v, w\}, 1)$ implies that $b, f(v), f(w) \leq f(\{v, w\}, 1)$ and from $f(v) \in V_1$ we conclude that $f(\{v, w\}, 1) \in E_1 \times \{1\}$. By the h-property of f we obtain that $f(v) \neq f(w)$, $\{f(v), f(w)\} \in E_1$ and $f(\{v, w\}, 1) = (\{f(v), f(w)\}, 1)$. Whence the domain-range restriction g of f to V_0 and V_1 is a graph homomorphism from $\mathbf{G}_0$ to $\mathbf{G}_1$ with $\Gamma g = f$, and hence (1) follows. $\square$

Corollary 5.4. *The functor* $\Gamma : \mathbb{GR} \to \mathbb{PH}(\mathrm{Var}(\mathbf{Q}_8))$ *is a $\mathcal{Z}$-relatively full embedding.*

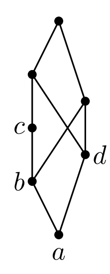
Let $B = \{\{c_i, c_5\} \mid i \in 5\} \cup \{\{c_{i+6}, c_{10}\} \mid i \in 4\}$. Then $\mathbf{B} = (D, B)$ is a bipartite graph with a bipartite decomposition $\{D_1 = \{c_i \mid i \in 5\} \cup \{c_{i+6} \mid i \in 4\}, D_2 = \{c_5, c_{10}\}\}$. Let us define a graph $\mathbf{F}_1 = (T_0, F_1)$ where $F_1 = F_0 \cup B$. Set $\mathcal{X} = \Gamma \mathbf{F}_1$, $Z = \{(e, 1) \mid e \in F_1\} = \mathrm{Max}(\Gamma \mathbf{F}_1)$, $U = \{c_i \mid i \in 11\} \cup \{b\} \cup \{(\{c_i, c_5\}, j) \mid i \in 5,\ j = 0, 1\} \cup \{(\{c_{i+6}, c_{10}\}, j) \mid i \in 4,\ j = 0, 1\}$, $V = \{c_{10}\} \cup \{(\{c_{i+6}, c_{10}\}, j) \mid i \in 4,\ j = 0, 1\}$, $C = [\{(\{c_i, c_5\}, 1) \mid i \in 4\} \cup \{(\{c_{i+6}, c_{10}\}, 1) \mid i \in 3\})$,

$$(u_i, v_i) = \begin{cases} (b, (\{c_{i+6}, c_{10}\}, 1)) & \text{for } i = 0, 1, 2, \\ (b, (\{c_{i-3}, c_5\}, 1)) & \text{for } i = 3, 4, 5, 6. \end{cases}$$

By a direct verification, $C \subseteq \mathrm{Max}(U)$ and $V \subseteq U$, V is an increasing subset of U and U is a convex subset of $\mathcal{X}$, thus U is functorial. Further $(V; \leq)$ and $(U \setminus V, \preceq)$ are order connected and (u_i, v_i) is a f-covering pair in $U(\mathcal{X})$ for all $i \in 7$, thus $(U(\mathcal{X}), V, \{(u_i, v_i) \mid i \in 7\})$ is a u-triple. By Lemma 5.1, $\mathcal{X}$ is automorphism free. Since $|\mathrm{Cov}(c)| \geq 2$ for every $c \in C$ we obtain, by Lemmas 5.1 and 5.3, that $\mathcal{X}$ is a (U, C)-representing object from $\mathrm{Var}(\mathbf{Q}_8)$. To prove that $\mathcal{X}$ is a (U, C, Z)-testing object from $\mathrm{Var}(\mathbf{Q}_8)$ it suffices to verify (t5). Observe that $\Gamma(T_0, F_0)$ is a h-subspace of $\mathcal{X} \setminus C$. Consider h-map $f : \Gamma(T_0, F_0) \to \mathcal{X}$. On the set $X \setminus U$ we recognize whether $f = \Gamma g$ for some graph homomorphism $g : (T_0, F_0) \to (T_0, F_1)$ (by Lemma 5.1, g is the identity) or $\mathrm{Im}(f) = \{a, b\}$ then for $z = (e, 1) \in Z$ we choose $u_z = (e, 0)$ or $b \notin \mathrm{Im}(f)$ then for $z \in Z$ we choose $u_z = b$. By Lemma 5.2, the other possibilities do not occur and for every h-map $g : \mathcal{X} \setminus C \to \mathcal{X}$ with $f \restriction X \setminus U = g \restriction X \setminus U$ we have $u_z \notin \mathrm{Im}\, g$ for all $z \in Z$. Thus (t5) holds and $\mathcal{X}$ is a (U, C, V)-testing object of $\mathrm{Var}(\mathbf{Q}_8)$. Therefore $\mathcal{X}$ is a finite universal testing object of $(\mathrm{Var}(\mathbf{Q}_8))$ with respect to $\mathrm{Var}(\mathbf{R}_8)$ and, by Corollary 3.10, $\mathcal{X}$ is also a standard Q-universal testing object. Hence, by Corollary 3.6 and Theorem 3.9 we obtain

Corollary 5.5. *The variety* $\mathrm{Var}(\mathbf{DQ}_8)$ *is* $\mathrm{Var}(\mathbf{DR}_8)$*-relatively ff-alg-universal and has an A-D family, so it is also Q-universal.*

Next we describe a $\mathcal{Z}$-relatively full embedding of $\mathbb{GR}$ into $\mathbb{PH}(\mathrm{Var}(\mathbf{Q}_{10}))$ for the h-space $\mathbf{Q}_{10}$ given in Figure 11.



$\mathbf{Q}_{10}$

Figure 11

For a graph $\mathbf{G} = (V, E) \in \mathbb{GR}$ define $\Xi \mathbf{G} = (Y_{\mathbf{G}}; \leq, \tau)$ where $Y_{\mathbf{G}} = \{a, b, c, d\} \cup V \cup E \cup \{(v, e) \mid v \in e \in E\}$ (we assume that a, b, c, and d are pairwise distinct elements and

$\{a,b,c,d\}$, V, E, and $\{(v,e) \mid v \in e \in E\}$ are pairwise disjoint sets), $\leq$ is the least partial order such that

- ($\bullet$) $a \leq b \leq c$ and $a \leq d$;
- ($\bullet$) $c \leq v \geq d$ for all $v \in V$ and $b \leq e \leq d$ for all $e \in E$;
- ($\bullet$) $v \leq (v,e) \geq e$ for all $e \in E$ and $v \in e$;

and τ is a discrete topology on $Y_{\mathbf{G}}$. For a graph homomorphism $f : (V,E) \to (V',E)$ define

$$\Xi f(u) = \begin{cases} u & \text{if } u \in \{a,b,c,d\}, \\ f(v) & \text{if } u = v \in V, \\ \{f(v), f(w)\} & \text{if } u = \{v,w\} \in E, \\ (f(v), \{f(v), f(w)\}) & \text{if } u = (v, \{v,w\}) \text{ for } \{v,w\} \in E. \end{cases}$$

By a direct inspection we obtain

Lemma 5.6. *For every graph* $\mathbf{G} = (V,E) \in \mathbb{GR}$*,*

1. $\Xi\mathbf{G}$ *is a finite h-space;*
2. $\operatorname{Max}(Y_{\mathbf{G}}) = \{(v,e) \mid v \in e \in E\}$*;*
3. $(x]$ *is isomorphic to* $\mathbf{Q}_{10}$ *for* $x \in Y_{\mathbf{G}}$ *if and only if* $x = (v,e)$ *for some* $e = \{v,w\} \in E$*;*
4. $\{a,b,c,d\} \subseteq ((v,e)]$ *for all* $e \in E$ *and* $v \in e$*;*
5. $|(e]| = 4$ *for all* $e \in E$ *and* $|(v]| = 5$ *for all* $v \in V$*;*
6. $(e]$ *is isomorphic to* $\mathbf{R}_8$ *from Figure 10 for all* $e \in E$*.*

For every graph homomorphism $f : \mathbf{G} \to \mathbf{G}' \in \mathbb{GR}$ *the mapping* $\Xi f : \Xi\mathbf{G} \to \Xi\mathbf{G}'$ *is an h-map. Thus* $\Xi : \mathbb{GR} \to \mathbb{PH}(\operatorname{Var}(\mathbf{Q}_{10}))$ *is a faithful functor.*

The proof of $\mathcal{Z}$-relative fulness is based on the following lemma.

Lemma 5.7. *If* $f : \Xi\mathbf{G} \to \Xi\mathbf{G}'$ *is an h-map for* $\mathbf{G} = (V,E), \mathbf{G}' = (V',E') \in \mathbb{GR}$ *then either there exists a graph homomorphism* $g : \mathbf{G} \to \mathbf{G}'$ *with* $\Xi g = f$ *or* $f(c) \in \{a,b,d\}$ *and* $\operatorname{Im}(f) \subseteq (E'] \cup \{c\}$ *or* $\operatorname{Im}(f) \subseteq \{a,b,c\}$*.*

Proof. Let $\mathbf{G} = (V,E)$ and $\mathbf{G}' = (V',E')$ be graphs from $\mathbb{GR}$ and let $f : (Y_{\mathbf{G}}; \leq, \tau) \to (Y_{\mathbf{G}'}; \leq, \tau)$ be an h-map where $\Xi\mathbf{G} = (Y_{\mathbf{G}}; \leq, \tau)$ and $\Xi\mathbf{G}' = (Y_{\mathbf{G}'}; \leq, \tau)$. From the h-property it follows that $f(a) = a$, $f(\{b,d\}) \subseteq \{a,b,d\}$ and, by Lemma 5.6(5), $f(c) \in \{a,b,c,d\}$.

First assume that $f(c) = c$ and $f(d) = d$. Then $f(b) = b$. Since E (or E') is the set of minimal elements in $[b) \cap [d)$ we conclude that $f(E) \subseteq E'$. Analogously, V (or V') is the set of minimal elements in $[c) \cap [d)$ and hence $f(V) \subseteq V'$. From $v, \{v,w\} \leq (v, \{v,w\})$ for all $\{v,w\} \in E$ it follows that $f(v), f(\{v,w\}) \leq f(v, \{v,w\})$ and whence $f(\{(v, \{v,w\}) \mid v \in \{v,w\} \in E\}) \subseteq \{(v', \{v',w'\}) \mid v' \in \{v',w'\} \in E'\}$ and if $f(v, \{v,w\}) = (v', \{v',w'\})$ then $f(v) = v'$, $f(\{v,w\}) = \{v',w'\}$ because

$$\begin{aligned} ((v, \{v,w\})] &= \{a,b,c,d,v,\{v,w\},(v,\{v,w\})\} \text{ and} \\ ((v', \{v',w'\})] &= \{a,b,c,d,v',\{v',w'\},(v',\{v',w'\})\}. \end{aligned}$$

If g is the domain-range restriction of f to V and V' then $\{g(v), g(w)\} = f(\{v,w\}) \in E'$ for all $\{v,w\} \in E$. Whence $g : (V,E) \to (V',E')$ is a graph homomorphism and $\Xi g = f$. If $f(c) \in \{a,b,d\}$ then $|(f(v)]| \leq 4$ for all $v \in V$ and hence, by Lemma 5.6(5), $f(V) \cup f(E) \subseteq (E'] \cup \{c\}$ and, by the h-property, $\operatorname{Im}(f) \subseteq (E'] \cup \{c\}$. If $f(c) = c$ and $f(d) \in \{a,b\}$ then clearly $\operatorname{Im}(f) \subseteq \{a,b,c\}$ and the proof is complete. $\square$

For a graph $\mathbf{G} = (V,E) \in \mathbb{GR}$ set $Z_{\mathbf{G}} = \{(v,e) \mid v \in e \in E\}$ and $\mathcal{Z} = \{Z_{\mathbf{G}} \mid \mathbf{G} \in \mathbb{GR}\}$.

Corollary 5.8. *The functor* $\Xi : \mathbb{GR} \to \mathbb{PH}(\operatorname{Var}(\mathbf{Q}_{10}))$ *is a* $\mathcal{Z}$*-relatively full embedding.*

Let $B = \{\{c_i, c_j\} \mid i \in 8,\ j = 8, 9\} \cup \{\{c_i, c_{10}\} \mid i \in 4\}$ then $\mathbf{B} = (D, B)$ is a bipartite graph with a bipartite decomposition $\{D_1 = \{c_i \mid i \in 8\}, D_2 = \{c_8, c_9, c_{10}\}\}$. Let us define $F_2 = F_0 \cup B$ and $\mathbf{F}_2 = (T_0, F_2)$. Set $\mathcal{X} = \Xi\mathbf{F}_2$, $Z = \{(v, e) \mid v \in e \in F_2\} = \operatorname{Max}(\Xi\mathbf{F}_2)$, $U = [D \cup B)$, $V = [\{c_{10}\})$, $C = \{(c_{10}, \{c_i, c_{10}\}) \mid i \in 3\} \cup \{(c_9, \{c_i, c_9\}) \mid i \in 4\}$,

$$(u_i, v_i) = \begin{cases} (\{c_i, c_{10}\}, (c_i, \{c_i, c_{10}\})) & \text{for } i = 0, 1, 2, \\ (\{c_{i-3}, c_9\}, (c_{i-3}, \{c_{i-3}, c_9\})) & \text{for } i = 3, 4, 5, 6. \end{cases}$$

By a direct verification, $C \subseteq \operatorname{Max}(U) \subseteq \operatorname{Max}(\mathcal{X}) = Z$ and $V \subseteq U$, V and U are increasing (thus U is convex), U is functorial, $(V; \leq)$ and $(U \setminus V; \preceq)$ are order connected and (u_i, v_i) are f-covering pairs for all $i \in 7$, thus $(U(\mathcal{X}), V, \{(u_i, v_i) \mid i \in 7\})$ is a u-triple. Clearly, $\Xi\mathbf{F}_0$ is a h-subspace of $\mathcal{X} \setminus C$. By Lemmas 5.1 and 5.7, if $f : \Xi\mathbf{F}_0 \to \mathcal{X}$ is an h-map then either there exists a graph homomorphism $g : \mathbf{F}_0 \to \mathbf{F}_2$ with $f = \Xi g$ (by Lemma 5.1, g is the identity) or f is not injective on $\{a, b, c, d\}$. By Lemma 5.6, $\{a, b, c, d\} \subseteq (z]$ for every $z \in Z$ and thus (t5) is satisfied. Thus $\mathcal{X}$ is a (U, C, Z)-testing object of $\operatorname{Var}(\mathbf{Q}_{10})$. Thus $\mathcal{X}$ is a finite universal testing object of $(\operatorname{Var}(\mathbf{DQ}_{10}))$ with respect to $\operatorname{Var}(\mathbf{DR}_8)$ and by Corollary 3.10 $\mathcal{X}$ is also a standard Q-universal testing object. Whence by Corollary 3.6 and Theorem 3.9 we obtain

Corollary 5.9. *The variety* $\operatorname{Var}(\mathbf{DQ}_{10})$ *is* $\operatorname{Var}(\mathbf{DR}_8)$*-relatively ff-alg-universal and has an A-D family, thus it is also Q-universal.*

Next we investigate the varieties of Heyting algebras determined by h-spaces $\mathbf{Q}_6$ and $\mathbf{Q}_7$ given in Figure 12. We apply some ideas from [9].

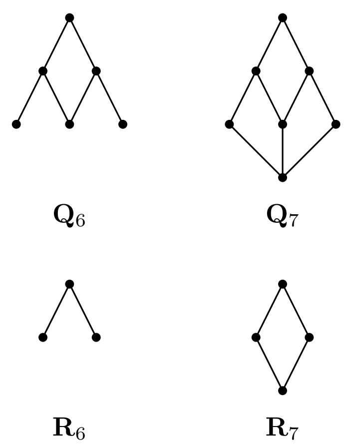


Figure 12

Let $\mathbb{V}$ be the variety of Heyting algebras generated by $\mathbf{DQ}_7$. First we shall construct a functor $\Omega : \mathbb{TGR} \to \mathbb{PHV}$. For a $\mathbb{TGR}$-object $(\mathbf{G} = (V, E)), f)$ set $\Omega(\mathbf{G}, f) = (Z_{\mathbf{G}}; \leq, \tau)$ where $Z_{\mathbf{G}} = \{t, 0, 1, 2\} \cup V \cup E$ (we assume that t is distinct from 0, 1 and 2, and that $\{t, 0, 1, 2\}$, V and E are pairwise disjoint), $\leq$ is the least partial order such that

- (•) $t < i$ for all $i \in \{0, 1, 2\}$;
- (•) $i < v$ for $i \in \{0, 1, 2\}$ and $v \in V$ just when $i \neq f(v)$;
- (•) $v < e$ for $v \in V$ and $e \in E$ just when $v \in e$;

and τ is the discrete topology on $Z_{\mathbf{G}}$. For a $\mathbb{TGR}$-morphism $g : ((V,E),f) \to ((V',E'),f')$ let us define

$$\Omega g(u) = \begin{cases} u & \text{if } u \in \{t,0,1,2\}, \\ g(v) & \text{if } u = v \in V, \\ \{g(v), g(w)\} & \text{if } u = \{v,w\} \in E. \end{cases}$$

By a direct verification we obtain that $\leq$ is a partial order, and because $Z_{\mathbf{G}}$ is finite we conclude

Lemma 5.10. *For every $\mathbb{TGR}$-object $\mathbf{G} = ((V,E),f)$ we have*

(1) *$Z_{\mathbf{G}}$ is finite and $\Omega(\mathbf{G}, f) = (Z_{\mathbf{G}}; \leq, \tau)$ is an h-space;*
(2) *$\mathrm{Max}(Z_{\mathbf{G}}) = E$ and $\mathrm{Min}(Z_{\mathbf{G}}) = \{t\}$;*
(3) *$(x]$ is isomorphic to $\mathbf{Q}_7$ for $x \in Z_{\mathbf{G}}$ if and only if $x \in E$.*

If $g : (\mathbf{G}, f) \to (\mathbf{G}', f')$ is a $\mathbb{TGR}$-morphism then Ωg is an h-map. Thus $\Omega : \mathbb{TGR} \to \mathbb{PH}(\mathrm{Var}(\mathbf{Q}_7))$ is a faithful functor.

Lemma 5.11. *Let $\mathbf{G} = ((V,E),f)$ and $\mathbf{G}' = ((V',E'),f')$ be $\mathbb{TGR}$-objects and let $g : \Omega G \to \Omega G'$ be an h-map. Then $g(t) = t$, $g(\{0,1,2\}) \subseteq \{t,0,1,2\}$ and one of the following cases occurs:*

(1) *there exists $i \in \{0,1,2\}$ such that $i \notin \{g(0), g(1), g(2)\}$ and then $\mathrm{Im}(g) \cap E' = \emptyset$ and $g^{-1}(\{v' \in V' \mid f'(v') \neq i\}) \subseteq E$;*
(2) *$g(E) \subseteq E'$, $g(V) \subseteq V'$, $g(\{0,1,2\}) = \{0,1,2\}$ and the domain-range restriction g' of g to $\{0,1,2\}$ and the domain-range restriction g'' of g to V and V' are such that $g'' : \mathbf{G} \to \mathbf{G}'$ is a graph homomorphism and $g' \circ f = f' \circ g''$.*

Proof. Since g is an h-map we have that $g(t) = t$, $g(\{0,1,2\}) \subseteq \{t,0,1,2\}$ and $g(V) \subseteq V' \cup \{t,0,1,2\}$. Assume that $i \notin \{g(0), g(1), g(2)\}$ for some $i \in \{0,1,2\}$. Consider $v' \in V'$ with $f'(v') \neq i$. Then $i < v'$. If $v \in V$ with $g(v) = v'$ then $i \in (g(v)] = g((v])$ but

$$g((v]) \subseteq g(\{v,t,0,1,2\}) \subseteq \{v',t\} \cup (\{0,1,2\} \setminus \{i\})$$

and this is a contradiction with the h-property of g. Hence $g^{-1}(v') \cap V = \emptyset$ and thus $g^{-1}(v') \subseteq E$. Since $|g(\{t,0,1,2\}| \leq 3$ we infer $|(e]| = 7 > |g(e])|$ for all $e \in E$ and thus $E' \cap \mathrm{Im}(g) = \emptyset$ and (1) is proved. Suppose that $g(\{0,1,2\}) = \{0,1,2\}$. Then the domain-range restriction g' of g to $\{0,1,2\}$ is a permutation of $\{0,1,2\}$. Since $\mathrm{Cov}(v) = \{i \in \{0,1,2\} \mid f(v) \neq i\}$ for all $v \in V$ we have $g(V) \subseteq V'$. For every $e = \{v,w\} \in E$, $\mathrm{Cov}(e) = \{v,w\}$ and $f(v) \neq f(w)$. Hence $g(v) \neq g(w)$ and therefore $g(e) \in E'$. Thus $g(E) \subseteq E'$. Let g'' be the restriction of g to V and V'. For every $e = \{v,w\} \in E$ we have $v, w < e$, hence $g(v), g(w) \leq g(e)$ and thus $g(e) = \{g(v), g(w)\} \in E'$. Therefore g'' is a graph homomorphism from $\mathbf{G}$ to $\mathbf{G}'$. Since $f'(g''(v)) \not< g''(v)$ for all $v \in V$ we have $g' \circ f(v) = g(f(v)) = f'(g(v)) = f' \circ g''(v)$ for all $v \in V$ and (2) is proved. ☐

Fix a graph homomorphism $f : \mathbf{F}_2 \to \mathbf{K}_3$ (it exists because the chromatic number of $\mathbf{F}_2$ is 3). By Lemma 5.1, f is also a graph homomorphism of $\mathbf{F}_0$ onto $\mathbf{K}_3$ and, by Lemmas 5.1 and 5.11, if $g : \Omega(\mathbf{F}_0, f') \to \Omega(\mathbf{F}_2, f)$ is an h-map then either $g = \Omega 1_{T_0}$ or $|g\{t,0,1,2\}| \leq 3$.

Thus we set $\mathcal{X} = \Omega(\mathbf{F}_2, f)$, $Z = F_2 = \mathrm{Max}(\Omega(\mathbf{F}_2, f))$, $U = [D)$, $V = [\{c_{10}\})$, $C = \{\{c_i, c_{10}\} \mid i \in 3\} \cup \{\{c_i, c_9\} \mid i \in 4\})$,

$$(u_i, v_i) = \begin{cases} (c_i, \{c_i, c_{10}\}) & \text{for } i = 0,1,2, \\ (c_{i-3}, \{c_{i-3}, c_9\}) & \text{for } i = 3,4,5,6. \end{cases}$$

Analogously as above we obtain that $(U(\mathcal{X}), V, \{(u_i, v_i) \mid i \in 7\})$ is a u-triple.

By Lemmas 5.1 and 5.11, we conclude that $\mathcal{X}$ is a (U, C, Z)-testing object of $\mathrm{Var}(\mathbf{DQ}_7)$. Thus $\mathcal{X}$ is a finite universal testing object of $(\mathrm{Var}(\mathbf{DQ}_7))$ with respect to $\mathrm{Var}(\mathbf{DR}_7)$ and, by Corollary 3.10, $\mathcal{X}$ is also standard Q-universal testing object. By Corollary 3.6 and Theorem 3.9 we obtain

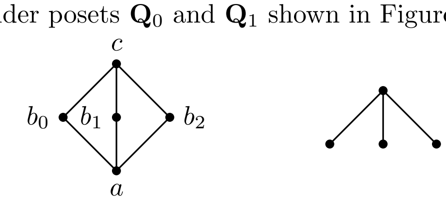
Corollary 5.12. *The variety* $\mathrm{Var}(\mathbf{DQ}_7)$ *is* $\mathrm{Var}(\mathbf{DR}_7)$*-relatively* ff*-alg-universal and has an A-D family, thus it is also* Q*-universal.*

By Lemma 5.10, $\mathcal{X} = \Omega(\mathbf{F}_2, f)$ has the least element t and $\mathcal{X} \setminus \{t\}$ is an h-space from the variety of Heyting algebras $\mathrm{Var}(\mathbf{DQ}_6)$. Clearly, $t \notin U$. Since $C \subseteq U$ we have $t \notin C$. Also $\Omega(\mathbf{F}_0, f)$ has the least element and hence $\Omega(\mathbf{F}_0, f) \setminus \{t\}$ is an h-space. If $g : \Omega(\mathbf{F}_0, f) \setminus \{t\} \to \Omega(\mathbf{F}_2, f) \setminus \{t\}$ is an h-map then the extension g' of g with $g'(t) = t$ is an h-map from $\Omega(\mathbf{F}_0, f)$ to $\Omega(\mathbf{F}_2, f)$. Conversely, if $g : \Omega(\mathbf{F}_0, f) \to \Omega(\mathbf{F}_2, f)$ is an h-map with $g^{-1}\{t\} = \{t\}$ then the domain-range restriction g' of g to $\Omega(\mathbf{F}_0, f) \setminus \{t\}$ and $\Omega(\mathbf{F}_2, f) \setminus \{t\}$ is an h-map. Thus $\mathcal{X} \setminus \{t\}$ is a finite universal testing object of $\mathrm{Var}(\mathbf{DQ}_6)$ with respect to $\mathrm{Var}(\mathbf{DR}_6)$ and a standard Q-universal testing object. Thus, by Corollary 3.6 and Theorem 3.9, we obtain

Theorem 5.13. *The variety* $\mathrm{Var}(\mathbf{DQ}_6)$ *is* $\mathrm{Var}(\mathbf{DR}_6)$*-relatively* ff*-alg-universal and has an A-D family, thus it is also* Q*-universal.*

6. Special cases

This section is devoted to three varieties of Heyting algebras that need to be treated separately. First we consider posets $\mathbf{Q}_0$ and $\mathbf{Q}_1$ shown in Figure 13 (or in Figure 1).



$\mathbf{Q}_0$ $\mathbf{Q}_1$

Figure 13

To begin with $\mathbf{Q}_0$, we consider h-spaces $\mathcal{X}_0 = (X_0; \leq, \tau)$, $\mathcal{X}_1 = (X_1; \leq, \tau)$ and $\mathcal{X}_2 = (X_2; \leq, \tau)$ such that $X_0 = \{a_0, a_1\} \cup \{b_i, c_i \mid i \in 41\}$, $X_1 = X_0 \cup \{a_2\}$, $X_2 = X_1 \cup \{a_3, a_4, a_5\}$ and $\leq$ is the least partial order such that

- ($\bullet$) $a_0 < b_i$ for all $i \in 41$;
- ($\bullet$) $b_i, b_{i+1}, b_{i+2} < c_i$ for $i \in 41$ where the addition is modulo 41;
- ($\bullet$) $b_0, b_1, b_{11} < a_1$;
- ($\bullet$) $b_{13+3i}, b_{14+3i}, b_{27+3i} < a_{2+i}$ for all $i \in 4$;

and discrete topology τ. Clearly, by Corollary 2.6, $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2 \in \mathrm{Var}(\mathbf{DQ}_0)$.

Lemma 6.1. *Let* $h : \mathcal{X}_0 \to \mathcal{X}_i$ *be an* h*-map for* $i = 1, 2$. *Then either* h *is the inclusion or* $\mathrm{Im}(h) \cap \mathrm{Max}(X_i) = \emptyset$ *and for every* $j \in 41$, $|\{h(b_j), h(b_{j+1}), h(b_{j+2})\}| \leq 2$ *and if* $|\{h(b_j), h(b_{j+1}), h(b_{j+2})\}| = 2$ *then* $a_0 \in \{h(b_j), h(b_{j+1}), h(b_{j+2})\}$ *where the addition is modulo* 41. *The inclusion from* X_0 *into* X_i *for* $i = 1, 2$ *is an* h*-map.*

Proof. Since X_0 is a decreasing subset in X_i for $i = 1, 2$, the inclusion is an h-map from $\mathcal{X}_0$ to $\mathcal{X}_i$.

Conversely, assume that $h : \mathcal{X}_0 \to \mathcal{X}_i$ for $i = 1, 2$ is an h-map. Let us denote $B = \{b_k \mid k \in 41\}$. Observe that $|(u]| \in \{1, 2, 5\}$ for every $u \in X_2$ and that $u \in \mathrm{Max}(X_i)$ if and only if $|(u]| = 5$. From the h-property of h it follows $h(a_0) = a_0$ and $h(B) \subseteq B \cup \{a_0\}$. Since $(c_j] = \{c_j, b_j, b_{j+1}, b_{j+2}, a_0\}$ for every $j \in 41$ we conclude that $h(c_j) \in \mathrm{Max}(X_i)$ if and only if $h(b_j)$, $h(b_{j+1})$ and $h(b_{j+2})$ are three distinct elements from the set B. Thus for every $j \in 41$ either $|\{h(b_j), h(b_{j+1}), h(b_{j+2})\} \cap B| = 3$ or there exists $l \in 41$ with

$\{h(b_j), h(b_{j+1}), h(b_{j+2}), h(c_j)\} \subseteq \{a_0, b_l\}$. Since $|(c_j] \cap (c_{j+1}]| = 3$ for every $j \in 41$ (the addition is modulo 41) we infer that either $h(\mathrm{Max}(X_0)) \subseteq \mathrm{Max}(X_i)$ or $\mathrm{Im}(h) \cap \mathrm{Max}(X_i) = \emptyset$ for every $j \in 41$ $|\{h(b_j), h(b_{j+1}), h(b_{j+2})\}| \leq 2$ and if $|\{h(b_j), h(b_{j+1}), h(b_{j+2})\}| = 2$ then $a_0 \in \{h(b_j), h(b_{j+1}), h(b_{j+2})\}$ where the addition is modulo 41.

It remains to restrict ourselves to the case of $h(\mathrm{Max}(X_0)) \subseteq \mathrm{Max}(X_i)$. Then $h(B) \subseteq B$. For $j \in 41$, let $g_{i,j} : X_i \setminus [b_j) \to \{a, b_0, b_1, b_2, c\}$ be a mapping such that $g_{i,j}(a_0) = a$, $g_{i,j}(\mathrm{Max}(X_i) \setminus [b_j)) = c$ and

$$g_{i,j}(b_k) = \begin{cases} b_{k \bmod 3} & \text{if } k < j, \\ b_{k+1 \bmod 3} & \text{if } k > j. \end{cases}$$

By a direct calculation, we obtain that $g_{i,j} : \mathcal{X}_i \setminus [b_j) \to \mathbf{Q}_0$ is an h-map for every $j \in 41$ and $i = 1, 2$. Moreover, if $x \in \mathrm{Max}(X_2) \setminus [b_j)$ then $g_{i,j}(x] = A = \{a, b_0, b_1, b_2, c\}$. Conversely assume that there exists a surjective h-map $g : \mathcal{X}_0 \to \mathbf{Q}_0$. Then for every $j \in 41$ either $g((c_j]) = A$ or there exists $l \in \{0, 1, 2\}$ with $g((c_j]) = \{a, b_l\}$. From $|(c_j] \cap (c_{j+1}]| = 3$ for all $j \in 41$ (the addition is modulo 41) it follows that either $g(\{b_j, b_{j+1}, b_{j+2}\}) = \{b_0, b_1, b_2\}$ for all $j \in 41$ or $c \notin \mathrm{Im}(g)$ and this is a contradiction. If

$$g(\{b_j, b_{j+1}, b_{j+2}\}) = g(\{b_{j+1}, b_{j+2}, b_{j+3}\}) = \{b_0, b_1, b_2\}$$

then necessarily $g(b_j) = g(b_{j+3})$. Hence $g(b_0) = g(b_{41 \bmod 3}) = g(b_2)$ and this is a contradiction. Thus there exists no surjective h-map from $\mathcal{X}_0$ onto $\mathbf{Q}_0$. If $h(\mathrm{Max}(X_0)) \subseteq \mathrm{Max}(X_i)$ and there exists $j \in 41$ with $b_j \notin h(B)$ then $g_{i,j} \circ h$ is a surjective h-map from $\mathcal{X}_0$ onto $\mathbf{Q}_0$ and this is a contradiction. Hence $h(B) = B$ and h is injective on B.

Consider $j \in 41$. If $h(c_j) = a_l$ for $l \in \{2, 3, 4, 5\}$ then, by the h-property of h we infer that $h(\{b_j, b_{j+1}, b_{j+2}\}) = \{b_{13+2(l-2)}, b_{14+2(l-2)}, b_{27+2(l-2)}\}$. Hence there exists $j' \in \{j, j+1\}$ with $b_{27+2(l-2)} \in h(\{b_{j'}, b_{j'+1}\})$. If $j' = j$ then set $k = j - 1$, if $j' = j + 1$ then set $k = j + 1$. In both cases $b_{27+2(l-2)} \in h(\{b_k, b_{k+1}, b_{k+2}\})$ and $\{b_{13+2(l-2)}, b_{14+2(l-2)}\} \cap h(\{b_k, b_{k+1}, b_{k+2}\}) \neq \emptyset$. So $h(c_k) = a_l$ and $h(\{b_k, b_{k+1}, b_{k+2}\}) = h(\{b_j, b_{j+1}, b_{j+2}\})$ and this contradicts the injectivity of h on B. If $h(c_j) = a_1$ then analogously we obtain that $h(\{b_j, b_{j+1}, b_{j+2}\}) = \{b_0, b_1, b_{11}\}$ and there exists $k \in \{j-1, j+1\}$ with $h(c_k) = a_1$ and $h(\{b_k, b_{k+1}, b_{k+2}\}) = h(\{b_j, b_{j+1}, b_{j+2}\})$ and this again contradicts the injectivity of h on B. Hence we conclude that for every $j \in 41$ there exists $k(j) \in 41$ such that $h(c_j) = c_{k(j)}$ and $h(\{b_j, b_{j+1}, b_{j+2}\}) = \{b_{k(j)}, b_{k(j)+1}, b_{k(j)+2}\}$.

If $h(b_{j+2}) = b_{k(j)+1}$ then set $l = j - 1$, if $h(b_j) = b_{k(j)+1}$ then set $l = j + 1$. By the assumption, we infer that $h(\{b_l, b_{l+1}, b_{l+2}\} \cap \{b_j, b_{j+1}, b_{j+2}\}) = \{b_{k(j)}, b_{k(j)+2}\}$ and this implies $h(\{b_l, b_{l+1}, b_{l+2}\}) = h(\{b_j, b_{j+1}, b_{j+2}\})$ and this is a contradiction with the injectivity of h on B. Thus either $h(b_j) = b_{k(j)}$, $h(b_{j+1}) = b_{k(j)+1}$, $h(b_{j+2}) = b_{k(j)+2}$ or $h(b_j) = b_{k(j)+2}$, $h(b_{j+1}) = b_{k(j)+1}$, $h(b_{j+2}) = b_{k(j)}$. In the first case, by the injectivity of h on B we infer that $h(b_{j+3}) = b_{k(j)+3}$ and $h(c_{j+1}) = c_{k(j)+1}$ in the second case we infer that $h(b_{j+3}) = b_{k(j)-1}$ and $h(c_{j+1}) = c_{k(j)-1}$. By induction we deduce that there exists $k(0) \in 41$ such that either $h(b_j) = b_{k(0)+j}$ for all $j \in 41$ or $h(b_j) = b_{k(0)-j}$ for all $j \in 41$ (the addition is modulo 41). Then $h(a_1) = a_1$ because the differences between indices are preserved and thus we obtain that $h(b_j) = b_j$ for all $j \in 41$. Therefore h is the inclusion. □

If we set $\mathcal{X} = \mathcal{X}_2$, $Z = \mathrm{Max}(X_2)$, $C = \{a_2, a_3, a_4, a_5\}$, $U = [\{b_i \mid i = 13, 14, \ldots, 33\})$, then $\mathcal{X} \setminus C = \mathcal{X}_0$ and, by Lemma 6.1, $\mathrm{Aut}(\mathcal{X})$ is a singleton group and if $h : \mathcal{X}_0 \to \mathcal{X}_2$ is an h-map then either h is the inclusion or $\mathrm{Im}(h) \cap \mathrm{Max}(X_2) = \emptyset$, for every $j \in 41$, $|\{h(b_j), h(b_{j+1}), h(b_{j+2})\}| \leq 2$ and if $|\{h(b_j), h(b_{j+1}), h(b_{j+2})\}| = 2$ then

$$a_0 \in \{h(b_j), h(b_{j+1}), h(b_{j+2})\}$$

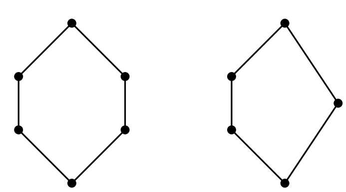
where the addition is modulo 41. Thus every h-map $h : \mathcal{X}_0 \to \mathcal{X}$ that is not the inclusion is non-injective on $X \setminus U$. For every $z \in Z$ there exists $i \in 41$ with $b_i, b_{i+1} < z$ and $b_i \notin \mathrm{Im}(h)$ or $b_{i+1} \notin \mathrm{Im}(h)$. Choose $u_z \in \{b_i, b_{i+1}\}$ such that $u_z \notin \mathrm{Im}(h)$. Then for every h-map $g : \mathcal{X}_1 \to \mathcal{X}$ with $h \restriction X \setminus U = g \restriction X \setminus U$ we have $u_z \notin \mathrm{Im}(g)$ for all $z \in Z$. Thus (t5) is fulfilled and hence $\mathcal{X}$ is a (U, C, Z)-testing object of $\mathrm{Var}(\mathbf{Q}_0)$. Let $V = [\{b_{31}, b_{32}, b_{33}\})$, then V is increasing, order connected set. Let us define $(u_0, v_0) = (b_{17}, a_4)$, $(u_1, v_1) = (b_{18}, a_4)$, $(u_2, v_2) = (b_{19}, a_5)$, $(u_3, v_3) = (b_{13}, a_2)$, $(u_4, v_4) = (b_{14}, a_2)$, $(u_5, v_5) = (b_{15}, a_3)$, $(u_6, v_6) = (b_{16}, a_3)$. Then $(U \setminus V, \preceq)$ is order connected and (u_i, v_i) for $i \in 7$ are f-covering pairs. Thus $(U, V, \{(u_i, v_i) \mid i \in 7\})$ is a u-triple. Let $\mathbb{H}_i$ be the variety of Heyting algebras generated by the Heyting algebra that is an i-element chain for $i = 2, 3$. If $x \in X_2 \setminus \mathrm{Max}(X_2)$ then $(x] \in \mathbb{PHH}_3$ and thus $\mathcal{X}$ is a finite universal testing object of $\mathrm{Var}(\mathbf{DQ}_0)$ with respect to $\mathbb{H}_3$ and, by Corollary 3.10, $\mathcal{X}$ is a standard Q-universal testing object. By Corollary 3.6 and Theorem 3.9 we obtain

Theorem 6.2. *The variety* $\mathrm{Var}(\mathbf{DQ}_0)$ *is* $\mathbb{H}_3$*-relatively* ff*-alg-universal and contains an A-D family, so it is* Q*-universal.*

Clearly, $\mathcal{X}$ has a unique minimal element a_0 and $\mathcal{X} \setminus \{a_0\}$ is an h-space from the variety $\mathrm{Var}(\mathbf{DQ}_1)$. Clearly $a_0 \notin U$ and thus $a_0 \notin C$. Also $\mathrm{Min}(X_0) = \{a_0\}$ and hence $\mathcal{X}_0$ has the least element and $\mathcal{X}_1 \setminus \{a_0\}$ is an h-space from $\mathrm{Var}(\mathbf{DQ}_1)$. If $g : \mathcal{X}_0 \setminus \{a_0\} \to \mathcal{X}_2 \setminus \{a_0\}$ is an h-map then the extension g' of g given by $g'(a_0) = a_0$ is an h-map from $\mathcal{X}_0$ to $\mathcal{X}_2$. Conversely, if $g : \mathcal{X}_0 \to \mathcal{X}_2$ is an h-map with $g^{-1}\{a_0\} = \{a_0\}$ then the domain-range restriction g' of g to $X_0 \setminus \{a_0\}$ and $X_2 \setminus \{a_0\}$ is an h-map. Thus $\mathcal{X} \setminus \{a_0\}$ is a finite universal testing object of $\mathrm{Var}(\mathbf{DQ}_1)$ with respect to the variety $\mathbb{H}_2$ of Boolean algebras and a standard Q-universal testing object. Thus, by Corollary 3.6 and Theorem 3.9, we obtain

Theorem 6.3. *The variety* $\mathrm{Var}(\mathbf{DQ}_1)$ *is* $\mathbb{H}_2$*-relatively* ff*-alg-universal and contains an A-D family, so that it is* Q*-universal.*

Finally, we consider the Heyting space $\mathbf{Q}_2$ given in Figure 14 (or in Figure 2).



$\mathbf{Q}_2$ $\mathbf{R}_2$

Figure 14

Although we do not know whether $\mathbb{V} = \mathrm{Var}(\mathbf{Q}_2)$ is $\mathrm{Var}(\mathbf{R}_2)$-relatively ff-alg-universal, we at least prove that $\mathbb{V} = \mathrm{Var}(\mathbf{Q}_2)$ is $\mathrm{Var}(\mathbf{R}_2)$-relatively alg-universal.

The proof is based on ideas from [10].

Let us denote $\mathbf{N}$ the set of all natural numbers, and let $\mathbf{Z}$ denote the set of all integers. Let us define disjoint posets $A = \{a_i \mid i \in \mathbf{N}\}$, $B = \{b_i \mid i \in \mathbf{N}\}$, $T = \{t_i \mid i \in \mathbf{Z}\}$ and $W = \{w_i \mid i \in \mathbf{Z}\}$ so that

$$a_{2i} < a_{2i+1} > a_{2i+2} \text{ and } b_{2i} < b_{2i+1} > b_{2i+2} \text{ for all } i \in \mathbf{N},$$
$$t_{2i} < t_{2i+1} > t_{2i+2} \text{ and } w_{2i} < w_{2i+1} > w_{2i+2} \text{ for all } i \in \mathbf{Z}.$$

Let $U = \{n_i \mid i \in 24\} \cup \{e_i \mid i \in 15\}$ be such that $n_{3i} > n_{3i+1} < n_{3i+2}$ for $i = 0, 1, \ldots, 7$ and $n_{3j+1} < e_{3j+i} > n_{3i+16}$ for $j \in 5$ and $i \in 3$. Set $V = [n_{22}) = \{n_{22}\} \cup \{e_{3j+2} \mid j \in 5\}$.

Then V is an increasing subset of U. Let us define $(u_i, v_i) = (n_{3i+1}, e_{3i+2})$ for $i \in 3$ and $(u_{i+3}, v_{i+3}) = (n_{3i+1}, e_{3i+1})$ for $i \in 4$ and set

$$C = \{e_{3i+j} \mid i \in 3,\ j = 1, 2\} \cup \{e_{10}\} = \{e_i \mid i \in 11 \text{ is not divisible by } 3\}.$$

By an easy calculation, we verify that $(U, V, \{(u_i, v_i) \mid i = 0, 1, \dots, 6\})$ with C satisfies the conditions (u1)–(u4) and hence $(U, V, \{(u_i, v_i) \mid i = 0, 1, \dots, 6\})$ is a u-triple. Let us define $S = \{s_i \mid i \in 8\}$. Let $(X; \leq)$ be the poset on the disjoint union X of A, B, T, W, U and S, and $\leq$ is the least partial order that is the union of partial orders on A, B, T, W, and U that also satisfies

- (•) $s_0 < s_1, s_2$;
- (•) $s_1 < s_3, s_6, n_{3k+1}, a_{4i+2}, b_{4i}, w_{4j+2}, t_{4j}$ for all $k \in 5$, $i \in \mathbf{N}$ and $j \in \mathbf{Z}$;
- (•) $s_2 < s_4, s_7, n_{3k+16}, a_{4i}, b_{4i+2}, w_{4j}, t_{4j+2}$ for all $k \in 3$, $i \in \mathbf{N}$ and $j \in \mathbf{Z}$;
- (•) $a_0 < s_6$, $b_0 < s_7$, $s_3 < s_5$, $s_4 < s_5$;
- (•) $w_{60i} < n_{3i}$, $t_{60i+2} < n_{3i+2}$ for $i \in 5$;
- (•) $w_{60i+2} < n_{3i}$, $t_{60i} < n_{3i+2}$ for $i = 5, 6, 7$.

A straightforward calculation gives

Lemma 6.4. *$(X; \leq)$ is a poset such that*

1. $\mathrm{Max}(U) = \{s_5, s_6, s_7\} \cup \{a_{2i+1}, b_{2i+1} \mid i \in \mathbf{N}\} \cup \{w_{2i+1}, t_{2i+1} \mid i \in \mathbf{Z}\} \cup \{n_{3i}, n_{3i+2} \mid i \in 8\} \cup \{e_i \mid i \in 15\}$;
2. *U is a finite increasing subset of X and $C \subseteq \mathrm{Max}(X) \cap U$;*
3. *for every $x \in \mathrm{Max}(X) \setminus \{s_6, s_7\}$, $(x]$ is isomorphic to $\mathbf{Q}_2$ while $(s_6]$ and $(s_7]$ are isomorphic to $\mathbf{R}_2$.*

Let $(Y; \leq)$ be the subposet of $(X; \leq)$ on the set

$$Y = X \setminus \{e_l \mid l \in 15\}$$

and let us denote

$$\begin{aligned} X_3 &= \{a_{4i+2}, b_{4i} \mid i \in \mathbf{N}\} \cup \{w_{4j+2}, t_{4j} \mid j \in \mathbf{Z}\} \cup \{n_{3k+1} \mid k \in 5\} \cup \{s_3\}, \\ X_4 &= \{a_{4i}, b_{4i+2} \mid i \in \mathbf{N}\} \cup \{w_{4j}, t_{4j+2} \mid j \in \mathbf{Z}\} \cup \{n_{3k+1} \mid k = 5, 6, 7\} \cup \{s_4\}. \end{aligned}$$

Observe also that the subposet $(S'; \leq)$ of $(X; \leq)$ on the set $S' = \{s_i \mid i \in 6\}$ is isomorphic to $\mathbf{Q}_2$.

Lemma 6.5. *Let $f : Y \to X$ be an order preserving mapping having the h-property. Then one of the following posibilities occurs:*

1. *$f(s_i) = s_i$ for $i = 0, 1, 2, 6, 7$, $f(a_0) = a_0$, $f(b_0) = b_0$, $f(A) \subseteq A \cup \{s_1, s_6\}$, $f(B) \subseteq B \cup \{s_2, s_7\}$ and $f(\mathrm{Max}(Y)) \subseteq \mathrm{Max}(X)$;*
2. *$f(s_0) = s_0$, $f(s_i) = s_{3-i}$ for $i = 1, 2$, $f(s_j) = s_{13-j}$ for $j = 6, 7$, $f(a_0) = b_0$, $f(b_0) = a_0$, $f(A) \subseteq B \cup \{s_2, s_7\}$, $f(B) \subseteq A \cup \{s_1, s_6\}$, $f(\mathrm{Max}(Y)) \subseteq \mathrm{Max}(X)$;*
3. *$f(s_0) = s_0$, $f(\{s_1, s_2\}) \subset \{s_0, s_1, s_2\}$,*
$$f(\mathrm{Max}(Y)) \cap \mathrm{Max}(X) \subseteq \{s_6, s_7\}$$
and $f(\{s_1, s_2\}) \neq \{s_1, s_2\}$, and if $s_1 \in f(\{s_1, s_2\})$ then $\mathrm{Im}(f) \subseteq X_3 \cup (s_6]$, if $s_2 \in f(\{s_1, s_2\})$ then $\mathrm{Im}(f) \subseteq X_4 \cup (s_7]$, if $\{s_0\} = f(\{s_1, s_2\})$ then $\mathrm{Im}(f) \subseteq X_3 \cup X_4 \cup \{s_0, s_1, s_2\}$.

Proof. Since $\mathrm{Min}(Y) = \mathrm{Min}(X) = \{s_0\}$ we obtain $f(s_0) = s_0$. Hence $f(\{s_1, s_2\}) \subseteq \{s_0, s_1, s_2\}$. If $f(\{s_1, s_2\}) = \{s_1, s_2\}$ then $f(\mathrm{Max}(Y)) \subseteq \mathrm{Max}(X)$ because $\{y \in Y \mid s_1, s_2 \in (y]\} = \mathrm{Max}(Y)$ and $\{x \in X \mid s_1, s_2 \in (x]\} = \mathrm{Max}(X)$. By Lemma 6.4(3), $f(\{s_6, s_7\}) \subseteq \{s_6, s_7\}$. Since $\mathrm{Cov}(s_6) = \{s_1, a_0\}$, $\mathrm{Cov}(s_7) = \{s_2, b_0\}$ it follows that $f(s_1) = s_1$ implies $f(s_i) = s_i$ for $i = 1, 2, 6, 7$, $f(a_0) = a_0$ and $f(b_0) = b_0$ and since

f has the h-property we deduce that $f(A) \subseteq A \cup \{s_1, s_6\}$ and $f(B) \subseteq B \cup \{s_2, s_7\}$ (if $f(a_{2i}) = a_{2j}$ for $j > 0$ then $f(\{a_{2i-1}, a_{2i+1}\}) \subseteq \{a_{2j-1}, a_{2j+1}\}$, if $f(a_{2i}) = a_0$ then $f(\{a_{2i-1}, a_{2i+1}\}) \subseteq \{s_6, a_1\}$ and an analogous claim holds for b_i). Similarly $f(s_1) = s_2$ implies $f(s_2) = s_1$, $f(s_6) = s_7$, $f(s_7) = s_6$, $f(a_0) = b_0$ and $f(b_0) = a_0$ and since f has the h-property we deduce that $f(B) \subseteq A \cup \{s_1, s_6\}$ and $f(A) \subseteq B \cup \{s_2, s_7\}$. Thus (1) and (2) hold. Finally assume that $f(\{s_1, s_2\}) \neq \{s_1, s_2\}$ then $|(f(x))]| \leq 5$ for all $x \in \mathrm{Max}(Y)$ and thus $f(\mathrm{Max}(Y)) \cap \mathrm{Max}(X) \subseteq \{s_6, s_7\}$. Observe that every 4-element chain in $(Y; \leq)$ contains either s_1 or s_2. Hence if $s_1 \in f(\{s_1, s_2\})$ then there exists no 4-element chain containing s_2 in $\mathrm{Im}(f)$ and thus $s_7 \notin \mathrm{Im}(f)$. Further for every $x \in \mathrm{Max}(Y)$ we have $f(x) \geq s_1$, thus $f(x) \in X_3 \cup \{s_6, s_1\}$ and we conclude that $\mathrm{Im}(f) \subseteq X_3 \cup (s_6]$. If $s_2 \in f(\{s_1, s_2\})$ then there exists no 4-element chain containing s_1 in $\mathrm{Im}(f)$ and thus $s_6 \notin \mathrm{Im}(f)$. Further, for every $x \in \mathrm{Max}(Y)$ we have $f(s) \geq s_2$, thus $f(x) \in X_4 \cup \{s_7, s_2\}$ and we conclude that $\mathrm{Im}(f) \subseteq X_3 \cup (s_7]$. If $\{s_0\} = f(\{s_1, s_2\})$ then there exists no 4-element chain in $\mathrm{Im}(f)$, thus $s_6, s_7 \notin \mathrm{Im}(f)$ and whence $\mathrm{Im}(f) \subseteq X_3 \cup X_4 \cup \{s_0, s_1, s_2\}$. We obtain (3). □

Let us denote $\mathbb{H}_4 = \mathrm{Var}(\mathbf{DR}_2)$ and $\mathbb{V} = \mathrm{Var}(\mathbf{DQ}_2)$. Our next goal is to define a topology τ on X so that $\mathcal{X} = (X; \leq, \tau)$ is an h-space, U is an open set and if $h : (Y; \leq, \tau) \to (X; \leq, \tau)$ is an h-map and $h(\{s_1, s_2\}) = \{s_1, s_2\}$ then h is the inclusion and satisfies (t5). Then $\mathcal{X}$ will be a (U, C, Z)-testing object of $\mathbb{V}$, for $Z = \mathrm{Max}(X) \setminus \{s_6, s_7\}$ since $(Y; \leq, \tau)$ is a h-subspace of $\mathcal{X} \setminus C$. Hence $\mathcal{X}$ will be a universal testing object of $\mathbb{V}$ with respect to $\mathbb{H}_4$ and Theorem 3.9 will conclude the proof that the variety $\mathbb{V}$ is a $\mathbb{H}_4$-relatively alg-universal.

To define a topology consider a decomposition $\mathcal{M} = \{M_j \mid j \in \mathbf{Z}\}$ of $\mathbf{N}$ such that

(o1) M_j is infinite for all $j \in \mathbf{Z}$;
(o2) if $j \equiv 0 \bmod 4$ then $n \equiv 0 \bmod 4$ for all $n \in M_j$;
(o3) if $j \equiv 2 \bmod 4$ then $n \equiv 2 \bmod 4$ for all $n \in M_j$;
(o4) if j is odd then n is odd for all $n \in M_j$;
(o5) the symmetric difference $\Delta(\{n \in \mathbf{N} \mid n-1 \in M_j \text{ or } n+1 \in M_j\}, M_{j-1} \cup M_{j+1})$ is finite for all $j \in \mathbf{Z}$.

For a set $Q \subseteq X$ let us denote $\mathcal{A}(Q) = \{i \in \mathbf{N} \mid a_i \in Q\}$ and $\mathcal{B}(Q) = \{i \in \mathbf{N} \mid b_i \in Q\}$. Let $\mathcal{C}_{\mathcal{M}}$ be the family of all subsets Q of X such that

(o6) if $w_j \in Q$ then $M_j \setminus \mathcal{A}(Q)$ is finite;
(o7) if $M_j \cap \mathcal{A}(Q)$ is infinite for $j \in \mathbf{Z}$ then $w_j \in Q$;
(o8) if $t_j \in Q$ then $M_j \setminus \mathcal{B}(Q)$ is finite;
(o9) if $M_j \cap \mathcal{B}(Q)$ is infinite for $j \in \mathbf{Z}$ then $t_j \in Q$;
(o10) if $s_3 \in Q$ then the sets $\{i \in \mathbf{Z} \mid i \equiv 2 \bmod 4,\ M_i \setminus \mathcal{A}(Q) \neq \emptyset\}$ and $\{i \in \mathbf{Z} \mid i \equiv 0 \bmod 4,\ M_i \setminus \mathcal{B}(Q) \neq \emptyset\}$ are finite;
(o11) if $\{i \in \mathbf{Z} \mid i \equiv 2 \bmod 4,\ M_i \cap \mathcal{A}(Q) \neq \emptyset\}$ is infinite or $\{i \in \mathbf{Z} \mid i \equiv 0 \bmod 4,\ M_i \cap \mathcal{B}(Q) \neq \emptyset\}$ is infinite then $s_3 \in Q$;
(o12) if $s_4 \in Q$ then the sets $\{i \in \mathbf{Z} \mid i \equiv 0 \bmod 4,\ M_i \setminus \mathcal{A}(Q) \neq \emptyset\}$ and $\{i \in \mathbf{Z} \mid i \equiv 2 \bmod 4,\ M_i \setminus \mathcal{B}(Q) \neq \emptyset\}$ are finite;
(o13) if $\{i \in \mathbf{Z} \mid i \equiv 0 \bmod 4,\ M_i \cap \mathcal{A}(Q) \neq \emptyset\}$ is infinite or $\{i \in \mathbf{Z} \mid i \equiv 2 \bmod 4,\ M_i \cap \mathcal{B}(Q) \neq \emptyset\}$ is infinite then $s_4 \in Q$;
(o14) if $s_5 \in Q$ then the sets $\{i \in \mathbf{Z} \mid i \text{ is odd},\ M_i \setminus \mathcal{A}(Q) \neq \emptyset\}$ and $\{i \in \mathbf{Z} \mid i \text{ is odd}, M_i \setminus \mathcal{B}(Q) \neq \emptyset\}$ are finite;
(o15) if $\{i \in \mathbf{Z} \mid i \text{ is odd},\ M_i \cap (\mathcal{A}(Q) \cup \mathcal{B}(Q)) \neq \emptyset\}$ is infinite then $s_5 \in Q$.

Lemma 6.6. *The family $\mathcal{C}_{\mathcal{M}}$ is closed under complements and finite unions and intersections and contains $\emptyset$ and X.*

Proof. Consider $Q \subseteq X$. Observe that $M_j \setminus \mathcal{A}(Q)$ is finite if and only if $M_j \cap \mathcal{A}(X \setminus Q)$ is finite and $M_j \setminus \mathcal{B}(Q)$ is finite if and only if $M_j \cap \mathcal{B}(X \setminus Q)$ is finite for all $j \in \mathbf{Z}$. Hence, by a standard calculation we obtain that if $Q \subseteq X$ satisfies (o6)-(o9) then also $X \setminus Q$ satisfies (o6)-(o9). Analogously, $M_j \subseteq \mathcal{A}(Q)$ if and only if $M_j \cap \mathcal{A}(X \setminus Q) = \emptyset$ and, by a standard calculation, we obtain that if $Q \subseteq X$ satisfies (o10)-(o15) then also $X \setminus Q$ satisfies (o10)-(o15). Whence $\mathcal{C}_{\mathcal{M}}$ is closed under complements. The proof that if $Q_1, Q_2 \subseteq X$ satisfies (o6)-(o15) then also $Q_1 \cup Q_2$ satisfies (o6)-(o15) is direct. Hence $\mathcal{C}_{\mathcal{M}}$ is closed under finite unions and, by deMorgan rules, it is also closed under finite intersections. Clearly, $\emptyset$ and X satisfy (o6)-(o15), and the proof is complete. □

Consider the topology τ on X with basis $\mathcal{C}_{\mathcal{M}}$. By Lemma 6.6, any set $Q \in \mathcal{C}_{\mathcal{M}}$ is clopen in τ. If Q and $X \setminus Q$ are open in τ then, by (o10), (o12) and (o14), we conclude that $Q \in \mathcal{C}_{\mathcal{M}}$. Thus $\mathcal{C}_{\mathcal{M}}$ is the Boolean algebra of all τ-clopen sets.

Proposition 6.7. *The triple $\mathcal{X} = (X; \leq, \tau)$ is an h-space belonging to $\mathbb{PHV}$.*

Proof. First we prove that the topology τ on X is compact. Let $\{O_i \mid i \in I\}$ be an open covering of X. Then there exist $i_0, i_1, i_2 \in I$ with $s_{3+j} \in O_{i_j}$ for $j = 0, 1, 2$. Since $\mathcal{C}_{\mathcal{M}}$ is a basis of τ there exist $Q_{i_j} \in \mathcal{C}_{\mathcal{M}}$ for $j = 0, 1, 2$ such that $s_{3+j} \in Q_{i_j} \subseteq O_{i_j}$. By (o10), (o12) and (o14), the sets $\{i \in \mathbf{Z} \mid i \equiv 2 \bmod 4,\ M_i \setminus \mathcal{A}(Q_{i_0}) \neq \emptyset\}$, $\{i \in \mathbf{Z} \mid i \equiv 0 \bmod 4,\ M_i \setminus \mathcal{B}(Q_{i_0}) \neq \emptyset\}$, $\{i \in \mathbf{Z} \mid i \equiv 0 \bmod 4,\ M_i \setminus \mathcal{A}(Q_{i_1}) \neq \emptyset\}$, $\{i \in \mathbf{Z} \mid i \equiv 2 \bmod 4,\ M_i \setminus \mathcal{B}(Q_{i_1}) \neq \emptyset\}$, $\{i \in \mathbf{Z} \mid i \text{ is odd},\ M_i \setminus \mathcal{A}(Q_{i_2}) \neq \emptyset\}$, $\{i \in \mathbf{Z} \mid i \text{ id odd},\ M_i \setminus \mathcal{B}(Q_{i_2}) \neq \emptyset\}$ are finite. If $M_i \subseteq \mathcal{A}(Q_{i_j})$ for some $i \in \mathbf{Z}$ and $j = 0, 1, 2$ then, by (o7), $w_i \in Q_{i_j}$, if $M_i \subseteq \mathcal{B}(Q_{i_j})$ for some $i \in \mathbf{Z}$ and $j = 0, 1, 2$ then, by (o9), $t_i \in Q_{i_j}$. Hence the sets $T \setminus \bigcup_{j=0}^{2} O_{i_j}$, $W \setminus \bigcup_{j=0}^{2} O_{i_j}$ are also finite. Thus there exists a finite subset $I_1 \subseteq I$ with $i_0, i_1, i_2 \in I_1$ and $T \cup W \subseteq \bigcup_{i \in I_1} O_i$. By (o6) and (o8), $A \setminus \bigcup_{i \in I_1} O_i$ and $B \setminus \bigcup_{i \in I_1} O_i$ are finite and thus $X \setminus \bigcup_{i \in I_1} O_i$ is finite. Whence (X, τ) is compact.

To prove that $(X; \leq, \tau)$ is a Priestley space it remains to show that for $x, y \in X$ with $y \not\leq x$ there exists a clopen decreasing set Q with $x \in Q$ and $y \notin Q$. First observe that $(x]$ is clopen for all $x \in A \cup B \cup (U \setminus \{n_{3i}, n_{3i+2} \mid i \in 8\}) \cup \{s_0, s_1, s_2, s_6, s_7\}$ and $[x)$ is clopen for all $x \in A \cup B \cup U \cup \{s_6, s_7\}$. The remaining case is that $x \in T \cup W \cup \{n_{3i}, n_{3i+2} \mid i \in 8\} \cup \{s_3, s_4, s_5\}$ and $y \in (S \setminus \{s_6, s_7\}) \cup T \cup W$. Observe that $\{t_j\} \cup \{b_i \mid i \in M_j\}$ and $\{w_j\} \cup \{a_i \mid i \in M_j\}$ are clopen for all $j \in \mathbf{Z}$. By (o5), the sets $[\{t_j\} \cup \{b_i \mid i \in M_j\})$, $(\{t_j\} \cup \{b_i \mid i \in M_j\}]$, $[\{w_j\} \cup \{a_i \mid i \in M_j\})$, and $(\{w_j\} \cup \{a_i \mid i \in M_j\}]$ are clopen for all $j \in \mathbf{Z}$, by (o5) because finite subsets of $A \cup B$ are clopen. Since $(\{t_j\} \cup \{b_i \mid i \in M_j\}] \cap S = (t_j] \cap S$ and $(\{w_j\} \cup \{a_i \mid i \in M_j\}] = (w_j] \cap S$ for all $j \in \mathbf{Z}$ we conclude that the required clopen decreasing set exists for $x \in T \cup W$ or $y \in T \cup W$. Since $(n_{3i}] \cup \{a_j \mid j \in M_{60i}\}$, $(n_{3i+2}] \cup \{b_j \mid j \in M_{60i+2}\}$ for $i \in 4$ and $(n_{3i}] \cup \{a_j \mid j \in M_{60i+2}\}$, $(n_{3i+2}] \cup \{b_j \mid j \in M_{60i}\}$ for $i = 4, 5, 6$ are clopen decreasing we can restrict to the case that $x \in \{s_3, s_4, s_5\}$ and $y \in S \setminus \{s_6, s_7\}$. The fact that the disjoint sets

$$\begin{aligned}
\{s_3, s_1, s_0\} \quad &\cup \quad \{w_j \mid j \equiv 2 \bmod 4\} \cup \{t_j \mid j \equiv 0 \bmod 4\} \cup \{a_i \mid i \in M_j \text{ for } j \equiv 2 \bmod 4\} \\
&\cup \quad \{b_i \mid i \in M_j \text{ for } j \equiv 0 \bmod 4\} \text{ and} \\
\{s_4, s_2, s_0\} \quad &\cup \quad \{w_j \mid j \equiv 0 \bmod 4\} \cup \{t_j \mid j \equiv 2 \bmod 4\} \cup \{a_i \mid i \in M_j \text{ for } j \equiv 0 \bmod 4\} \\
&\cup \quad \{b_i \mid i \in M_j \text{ for } j \equiv 2 \bmod 4\}
\end{aligned}$$

are clopen and decreasing and the fact that $(s_5] = \{s_0, s_1, s_2, s_3, s_4, s_5\}$ is isomorphic to $\mathbf{Q}_2$ complete the proof that $(X; \leq, \tau)$ is a Priestley space.

Next we prove that $(X; \leq, \tau)$ is an h-space. Consider $Q \in \mathcal{C}_{\mathcal{M}}$. For $k = 0, 2$ observe that $[\{t_j \mid j \equiv k \bmod 4\}) = \{t_j \mid j \not\equiv (k+2) \bmod 4\}$ and $[\{w_j \mid j \equiv k \bmod 4\}) = \{w_j \mid j \not\equiv (k+2) \bmod 4\}$. By a routine calculation, we obtain that

$$\begin{aligned}
[s_1) = X \setminus (\{s_2, s_4\} \quad &\cup \quad \{n_{3k+1} \mid k = 4, 5, 6\} \cup \{a_i, b_{i+2} \mid i \in \mathbf{N},\ i \equiv 0 \bmod 4\} \\
&\cup \quad \{w_i, t_{i+2} \mid i \in \mathbf{Z},\ i \equiv 0 \bmod 4\})
\end{aligned}$$

and

$$\begin{aligned} [s_2) = X \setminus (\{s_1, s_3\} &\cup \{n_{3k+1} \mid k \in 4\} \cup \{a_{i+2}, b_i \mid i \in \mathbf{N},\, i \equiv 0 \bmod 4\} \\ &\cup \{w_{i+2}, t_i \mid i \in \mathbf{Z},\, i \equiv 0 \bmod 4\}). \end{aligned}$$

Thus $[Q)$ satisfies (o10)–(o15), and, by (o5), $[Q)$ satisfies also (o6)–(o9). Whence $[Q) \in \mathcal{C}_{\mathcal{M}}$ and $(X; \leq, \tau)$ is an h-space since $\mathcal{C}_{\mathcal{M}}$ is a base of τ consisting of clopen sets. Since $\mathbf{R}_2 \in \mathbb{PHV}$ we obtain, by Theorem 2.5 and Lemma 6.4(3), that $\mathcal{X} \in \mathbb{PHV}$. □

It is clear that the ordered space $\mathcal{Y} = (Y; \leq, \tau)$ obtained from $\mathcal{X}$ by removing its clopen subset $\{e_l \mid l \in 15\}$ of $\mathrm{Max}(\mathcal{X})$ is an h-space and that $\mathcal{Y} \in \mathbb{PHV}$.

Lemma 6.8. *Let $f : \mathcal{Y} \to \mathcal{X}$ be an h-map such that $f(\{s_1, s_2\}) = \{s_1, s_2\}$. Then $f(s_5) = s_5$, $f^{-1}\{x\}$ is finite for all $x \in X$ and*

(1) *if $f(s_1) = s_1$ then $f(s_i) = s_i$ for all $i \in 8$, $A \subseteq f(A)$, $B \subseteq f(B)$, $W = f(W)$, and $T = f(T)$;*
(2) *if $f(s_1) = s_2$ then $f(s_5) = s_5$, $f(s_i) = s_{3-i}$ for $i = 1, 2$, $f(s_i) = s_{7-i}$ for $i = 3, 4$, $f(s_i) = s_{13-i}$ for $i = 6, 7$, $A \subseteq f(B)$, $B \subseteq f(A)$, $W = f(T)$, and $T = f(W)$.*

Proof. Let $f : \mathcal{Y} \to \mathcal{X}$ be an h-map with $f(\{s_1, s_2\}) = \{s_1, s_2\}$. First let $f(s_1) = s_1$. Then, by Lemma 6.5(1), $f(A) \subseteq A \cup \{s_1, s_6\}$ and $f(B) \subseteq B \cup \{s_2, s_7\}$. Observe that the singletons $\{s_1\}$, $\{s_2\}$, $\{s_6\}$, and $\{s_7\}$ are clopen sets. From the definition of $\mathcal{C}_{\mathcal{M}}$ it follows that $\bigcap\{Q \mid Q \in \mathcal{C}_{\mathcal{M}},\, A \subseteq Q\} = A \cup W \cup \{s_3, s_4, s_5\}$ and $\bigcap\{Q \mid Q \in \mathcal{C}_{\mathcal{M}},\, B \subseteq Q\} = B \cup T \cup \{s_3, s_4, s_5\}$. Thus the closure of A is the set $\{s_3, s_4, s_5\} \cup W \cup A$ and the closure of B is the set $\{s_3, s_4, s_5\} \cup T \cup B$. Hence $f(A \cup W \cup \{s_3, s_4, s_5\}) \subseteq A \cup W \cup \{s_1, s_3, s_4, s_5, s_6\}$ and $f(B \cup T \cup \{s_3, s_4, s_5\}) \subseteq B \cup T \cup \{s_2, s_3, s_4, s_5, s_7\}$. Thus $f(\{s_3, s_4, s_5\}) \subseteq \{s_3, s_4, s_5\}$ and, by Lemma 6.5(1), we deduce that $f(s_i) = s_i$ for all $i \in 8$. By Lemma 6.5, $f(\mathrm{Max}(Y)) \subseteq \mathrm{Max}(Y)$ and hence

$$\begin{aligned} f(\{a_i \mid i \in \mathbf{N},\, i \text{ is odd}\}) &\subseteq \{s_5, s_6\} \cup \{a_i \mid i \in \mathbf{N},\, i \text{ is odd}\} \cup \{w_i \mid i \in \mathbf{Z},\, i \text{ is odd}\} \\ f(\{b_i \mid i \in \mathbf{N},\, i \text{ is odd}\}) &\subseteq \{s_5, s_7\} \cup \{b_i \mid i \in \mathbf{N},\, i \text{ is odd}\} \cup \{t_i \mid i \in \mathbf{Z},\, i \text{ is odd}\}. \end{aligned}$$

If for some odd $i \in \mathbf{N}$ we have $f(a_i) = s_5$ then $f(\{a_{i-1}, a_{i+1}\}) = \{s_3, s_4\}$ and thus $f(a_{i+2}), f(a_{i-2}) = s_5$. From this it follows that $f(a_0) \in \{s_3, s_4\}$ and this is a contradiction with Lemma 6.5(1). Analogously, we obtain that $f(a_i) \notin \{w_j \mid j \in \mathbf{Z},\, j \text{ is odd}\}$ and $f(b_i) \notin \{s_5\} \cup \{t_j \mid j \in \mathbf{Z},\, j \text{ is odd}\}$. Since every finite subset of A or B is closed we conclude $f(A)$ and $f(B)$ are infinite. By Lemma 6.5(1), we infer that $A \subseteq f(A)$ and $B \subseteq f(B)$ because A and B are one-way infinite zig-zags. From this it follows that $f(W) = W$ and $f(T) = T$ because for every $j \in \mathbf{Z}$ the closure of $\{a_i \mid i \in M_j\}$ (or $\{b_i \mid i \in M_j\}$) is $\{a_i \mid i \in M_j\} \cup \{w_j\}$ (or $\{b_i \mid i \in M_j\} \cup \{t_j\}$, respectively).

Let $f(s_1) = s_2$. Then, by Lemma 6.5(2), $f(A) \subseteq B \cup \{s_2, s_7\}$ and $f(B) \subseteq A \cup \{s_1, s_6\}$. Hence $f(A \cup W \cup \{s_3, s_4, s_5\}) \subseteq B \cup T \cup \{s_2, s_3, s_4, s_5, s_7\}$ and $f(B \cup T \cup \{s_3, s_4, s_5\}) \subseteq A \cup W \cup \{s_1, s_3, s_4, s_5, s_6\}$. Thus $f(\{s_3, s_4, s_5\}) \subseteq \{s_3, s_4, s_5\}$ and, by Lemma 6.5(2), $f(s_5) = s_5$, $f(s_i) = s_{3-i}$ for $i = 1, 2$, $f(s_i) = s_{7-i}$ for $i = 3, 4$, and $f(s_i) = s_{13-i}$ for $i = 6, 7$. From the closedness of finite sets we obtain again $A \subseteq f(B)$ and $B \subseteq f(A)$ and whence $f(W) = T$ and $f(T) = W$.

Consider $x \in A \cup B$. Since every element of $A \cup B$ is clopen we obtain that any subset of $A \cup B$ is open and compactness of $\mathcal{X}$ implies that the closure of any infinite subset of $A \cup B$ contains an element of $W \cup T \cup \{s_3, s_4, s_5\}$. We have $f^{-1}\{x\} \subseteq A \cup B \cup \{s_1, s_2, s_6, s_7\}$ and claim that this set is finite. From $\bigcap\{Q \mid Q \in \mathcal{C}_{\mathcal{M}},\, W \subseteq Q\} = W \cup \{s_3, s_4, s_5\}$ and $\bigcap\{Q \mid Q \in \mathcal{C}_{\mathcal{M}},\, T \subseteq Q\} = T \cup \{s_3, s_4, s_5\}$ it follows that the closure of W is the set $W \cup \{s_3, s_4, s_5\}$ and the closure of T is the set $T \cup \{s_3, s_4, s_5\}$. Since for every $x \in W \cup T$ there exists an open set O such that $\{x\} = O \cap (W \cup T)$, the compactness of $\mathcal{X}$ implies that the closure of any infinite subset of $W \cup T$ contains an element of $\{s_3, s_4, s_5\}$. Thus $f^{-1}\{x\}$ is finite for all $x \in W \cup T$ because $f^{-1}\{x\} \subseteq W \cup T$. Thus $f^{-1}\{x\} \cap (A \cup B \cup W \cup T)$ is finite for all $x \in X$ and hence $f^{-1}\{x\}$ is finite for all $x \in X$. $\square$

Lemma 6.9. *Let $f : \mathcal{Y} \to \mathcal{X}$ be an h-map such that $\{f(s_1), f(s_2)\} = \{s_1, s_2\}$. Then*

(1) *if $f(s_1) = s_1$ then for every $j \in \mathbf{Z}$ there exists $\nu_j \in \mathbf{N}$ such that if $f(w_j) = w_{j'}$ then for every $i \in M_j$ with $i \geq \nu_j$ there exists $i' \in M_{j'}$ with $f(a_i) = a_{i'}$ and if $f(t_j) = t_{j'}$ then for every $i \in M_j$ with $i \geq \nu_j$ there exists $i' \in M_{j'}$ with $f(b_i) = b_{i'}$;*
(2) *if $f(s_1) = s_1$ then for every $j \in \mathbf{Z}$ there exists $\mu_j \in \mathbf{N}$ such that for every $i \in M_j$ with $i \geq \mu_j$ we have $\mathcal{A}(f^{-1}\{a_i\}) \subseteq \bigcup\{M_{j'} \mid f(w_{j'}) = w_j\}$ and $\mathcal{B}(f^{-1}\{b_i\}) \subseteq \bigcup\{M_{j'} \mid f(t_{j'}) = t_j\}$;*
(3) *if $f(s_1) = s_2$ then for every $j \in \mathbf{Z}$ there exists $\nu_j \in \mathbf{N}$ such that if $f(w_j) = t_{j'}$ then for every $i \in M_j$ with $i \geq \nu_j$ there exists $i' \in M_{j'}$ with $f(a_i) = b_{i'}$ and if $f(t_j) = w_{j'}$ then for every $i \in M_j$ with $i \geq \nu_j$ there exists $i' \in M_{j'}$ with $f(b_i) = a_{i'}$;*
(4) *if $f(s_1) = s_2$ then for every $j \in \mathbf{Z}$ there exists $\mu_j \in \mathbf{N}$ such that for every $i \in M_j$ with $i \geq \mu_j$ we have $\mathcal{A}(f^{-1}\{b_i\}) \subseteq \bigcup\{M_{j'} \mid f(w_{j'}) = t_j\}$ and $\mathcal{B}(f^{-1}\{a_i\}) \subseteq \bigcup\{M_{j'} \mid f(t_{j'}) = w_j\}$;*
(5) *if $f(a_i) = a_j$ (or $f(a_i) = b_j$) then $j \leq i$, if $f(b_i) = b_j$ (or $f(b_i) = a_j$) then $j \leq i$;*
(6) *if $f(a_i) = a_k$ (or $f(a_i) = b_k$) and $f(a_j) = a_l$ (or $f(a_j) = b_l$) for $i \leq j$ then $\{a_n \mid k \leq n \leq l \text{ or } l \leq n \leq k\} \subseteq \{f(a_n) \mid i \leq n \leq j\}$ (or $\{b_n \mid k \leq n \leq l \text{ or } l \leq n \leq k\} \subseteq \{f(a_n) \mid i \leq n \leq j\}$) and hence $|k - l| \leq |i - j|$, if $f(b_i) = b_k$ (or $f(b_i) = a_k$) and $f(b_j) = b_l$ (or $f(b_j) = a_l$) then $\{b_n \mid k \leq n \leq l \text{ or } l \leq n \leq k\} \subseteq \{f(b_n) \mid i \leq n \leq j\}$ (or $\{a_n \mid k \leq n \leq l \text{ or } l \leq n \leq k\} \subseteq \{f(b_n) \mid i \leq n \leq j\}$) and hence $|k - l| \leq |i - j|$.*

Proof. Let $j \in \mathbf{Z}$. Then by (o6) and (o7), w_j is a member of the closure of $Q \subseteq A$ if and only if $\mathcal{A}(Q) \cap M_j$ is infinite. By (o10)–(o15), the intersection of $\{s_3, s_4, s_5\}$ with the closure of a set $Q \subseteq A$ is non-empty if and only if the set $\{j \in \mathbf{Z} \mid M_j \cap \mathcal{A}(Q) \neq \emptyset\}$ is infinite. Analogously, t_j is a member of the closure of $Q \subseteq B$ if and only if $\mathcal{B}(Q) \cap M_j$ is infinite and the intersection of $\{s_3, s_4, s_5\}$ with the closure of a set $Q \subseteq B$ is non-empty if and only if the set $\{j \in \mathbf{Z} \mid M_j \cap \mathcal{B}(Q) \neq \emptyset\}$ is infinite.

Since every h-map $f : \mathcal{Y} \to \mathcal{X}$ is closed, for every $j \in \mathbf{Z}$ the set $f(\{w_j\} \cup \{a_i \mid i \in M_j\})$ is closed. Since $\{f(w_j)\} = f(\{w_j\} \cup \{a_i \mid i \in M_j\}) \setminus (A \cup B)$ we conclude that if $f(w_j) = w_{j'}$ (or $f(w_j) = t_{j'}$) then the set $\{i \in M_j \mid f(a_i) \notin \{a_k \mid k \in M_{j'}\}\}$ (or $\{i \in M_j \mid f(a_i) \notin \{b_k \mid k \in M_{j'}\}\}$) is finite and hence, by Lemma 6.8, the clauses (1) and (3) are proved. Since each h-map is continuous and $\{w_j\} \cup \{a_i \mid i \in M_j\}$ and $\{t_j\} \cup \{b_i \mid i \in M_j\}$ are clopen sets for each $j \in \mathbf{Z}$, by Lemma 6.8 and (o6)-(o15) we obtain that the difference sets of $f^{-1}(\{a_i \mid i \in M_j\})$ and $\{a_i \mid i \in M_{j'},\, f(w_{j'}) = w_j\} \cup \{b_i \mid i \in M_{j'},\, f(t_{j'}) = w_j\}$, and of $f^{-1}(\{b_i \mid i \in M_j\})$ and $\{a_i \mid i \in M_{j'},\, f(w_{j'}) = t_j\} \cup \{b_i \mid i \in M_{j'},\, f(t_{j'}) = t_j\}$ are finite. Whence (2) and (4) follow.

By Lemma 6.5(1) and (2), we obtain (5) because any order preserving mapping maps a zig-zag of length k onto a zig-zag of length at most k. Since every order preserving mapping preserves connectedness we obtain (6). $\square$

To complete the proof we now specify the sets M_j with $j \in \mathbf{Z}$. Let $\{n_i\}_{i=0}^{\infty}$ and $\{m_i\}_{i=0}^{\infty}$ be two increasing sequences of natural numbers and set

$$I = \{(i, j) \mid i \in \mathbf{N},\, j \in \mathbf{Z},\, -n_i \leq j \leq m_i\}.$$

For $(i, j) \in I$, let us define

$$\operatorname{suc}(i, j) = \begin{cases} (i, j + 1) & \text{if } j < m_i, \\ (i + 1, -n_{i+1}) & \text{if } j = m_i. \end{cases}$$

Consider the lexicographical order $\leq$ on I (in which $(i,j) \leq (i',j')$ just when either $i < i'$ or $i = i'$ and $j \leq j'$). Then suc is the successor function on I with respect to $\leq$. For a finite interval J of natural numbers, let $l(J) = \min J$ and $u(J) = \max J$; then $J = \{i \in \mathbf{N} \mid l(J) \leq i \leq u(J)\}$. Let $\{R_{(i,j)} \mid (i,j) \in I\}$ be a family of finite non-empty intervals of natural numbers such that $l(i,j) \leq u(i,j) = l(\mathrm{suc}(i,j)) - 1$ for all $(i,j) \in I$ and $l(0, -n_0) = 0$. Then we define M_{4j+r} for $j \in \mathbf{Z}$ by $M_{4j} = \{4k \mid \exists i, (i,j) \in I,\ k \in R_{(i,j)}\}$, $M_{4j+2} = \{4k+2 \mid \exists i, (i,j) \in I,\ k \in R_{(i,j)}\}$, $M_{4j+1} = \{k \mid k-1, k+1 \in M_{4j} \cup M_{4j+2}\}$, $M_{4j+3} = \{k \mid k-1 \in M_{4j+2},\ k+1 \notin M_{4j}\}$.

Lemma 6.10. *The family $\{M_j \mid j \in \mathbf{Z}\}$ satisfies the conditions* (o1)–(o5).

Proof. Since for every $j \in \mathbf{Z}$ there exist infinitely many $i \in \mathbf{N}$ with $(i,j) \in I$ and since $R_{(i,j)} \neq \emptyset$ for all $(i,j) \in I$, the set M_j is infinite for every $j \in \mathbf{Z}$ and (o1) is true. The conditions (o2)–(o4) immediately follow from the definition. Clearly, the sets M_{4j+1} and M_{4j+2} satisfy (o5) for all $j \in \mathbf{Z}$. Consider $n \in M_{4j+3}$. Then $n-1 \in M_{4j+2}$ and $n+1 \notin M_{4j+4}$ if and only if there exists $i \in \mathbf{N}$ such that $j = m_i$ and $n = u(R_{i,j})$. But for given $j \in \mathbf{Z}$ there exists at most one $i \in \mathbf{N}$ with $j = m_i$ because $\{m_i\}_{i \in \mathbf{N}}$ is an increasing sequence. Analogously, if $n \in M_{4j}$ then $n+1 \in M_{4j+1}$ and $n-1 \notin M_{4j-1} = M_{(4j-1)+3}$ if and only if there exists $i \in \mathbf{N}$ such that $j = -n_i$ and $n = l(R_{i,j})$. Again for a given $j \in \mathbf{Z}$ there exists at most one $i \in \mathbf{N}$ with $j = -n_i$ because $\{n_i\}_{i \in I}$ is an increasing sequence. Thus (o5) holds. □

Next we define intervals $R_{i,j}$ for $(i,j) \in I$. Let $\{p_i\}_{i \in \mathbf{N}}$ be a sequence of integers such that

- (s1) $-n_i \leq p_i \leq m_i$ for every $i \in \mathbf{N}$;
- (s2) for every finite set $K \subseteq \mathbf{Z}$ and for $l, q \in \mathbf{N}$ there exists $i > l$ such that $p_{i+m} \notin K$ for all $m = 0, 1, \ldots, q$;
- (s3) for every $k_1, k_2, j \in \mathbf{Z}$ with $k_1 < k_2$ and $l, q \in \mathbf{N}$ there exists $i > l$ such that $p_i \neq j$ and $k_1 \leq p_{i+m} \leq k_2$ for all $m = 0, 1, \ldots, q$;
- (s4) for every $j, k \in \mathbf{Z}$ and $l, q \in \mathbf{N}$ there exists $i > l$ such that $p_i = j$ and $p_{i+m} \neq k$ for all $m = 1, 2, \ldots, q$.

From (s4) it follows that for every $j \in \mathbf{Z}$ there exist infinitely many $i \in \mathbf{N}$ with $p_i = j$. Choose a natural number $\alpha > 0$. Then for $(i,j) \in I$ with $j \neq p_i$ we set $u(R_{i,j}) - l(R_{i,j}) = \alpha - 1$ and for $(i,j) \in I$ with $j = p_i$ we set $u(R_{i,j}) - l(R_{i,j}) = \prod_{k=0}^{i^2}(m_k + n_k)\alpha - 1$. Thus α and the sequence $\{p_i\}_{i \in \mathbf{N}}$ uniquely determine a family $\{R_{i,j} \mid (i,j) \in I\}$ of intervals of natural numbers.

Lemma 6.11. *If $f : \mathcal{Y} \to \mathcal{X}$ is an h-map such that $\{f(s_1), f(s_2)\} = \{s_1, s_2\}$ then either*

1. *$f(a_i) = a_i$, $f(b_i) = b_i$ for all $i \in \mathbf{N}$ and $f(w_i) = w_i$, $f(t_i) = t_i$ for all $i \in \mathbf{Z}$ or*
2. *$f(a_i) = b_i$, $f(b_i) = a_i$ for all $i \in \mathbf{N}$ and $f(w_i) = t_i$, $f(t_i) = w_i$ for all $i \in \mathbf{Z}$.*

Proof. By Lemma 6.5(1) and (2), it suffices to prove that

- (i) if $f(a_0) = a_0$ then $f(a_i) = a_i$ for all $i \in \mathbf{N}$ and $f(w_i) = w_i$ for all $i \in \mathbf{Z}$;
- (ii) if $f(a_0) = b_0$ then $f(a_i) = b_i$ for all $i \in \mathbf{N}$ and $f(w_i) = t_i$ for all $i \in \mathbf{Z}$;
- (iii) if $f(b_0) = b_0$ then $f(b_i) = b_i$ for all $i \in \mathbf{N}$ and $f(t_i) = t_i$ for all $i \in \mathbf{Z}$;
- (iv) if $f(b_0) = a_0$ then $f(b_i) = a_i$ for all $i \in \mathbf{N}$ and $f(t_i) = w_i$ for all $i \in \mathbf{Z}$.

Observe that if $f(a_0) = a_0$ then for every $i \in \mathbf{N}$ there exists $i' \in \mathbf{N}$ with $f(a_{4i}) = a_{4i'}$. Let us define $g(i) = i'$ and for every $j \in \mathbf{Z}$ there exists $j' \in \mathbf{Z}$ with $f(w_{4j}) = w_{4j'}$, let us define $h(j) = j'$. If $f(a_0) = b_0$ then for every $i \in \mathbf{N}$ there exists $i' \in \mathbf{N}$ with $f(a_{4i}) = b_{4i'}$, let us define $g(i) = i'$ and for every $j \in \mathbf{Z}$ there exists $j' \in \mathbf{Z}$ with $f(w_{4j}) = t_{4j'}$, let us define $h(j) = j'$. Once we prove that both g and h are the identity then (i) and (ii) are proved, The proof of (iii) and (iv) is by symmetry. To prove that both g and h are the identity mapping, denote $R_j = \bigcup\{R_{(i,j)} \mid \exists i \in I \text{ with } (i,j) \in I\}$. We claim the following properties of g and h.

(1) if $g(i) = i'$ for $i \in \mathbf{N}$ then $g(i+1) \in \{i'-1, i', i'+1\}$, if $h(j) = j'$ for $j \in \mathbf{Z}$ then $h(j-1), h(j+1) \in \{j'-1, j', j'+1\}$;
(2) $g(0) = 0$ and if $g(i) = i'$ for $i \in \mathbf{N}$ then $i' \leq i$;
(3) if $i < j$ and $g(i) = i'$, $g(j) = j'$ then $\{l \mid i' \leq l \leq j' \text{ or } j' \leq l \leq i'\} \subseteq g(\{l \mid i \leq l \leq j\})$;
(4) $g(\mathbf{N}) = \mathbf{N}$, $h(\mathbf{Z}) = \mathbf{Z}$;
(5) $g^{-1}\{i\}$ and $h^{-1}\{j\}$ are finite sets for all $i \in \mathbf{N}$ and $j \in \mathbf{Z}$;
(6) if $h(j) = j'$ then $g(R_j) \setminus R_{j'}$ is a finite set for all $j \in \mathbf{Z}$;
(7) $g^{-1}(R_j) \setminus (\bigcup_{k \in h^{-1}(j)} R_k)$ is a finite set for all $j \in \mathbf{Z}$.

Indeed, the h-property of f implies (1), Lemmas 6.5 and 6.9(5) imply (2), Lemma 6.9(6) implies (3), Lemma 6.8 implies (4) and {5), Lemma 6.9(1) and (3) implies (6) and Lemma 6.9(2) and (4) implies (7).

Choose $j \in \mathbf{Z}$. Since n_i and m_i form increasing sequences, there exists $\iota_0 \in \mathbf{N}$ such that $-n_i < j-1 < j+1 < m_i$ for every $i \geq \iota_0$, thus $(i, j-1), (i,j), (i, j+1) \in I$ for every $i \geq \iota_0$. By (7), there exists $\iota_1 \in \mathbf{N}$ with $\iota_1 \geq \iota_0$ and $g^{-1}(R_{i,j}) \subseteq \bigcup_{k \in h^{-1}(j)} R_k$ for all $i \geq \iota_1$. By (5), there exists $\iota_2 \in \mathbf{N}$ such that for every $i \geq \iota_2$ and for every $k \in h^{-1}\{j\}$ we have $(i, k-1), (i,k), (i,k+1) \in I$ and, by (1), (5) and (6), we can assume that $g(R_{i,k-1}) \subseteq R_{h(k-1)}$, $g(R_{i,k+1}) \subseteq R_{h(k+1)}$ and $g(R_{i,k}) \subseteq \bigcup_{i \geq \iota_1} R_{i,j}$. Since $(i, j-1), (i, j+1) \in I$ for all $i \geq \iota_0$ and $\iota_1 \geq \iota_0$ and since $R_{i,j-1}$ and $R_{i,j+1}$ are non-empty intervals we conclude that for every $i \geq \iota_2$ and every $k \in h^{-1}\{j\}$ there exists $\phi(i,k) > \iota_1$ with $g(R_{i,k}) \subseteq R_{\phi(i,k),j}$. By (2), $\phi(i,k) \leq i$ because $g(i) \leq i$ for all $i \in \mathbf{N}$. If $h(k) = j$ and $i > \iota_2$ then $h(k-1) \neq j$ (or $h(k+1) \neq j$) implies that $R_j \cap R_{h(k-1)} = \emptyset$ and $g(R_{i,k-1}) \subseteq R_{h(k-1)}$ (or $R_j \cap R_{h(k+1)} = \emptyset$ and $g(R_{i,k+1}) \subseteq R_{h(k+1)}$). Hence if $i, i' \geq \iota_2$ and $k, k' \in h^{-1}\{j\}$ are such that $(i,k) < (i', k')$ and there exists no $(i'', k'') \in I$ with $(i,k) < (i'', k'') < (i', k')$ and $k'' \in h^{-1}\{j\}$ then one of the following possibilities occurs:

(a) if $(i', k') = \operatorname{suc}(i,k)$ then $\phi(i,k) = \phi(i', k')$;
(b) if $(i', k') \neq \operatorname{suc}(i,k)$, $h(k+1) = j-1$ and $h(k'-1) = j+1$ then $\phi(i', k') = \phi(i,k) - 1$;
(c) if $(i', k') \neq \operatorname{suc}(i,k)$, $h(k+1) = j-1$ and $h(k'-1) = j-1$ then $\phi(i', k') = \phi(i,k)$;
(d) if $(i', k') \neq \operatorname{suc}(i,k)$, $h(k+1) = j+1$ and $h(k'-1) = j+1$ then $\phi(i', k') = \phi(i,k)$;
(e) if $(i', k') \neq \operatorname{suc}(i,k)$, $h(k+1) = j+1$ and $h(k'-1) = j-1$ then $\phi(i', k') = \phi(i,k) + 1$.

Hence there exists an integer β such that $\phi(i+1, k) = \phi(i,k) + \beta$ for all $i \geq \iota_2$ and $k \in h^{-1}\{j\}$. By (4), $g(\mathbf{N}) = \mathbf{N}$ and $\beta > 0$.

Since $\{n_i\}_{i=0}^{\infty}$ and $\{m_i\}_{i=0}^{\infty}$ are increasing sequences, we deduce that $\{m_i + n_i\}_{i=0}^{\infty}$ is an increasing sequence of natural numbers. Since $\prod_{k=0}^{i^2}(m_k + n_k)\alpha - 1$ depends only on i we obtain that

$$\begin{aligned}(m_i + n_i)\alpha + \prod_{k=0}^{i^2}(m_k + n_k)\alpha - 1 \quad &\leq \quad l(R_{i+1,j}) - l(R_{i,j}) \\ &\leq \quad (m_{i+1} + n_{i+1})\alpha + \prod_{k=0}^{(i+1)^2}(m_k + n_k)\alpha - 1,\end{aligned}$$

whenever $-n_i \leq j \leq m_i$. Hence we deduce that if $-n_i \leq j \leq m_i$ then for every integer $k > 0$ we have $l(R_{i+k+1,j}) - l(R_{i+k,j}) > l(R_{i+1,j}) - l(R_{i,j})$. Thus if $k \in h^{-1}\{j\}$ is such that $\beta > 1$ then there exists i with $\phi(i,k) > i$. Then $l(R_{i+1,k}) - l(R_{i,k}) < l(R_{\phi(i+1,k),j}) - l(R_{\phi(i,k),j})$ and this contradicts (3). Hence $\beta = 1$. Since $\phi(i,k) \leq i$ for all $i \geq \iota_2$ and $k \in h^{-1}\{j\}$, there exists a natural number γ_k such that $\phi(i,k) = i - \gamma_k$ for all $i \geq \iota_2$.

First assume that there exist $k, m \in h^{-1}\{j\}$ such that $k < m$, there exists no $k' \in h^{-1}\{j\}$ with $k < k' < m$ and $\phi(i,m) = \phi(i,k)+1$ for every $i \geq \iota_2$. By (s2), there exists $i_1 \in \mathbf{N}$ such that $i_1 = \phi(i,k)$ for some $i \geq \iota_2$, $p_{i_1} \neq j$, and either $p_i < k$ or $p_i > m$. Hence $i_1 \geq \iota_1 \geq \iota_0$. Then $g(R_{i,k}) \subseteq R_{i_1,j}$ and $g(R_{i,m}) \subseteq R_{i_1+1,j}$ and, by (3),

$$\{l \mid u(R_{i_1,j}) < l < l(R_{i_1+1,j})\} = \bigcup\{R_{\alpha,\beta} \mid (\alpha,\beta) \in I,\, (i_1,j) < (\alpha,\beta) < (i_1+j)\}$$
$$\subseteq \{g(l) \mid u(R_{i,k}) < l < l(R_{i,m})\} = g\big(\bigcup\{R_{\alpha,\beta} \mid \exists(\alpha,\beta) \in I,\, (i,k) < (\alpha,\beta) < (i,m)\}\big).$$

By the construction of the family $\{R_{i,j} \mid (i,j) \in I\}$, we have

$$l(R_{i_1+1,j}) - u(R_{i_1,j}) = \Big(m_{i_1} + n_{i_1+1} - 2 + \prod_{l=0}^{i_1^2}(m_l+n_l)\Big)\alpha - 1$$

and $l(R_{i,m}) - u(R_{i,k}) = (m-k-1)\alpha - 1$. Then $m-k < m_{\iota_0} + n_{\iota_0} < m_{i_1} + n_{i_1+1}$, and hence $l(R_{i_1+1,j}) - u(R_{i_1,j}) > l(R_{i,m}) - u(R_{i,k})$ and this is a contradiction. Hence $\phi(i,k) = \phi(i,m)$ for all $k, m \in h^{-1}\{j\}$ and $\phi(i+1, \min h^{-1}\{j\}) = \phi(i, \max h^{-1}\{j\}) + 1$ because $\beta = 1$. Thus $\gamma_k = \gamma_m$ for all $k, m \in h^{-1}\{j\}$.

Assume that $|h^{-1}\{j\}| \geq 2$ and set $k = \max h^{-1}\{j\}$, $m = \min h^{-1}\{j\}$. By (s3), there exists $i_1 \in \mathbf{N}$ such that $\phi(i,k) = i_1$ for some $i \geq \iota_2$, $i_1 > \gamma_k$, $p_{i_1} \neq j$ and $m \leq p_i \leq k$. Then $g(R_{i,k}) \subseteq R_{i_1,j}$ and $g(R_{i+1,m}) \subseteq R_{i_1+1,j}$ and, by (3),

$$\{l \mid u(R_{i_1,j}) < l < l(R_{i_1+1,j})\} = \bigcup\{R_{\alpha,\beta} \mid (\alpha,\beta) \in I,\, (i_1,j) < (\alpha,\beta) < (i_1+j)\}$$
$$\subseteq \quad g(\{l \mid u(R_{i,k}) < l < l(R_{i+1,m}\}) = g(\bigcup\{R_{\alpha,\beta} \mid (i,k) < (\alpha,\beta) < (i+1,m)\}).$$

By the construction of the family $\{R_{i,j} \mid (i,j) \in I\}$, we have

$$l(R_{i_1+1,j}) - u(R_{i_1,j}) = \Big(m_{i_1} + n_{i_1+1} - 2 + \prod_{l=0}^{i_1^2}(m_l+n_l)\Big)\alpha - 1$$

and $l(R_{i+1,m}) - u(R_{i,k}) < (m_i + n_{i+1} - 1)\alpha - 1$. Since $i_1 = i - \gamma_k$ and $i_1 > \gamma_k$ we conclude that $i_1^2 > i+1$. Then $(m_i + n_{i+1} - 1)\alpha < \prod_{l=0}^{i_1^2}(m_l+n_l)\alpha$ and hence $l(R_{i_1+1,j}) - u(R_{i_1,j}) > l(R_{i+1,m}) - u(R_{i,k})$ and this is a contradiction. Thus $|h^{-1}\{j\}| = 1$.

Finally we prove that $\gamma = 0$ and $h^{-1}\{j\} = \{j\}$. Assume the contrary. Thus if $h^{-1}\{j\} = \{k\}$ then $k \neq j$ or $\gamma_k \neq 0$. By (s4), there exists $i_1 \in \mathbf{N}$ such that $\phi(i,k) = i_1$ for some $i \geq \iota_2$, $p_{i_1} = j$ and $p_i \neq k$. Then, by (3), $g(R_{i,k}) = R_{i_1,j}$. But $|R_{i_1,j}| = \prod_{l=0}^{i_1^2}(m_l+n_l)\alpha > \alpha = |R_{i,k}|$ – this is a contradiction. Thus $h(j) = j$ and for every $i \geq \iota_2$ we have $g(R_{i,j}) = R_{i,j}$. By (2), for every $l \in \mathbf{N}$ there exists $i \in \mathbf{N}$ with $i > l$ and $g(i) = i$. From (2) and (3) it follows that $g(l) = l$ for all $l \in \mathbf{N}$. Since j is arbitrary we have $h(j) = j$ for all $j \in \mathbf{Z}$. $\square$

Lemma 6.12. *If $f : \mathcal{Y} \to \mathcal{X}$ is an h-map with $\{f(s_1), f(s_2)\} = \{s_1, s_2\}$ then f is the inclusion.*

Proof. If $f(s_1) = s_1$ then, by Lemma 6.11, f is the inclusion. Assume that $f(s_2) = s_1$. By Lemma 6.11, $f(w_j) = t_j$ and $f(t_j) = w_j$ for all $j \in \mathbf{Z}$. Then $w_{60i} < n_{3i} > n_{3i+1} < n_{3i+2} > t_{60i+2}$. Thus $t_{60i} < f(n_{3i}) > f(n_{3i+1}) < f(n_{3i+2}) > w_{60i+2}$, but no such zig-zag exists in $(X; \leq)$ – a contradiction. Thus $f(s_1) = s_2$ is impossible, and the proof is complete. $\square$

Finally, it remains to construct a sequence $\{p_i\}_{i \in \mathbf{N}}$ satisfying the conditions (s1)–(s4). We shall construct this sequence by induction. Let κ and λ be natural numbers. At the initial step we set $\lambda = 1$, $\kappa = m_0$ and $p_0 = -n_0$. If δ is the greatest natural number such that p_i was constructed for all $i \in \delta$ then we apply the following step:

> if $p_{\delta-1} < \kappa$ then $p_\delta = p_{\delta+1} = \cdots = p_{\delta+\lambda} = p_{\delta-1} + 1$, if $p_{\delta-1} = \kappa$ then we increase λ by 1, set $\kappa = m_\delta$ and $p_\delta = p_{\delta+1} = \cdots = p_{\delta+\lambda} = -n_\delta$.

From the construction it immediately follows that the sequence $\{p_i\}_{i\in\mathbf{N}}$ satisfies the conditions (s1)-(s4).

Thus, by Lemmas 6.5 and 6.12, $\mathcal{X}$ is a (U,C,Z)-testing object of $\mathbb{V}$ because $\{s_0,s_1,s_2\} \subseteq (z]$ for all $z\in Z$. By Lemma 6.4, $(x]\in \mathbb{PHH}_4$ for all $x\in X\setminus Z$, and thus $\mathcal{X}$ is a universal testing object of $\mathbb{V}$ with respect to $\mathbb{H}_4$. Since $\mathcal{X}$ is automorphism-free, Theorem 3.9 gives

Corollary 6.13. *The variety $\mathbb{V}$ of Heyting algebras is $\mathbb{H}_4$-relatively alg-universal.*

To prove the Q-universality of the variety $\mathbb{V}$, we use the technique from [18] in order to construct a standard Q-universal testing object for $\mathbb{V}$.

Let $(A;\leq)$ be a poset where $A=\{a_i \mid i\in 80\}$ and $a_{2i}<a_{2i+1}>a_{2i+2}$ for all $i\in 40$ where the addition is modulo 80. For $j\in 4$, let $(D_j;\leq)$ be a poset with $D_j=\{d_{i,j}\mid i\in 21\}$ and $d_{2i,j}>d_{2i+1,j}<d_{2i+2,j}$ for all $i\in 10$ and each $j\in 4$ (we assume that $D_j\cap D_{j'}=\emptyset$ for distinct $j,j'\in 4$ and $A\cap D_j=\emptyset$ for all $j\in 4$). Let $(X;\leq)$ be a new poset that is the union of posets $(A;\leq)$, $(D_j;\leq)$ for $j\in 4$ and the set $S=\{s_i\mid i\in 10\}$, whose members are related as follows:

- (•) $s_0<s_1,s_2$; $a_0<s_3>s_2$; $a_{26}<s_4>s_1$; $a_{56}<s_5>s_2$;
- (•) $s_1<a_{4i},d_{4j+1,k},d_{4j+3,l}$ for $i\in 20$, $j\in 4$, $k=0,2$, $l=1,3$;
- (•) $s_2<a_{4i+2},d_{4j+3,k},d_{4j+1,l}$ for $i\in 20$, $j\in 4$, $k=0,2$, $l=1,3$;
- (•) $a_{30+2i}<d_{0,i}$ and $a_{76-2i}<d_{20,i}$ for $i\in 4$;
- (•) $d_{9,0}<s_6>d_{9,1}$; $d_{9,0}<s_7>d_{9,3}$; $d_{9,2}<s_8>d_{9,1}$; $d_{9,2}<s_9>d_{9,3}$.

Let us denote $U=[\{d_{9,i}\mid i\in 4\})=\{d_{j,i}\mid j=8,9,10,\ i\in 4\}\cup\{s_j\mid j=6,7,8,9\}$, $C=\{s_9\}$ and $(a,b)=(d_{9,3},s_9)$. The topology τ is discrete. By a direct verification we obtain

Lemma 6.14. *For $\mathcal{X}=(X;\leq,\tau)$ we have*

1. $\mathrm{Max}(X)=\{a_{2i+1}\mid i\in 39\}\cup\{d_{2i,j}\mid i\in 11,\ j\in 4\}\cup\{s_k\mid k=3,4,\dots,9\}$*;*
2. $(x]$ *is isomorphic to* $\mathbf{Q}_2$ *for all* $x\in\mathrm{Max}(X)\setminus\{s_3,s_4,s_5\}$ *and* $(x]$ *is isomorphic to* $\mathbf{R}_2$ *for* $x\in\{s_3,s_4,s_5\}$*;*
3. $[s_1)\cap[s_2)=\mathrm{Max}(X)$*;*
4. $\mathcal{X}$ *belongs to* $\mathbb{PHV}$*;*
5. $U\subseteq X$ *is functorial,* $C\subseteq\mathrm{Max}(U)$ *and* $(d_{9,3},s_9)$ *is an* f*-covering pair.*

Set $Y=X\setminus\{s_6,s_7,s_8,s_9\}$. Since X is finite and $s_6,s_7,s_8,s_9\in\mathrm{Max}(X)$ we conclude that $\mathcal{Y}=(Y;\leq,\tau)$ is an h-space and $\mathcal{Y}\in\mathbb{PHV}$. We will investigate h-maps from $\mathcal{Y}$ to $\mathcal{X}$.

Lemma 6.15. *Let $f:\mathcal{Y}\to\mathcal{X}$ be an h-map. Then $f(s_0)=s_0$, $f(\{s_i\mid i=1,2\})\subseteq\{s_0,s_1,s_2\}$ and either $f(\{s_1,s_2\})=\{s_1,s_2\}$, $f(\mathrm{Max}(Y))=\mathrm{Max}(X)$ and $f(\{s_3,s_4,s_5\})\subseteq\{s_3,s_4,s_5\}$ or there exists $i=1,2$ such that $s_i\notin f(\{s_j\mid j\in 3\})$ and $\mathrm{Im}(f)\cap\mathrm{Max}(Y)\subseteq\{s_3,s_4,s_5\}$.*

Proof. Since $\mathrm{Min}(X)=\mathrm{Min}(Y)=\{s_0\}$, we have $f(s_0)=s_0$. To demonstrate that $f(\{s_i\mid i=1,2\})\subseteq\{s_i\mid i\in 3\}$ observe that $\mathrm{Cov}(y)=\{s_0\}$ in $\mathcal{Y}$ for $y\in Y$ if and only if $y\in\{s_1,s_2\}$, and $\mathrm{Cov}(x)=\{s_0\}$ in $\mathcal{X}$ for $x\in X$ if and only if $x\in\{s_1,s_2\}$. Thus $f(\{s_i\mid i=1,2\})\subseteq\{s_i\mid i\in 3\}$. Next observe that $s_1,s_2\in(x]$ for $x\in X$ (or $x\in Y$) if and only if $x\in\mathrm{Max}(X)$ (or $x\in\mathrm{Max}(Y)$). Hence if $f(\{s_1,s_2\})=\{s_1,s_2\}$ then $f(\mathrm{Max}(Y))\subseteq\mathrm{Max}(X)$ and, by Lemma 6.14(2), $f(\{s_3,s_4,s_5\})\subseteq\{s_3,s_4,s_5\}$. If $|f(\{s_i\mid i\in 3\})|\leq 2$ then f is not injective on $(x]$ for all $x\in\mathrm{Max}(Y)$ because $\{s_i\mid i\in 3\}\subseteq(x]$ for all $x\in\mathrm{Max}(Y)$. By Lemma 6.14(2), $|(x]|=6$ if and only if $x\in\mathrm{Max}(X)\setminus\{s_3,s_4,s_5\}$ and hence $\mathrm{Im}(f)\cap(\mathrm{Max}(X)\setminus\{s_3,s_4,s_5\}=\emptyset$ and the proof is complete. ☐

Finally, we investigate h-maps $f:\mathcal{Y}\to\mathcal{X}$ with $\{f(s_1),f(s_2)\}=\{s_1,s_2\}$ in more detail. Let us denote $S_0=\{s_i\mid i=6,7,8,9\}\cup\{d_{9,k}\mid k\in 4\}$.

Lemma 6.16. *Let $f : \mathcal{Y} \to \mathcal{X}$ be an h-map such that $\{f(s_1), f(s_2)\} = \{s_1, s_2\}$. Then one of the following possibilities occurs:*

(1) *f is the inclusion;*
(2) *f is not injective on the set $\{a_i \mid i \in 56\}$ and $\mathrm{Im}(f) \cap S_0 = \emptyset$.*

Proof. First observe that $\mathrm{Cov}(s_3) \cap \{s_1, s_2\} = \mathrm{Cov}(s_5) \cap \{s_1, s_2\} = \{s_2\}$ and $\mathrm{Cov}(s_4) \cap \{s_1, s_1\} = s_2$. From $f(\{s_3, s_4, s_5\}) \subseteq \{s_3, s_4, s_5\}$ it follows that f is injective on $(s_i]$ for $i = 3, 4, 5$ and hence either $f(\{s_3, s_5\}) \subseteq \{s_3, s_5\}$ and $f(s_4) = s_4$ or $f(s_3) = f(s_5) = s_4$ and $f(s_4) \in \{s_3, s_5\}$. Moreover, if $f(\{s_3, s_5\}) \subseteq \{s_3, s_5\}$ then $f(s_1) = s_1$ and $f(s_2) = s_2$, if $f(s_3) = f(s_5) = s_4$ then $f(s_1) = s_2$ and $f(s_2) = s_1$. By Lemma 6.15, if $x \in \mathrm{Max}(Y) \setminus \{s_3, s_4, s_5\}$ and $f(x) \neq s_3, s_4, s_5$ then f is injective on $(x]$ because, by Lemma 6.14(2), $|(x]| = 6 = |(f(x)]|$. If $x \in \mathrm{Max}(Y) \setminus \{s_3, s_4, s_5\}$ and $f(x) \in \{s_3, s_5\}$ then f is injective on $(x] \setminus f^{-1}(s_2)$ and $|(x] \cap f^{-1}(s_2)| = 2$, and from $f(x) = s_4$ it follows that f satisfies $f(\mathrm{Cov}(x)) = \mathrm{Cov}(f(x))$ for every $x \in \mathrm{Max}(Y)$.

Define posets $(P_1, \leq)$ and $(P_2, \leq)$ where $P_1 = A \cup (\bigcup_{j=0}^{3} D_j) \cup \{s_3, s_4, s_5\}$ and $P_2 = P_1 \cup \{s_6, s_7, s_8, s_9\}$ and $p \leq q$ in P_1 or P_2 just when $p \in \mathrm{Cov}(q)$ in $(X, \leq)$. Then $f(P_1) \subseteq P_2$ and the domain-range restriction g of f to P_1 and P_2 is a h-map from $(P_1, \leq)$ to $(P_2, \leq)$ such that $g(\mathrm{Max}(P_1)) \subseteq \mathrm{Max}(P_2)$ and either $g(\{s_3, s_5\}) \subseteq \{s_3, s_5\}$ and $g(s_i) = s_i$ for $i = 1, 2, 4$ or else $g(s_3) = g(s_5) = s_4$, $g(s_4) \in \{s_3, s_5\}$, $g(s_1) = s_2$, and $g(s_2) = s_1$.

For $i = 1, 2$ and $p, q \in P_i$ let $\mathrm{dist}_{P_i}(p, q)$ be length of the shortest sequence $x_0, x_1, \dots, x_k$ such that $p = x_0$, $q = x_k$ and x_j and x_{j+1} are comparable for all $j = 0, 1, \dots, k-1$. Such a sequence $x_0, x_1, \dots, x_k$ is called a <u>path</u> between p and q. Then $\mathrm{dist}_{P_2}(g(p), g(q)) \leq \mathrm{dist}_{P_1}(p, q)$ for all $p, q \in P_1$.

Observe that $\mathrm{dist}_{P_i}(s_3, s_4) = 29$ and $\mathrm{dist}_{P_i}(s_5, s_4) = 31$ for $i = 1, 2$. Thus either $s_3 \in g(\{s_3, s_5\}) \subseteq \{s_3, s_5\}$ and $g(s_4) = s_4$ or $g(s_3) = g(s_5) = s_4$ and $g(s_4) = s_3$. Since $s_3, a_0, a_1, \dots, a_{26}, s_4$ is the unique shortest path between s_3 and s_4 then in the first case $g(a_i) = a_i$ for $i \in 27$ and in the second case $g(a_i) = a_{26-i}$ for $i \in 27$. Observe that $s_5, s_2, s_3, a_0, a_1, \dots, a_{26}, s_4$ is the unique shortest path between s_5 and s_4 and that

$$s_5, a_{56}, a_{55}, \dots, a_{26}, s_4$$

is the unique path between s_5 and s_4 of length 33. If $g(s_5) = s_3$ then $g(s_2) = s_2$ and hence $g(a_{56}) = a_0$. Thus $g(\{a_{56-i} \mid i \in 30\}) \subseteq \{a_j \mid j \in 28\} \cup \{s_1, s_2, s_3, s_4, a_{78}, a_{79}\}$. If $g(s_5) = s_5$ then $g(a_{56}) = a_{56}$ and hence $g(a_{55}) \in \{a_{55}, a_{57}, s_5\}$. Since $31 = \mathrm{dist}_{P_2}(s_5, s_4) = \mathrm{dist}_{P_2}(a_{55}, s_4) < \mathrm{dist}_{P_2}(a_{57}, s_4)$ and since $a_{55}, a_{54}, \dots, a_{25}, s_4$ is the unique shortest path between a_{55} and s_4 we conlcude that either $g(a_{55}) = s_5$, $g(a_{54}) = s_2$, $g(a_{53}) = s_3$, $g(a_{52-i}) = a_i$ for $i \in 27$ or $g(a_{55-i}) = a_{55-i}$ for $i \in 30$. If $g(s_5) = g(s_3) = s_4$ then $g(a_{56}) = a_{26}$ and $g(a_{26}) = a_0$. Since between a_0 and a_{26} every path of length at most 30 contains a_i for $i \in 26$ we conlcude that $g(\{a_{56-i} \mid i \in 30\}) \subseteq \{a_i \mid i \in 28\} \cup \{s_1, s_2, s_3, s_4, a_{79}, a_{78}\}$. From the above we infer that one of the following four cases occurs:

(1) $g(a_i) = a_i$ for $i \in 57$ and $g(s_i) = s_i$ for $i = 1, 2, 3, 4, 5$;
(2) $g(s_i) = s_i$ for $i = 1, 2, 3, 4, 5$, $g(a_i) = a_i$ for $i \in 27$, $g(a_{56}) = a_{56}$, $g(a_{55}) = s_5$, $g(a_{54}) = s_2$, $g(a_{53}) = s_3$ and $g(a_{52-i}) = a_i$ for $i \in 27$;
(3) $g(s_i) = s_i$ for $i = 1, 2, 3, 4$, $g(s_5) = s_3$, $g(a_i) = a_i$ for $i \in 27$, $g(\{a_{56-i} \mid i \in 30\}) \in \{a_j \mid j \in 28\} \cup \{s_1, s_2, s_3, s_4, a_{78}, a_{79}\}$;
(4) $g(s_3) = g(s_5) = s_4$, $g(s_4) = s_3$, $g(s_2) = s_1$, $g(s_1) = s_2$, $g(a_{26-i}) = a_i$ for $i \in 27$ and
$$g(\{a_{56-i} \mid i \in 30\}) = \{a_j \mid j \in 28\} \cup \{s_1, s_2, s_3, s_4, a_{78}, a_{79}\}.$$

We now consider these four cases.

Case (1). Direct observation shows that

$$s_3, a_{79}, a_{78}, a_{77}, a_{76}, d_{20,0}, d_{19,0}, \dots, d_{0,0}, a_{30}, a_{29}$$

is the unique shortest path between s_3 and a_{29} and hence $g(a_i) = a_i$ for $i = 76, 77, 78, 79$ and $g(d_{i,0}) = d_{i,0}$ for all $i \in 21$. Since

$$a_{77-2i}, a_{76-2i}, d_{20,i}, d_{19,i}, \dots, d_{0,i}, a_{30+2i}, a_{31+2i}$$

is the unique shortest path between a_{77-2i} and a_{31+2i} for $i = 1, 2, 3$ we infer that $g(a_i) = a_i$ for $i = 75, 74, \dots, 70$ and $g(d_{i,j}) = d_{i,j}$ for all $i \in 21$ and $j = 1, 2, 3$. There are exactly two shortest paths $s_5, s_2, s_3, a_0, a_{79}, a_{78}, \dots, a_{69}$ and $s_5, a_{56}, a_{57}, \dots, a_{69}$ between s_5 and a_{69}. Since $g(a_{56}) = a_{56}$ we conclude that $g(a_{57}) \neq s_3$ and therefore g maps the path $s_5, a_{56}, a_{57}, \dots, a_{69}$ into itself. Thus $g(a_i) = a_i$ for all $i \in 80$ and $g(d_{i,j}) = d_{i,j}$ for all $i \in 21$ and $j \in 4$. In this case f is the inclusion.

Case (2). We know that g is not injective on $\{a_i \mid i \in 56\}$, and hence f is also not injective on $\{a_i \mid i \in 57\}$. By the assumption, $g(a_{29}) = a_{23}$. Since $\mathrm{dist}_{P_1}(a_{29}, a_{77}) = 25$ and $\mathrm{dist}_{P_1}(s_3, a_{77}) = 5$ we conclude that $g(a_{77}) \in \{a_{79}, a_1, a_3, s_3\}$. From $\mathrm{dist}_{P_1}(a_{69}, a_{77}) = 9$ it follows that $g(\{a_i \mid i = 79, 77, \dots, 69\})$ is a subset of $B_0 \cup D_0$ where $B_0 = \{a_i \mid i = 1, 3, \dots, 11\} \cup \{51, 53, \dots, 61\} \cup \{71, 73, 75, 77, 79\}\}$ and

$$D_0 = \{d_{20,0}, d_{18,0}, d_{16,0} d_{20,2}, d_{18,2}, d_{20,4}, s_3, s_5\}.$$

Thus $g(\{a_i \mid i \in 57, 59, \dots, 69\})$ is a subset $B_0 \cup D_0 \cup \{a_i \mid i = 63, 49\}$. Hence $S_0 \cap \mathrm{Im}(g) = \emptyset$ and thus also $S_0 \cap \mathrm{Im}(f) = \emptyset$.

Case (3). Again, g is not injective on $\{a_i \mid i \in 57\}$ and hence also f is not injective on $\{a_i \mid i \in 57\}$. From $g(s_5) = s_3$ we infer that $g(a_{56}) = a_0$ and hence $g(a_{55}) \in \{a_{79}, a_1, s_3\}$. If $g(a_{55}) = a_{79}$ then $g(a_{54}) = a_{78}$ and hence $g(a_{53}) \in \{a_{77}, a_{79}\}$. Since $\mathrm{dist}_{P_1}(a_{53}, s_4) = 29$ and $\mathrm{dist}_{P_2}(a_{77}, s_4) = 31$, we obtain $g(a_{53}) = a_{79}$, $g(a_{52-i}) = a_i$ for $i \in 27$. If $g(a_{55}) = s_3$ then $g(a_{54}) = s_2$ and hence $g(a_{53}) \in \{s_3, s_5\}$. From $\mathrm{dist}_{P_1}(a_{53}, s_4) = 29$ and $\mathrm{dist}_{P_2}(s_5, s_4) = 31$ we infer that $g(a_{53}) = s_3$ and $g(a_{52-i}) = a_i$ for $i \in 27$. If $g(a_{55}) = a_1$ then $g(a_{54}) = a_2$ and $g(a_{53}) \in \{a_1, a_3\}$. Hence $g(a_{52-i}) \in \{a_i, a_{i+4}\}$ for $i \in 27$ which is even and $g(a_{52-i}) \in \{a_i, a_{i+2}, a_{i+4}\}$ for $i \in 27$ which is odd. In particular, $g(a_{29}) \in \{a_{23}, a_{25}, a_{27}, s_4\}$. From $\mathrm{dist}_{P_1}(a_{29}, a_{77}) = 25$ and $\mathrm{dist}_{P_1}(s_3, a_{77}) = 5$ we get $g(a_{77}) \in \{a_{79}, a_1, a_3, s_3\}$. ¿From $\mathrm{dist}_{P_1}(a_{69}, a_{77}) = 9$ it follows that $g(\{a_i \mid i = 79, 77, \dots, 69\})$ is a subset of $B_0 \cup D_0$ where $B_0 = \{a_i \mid i = 1, 3, \dots, 11\} \cup \{51, 53, \dots, 61\} \cup \{71, 73, 75, 77, 79\}\}$ and $D_0 = \{d_{20,0}, d_{18,0}, d_{16,0}, d_{20,2}, d_{18,2}, d_{20,4}, s_3, s_5\}$. Thus $g(\{a_i \mid i \in 57, 59, \dots, 69\})$ is a subset of $B_0 \cup D_0$ because $g(s_5) = s_3$. Hence $S_0 \cap \mathrm{Im}(g) = \emptyset$ and thus $S_0 \cap \mathrm{Im}(f) = \emptyset$.

Case (4). We know that g, and also f, is not injective on $\{a_i \mid i \in 57\}$. From $g(s_5) = s_4$ we get that $g(a_{56}) = a_{26}$ and thus $g(a_{55}) \in \{a_{25}, a_{27}, s_4\}$. If $g(a_{55}) \in \{s_4, a_{27}\}$ then $g(a_{54}) \in \{a_{28}, s_1\}$, but

$$\mathrm{dist}_{P_2}(a_{28}, s_3) = \mathrm{dist}_{P_2}(s_1, s_3) = \mathrm{dist}_{P_1}(a_{54}, s_4) = 30$$

and hence $g(a_{53}) = g(a_{55})$ and $g(a_{52-i}) = a_i$ for $i \in 27$. If $g(a_{55}) = a_{25}$ then $g(a_{54}) = a_{24}$ and $g(a_{53}) \in \{a_{23}, a_{25}\}$. Hence $g(a_{52-i}) \in \{a_i, a_{i-4}\}$ for $i \in 27$ which is even and $g(a_{52-i}) \in \{a_i, a_{i-2}, a_{i-4}\}$ for $i \in 27$ which is odd. From $\mathrm{dist}_{P_1}(a_{29}, a_{77}) = 25$ and $\mathrm{dist}_{P_1}(s_3, a_{77}) = 5$ it follows that $g(a_{77}) \in \{a_{27}, a_{25}, a_{23}, s_4\}$. From $\mathrm{dist}_{P_1}(a_{69}, a_{77}) = 9$ it follows that $g(\{a_i \mid i = 79, 77, \dots, 69\})$ is a subset of $B_1 \cup D_1$ where $B_1 = \{a_i \mid i = 15, 17, \dots, 35\}$ and $D_1 = \{d_{0,0}, d_{2,0}, d_{4,0}, d_{0,2}, d_{2,2}, d_{0,4}, s_4\}$. Thus $g(\{a_i \mid i \in 57, 59, \dots, 69\})$ is a subset of $B_1 \cup D_1$ because $g(s_5) = s_4$. Hence $S_0 \cap \mathrm{Im}(g) = \emptyset$ and thus $S_0 \cap -\mathrm{Im}(f) = \emptyset$.

Thus in the case (1), we conclude that 6.16(1) holds and in the cases (2), (3) and (4) that 6.16(2) holds. □

If we set $Z = \{s_9\}$ then $C \subseteq Z$ and to obtain that $\mathcal{X}$ is a standard Q-testing object it remains to prove that $\mathcal{X}$ is a finite (U, C, Z)-testing object. Since $\mathcal{Y}$ is an h-subspace of $\mathcal{X} \setminus C$ and $\bigcup_{c \in C} \mathrm{Cov}(c) \subseteq S_0$, we obtain, by Lemmas 6.15 and 6.16 that $\mathcal{X}$ is a (U, C, Z)-testing object. Since $\mathcal{X}$ is finite the proof is complete. Then, by Corollary 3.6, we obtain

Corollary 6.17. *The variety $\mathbb{V}$ contains an A-D family and hence it is Q-universal.*

7. Conclusion

Theorem 1.3 follows from Theorems 4.7, 4.13, 5.13, 6.2, 6.3 and Corollaries 4.17, 5.5, 5.9, 5.12, 6.13, 6.16.

Next we give the proof of Corollaries 1.4 and 1.5. First we give an auxiliary notion. From Priestley duality it follows that if $\mathcal{H} \in \mathbb{PHV}$ and if $f : \mathcal{H}' \to \mathcal{H}$ is an injective h-map and $g : \mathcal{H} \to \mathcal{H}''$ is a surjective h-map then $\mathcal{H}', \mathcal{H}'' \in \mathbb{PHV}$. This fact motivates the following notion. Let $(X; \leq, \tau)$ be an h-space then an equivalence θ on X is called an h-congruence if a quotient space $(X; \leq, \tau)/\theta = (X/\theta, \sqsubseteq, \sigma)$ (it means that $(X/\theta, \sqsubseteq)$ is a quotient poset of $(X, \leq)$ by θ and (X, σ) is a quotient topological space of (X, τ) by θ) is an h-space and the associated canonical quotient mapping is an h-map. This implies that every class of θ is closed and convex. If $(X; \leq, \tau)$ is finite then, by a standard calculation, an equivalence θ is an h-congruence if and only if

(1) every class of θ is convex;
(2) if $x, y \in X$ with $x\theta y$ and $z \leq x$ for $z \in X$ then there exists $u \in X$ with $u \leq y$ and $u\theta z$.

If $\mathbb{V}$ is a variety and $\mathcal{H} \in \mathbb{PHV}$ and θ is an h-congruence of $\mathcal{H}$ then $\mathcal{H}/\theta \in \mathbb{PHV}$.

Proof of Corollary 1.4 Referring to Fig. 3 and Fig. 1, we show that every variety $\mathrm{Var}(\mathbf{DS}_i)$ with $i = 0, 1, \dots, 11$ contains one of the varieties $\mathrm{Var}(\mathbf{DQ}_j)$ with $j = 0, 1, \dots, 10$.

(0) on $\mathbf{S}_0$, the equivalence θ collapsing only the minimal elements in the middle and on the right is an h-congruence with $\mathbf{S}_0/\theta \cong \mathbf{Q}_3$;
(1) on $\mathbf{S}_1$, the h-congruence θ collapsing only all minimal elements gives $\mathbf{S}_1/\theta \cong \mathbf{Q}_0$;
(2) on $\mathbf{S}_2$, the h-congruence θ collapsing only the least element and its three covers gives $\mathbf{S}_2/\theta \cong \mathbf{Q}_0$;
(3) for $\mathbf{S}_3$, its subposet $(x]$ where x is the cover of the minimal element on the right is isomorphic to $\mathbf{Q}_3$;
(4) on $\mathbf{S}_4$, the h-congruence θ that collapses only the minimal element on the right and its two covers gives $\mathbf{S}_4/\theta \cong \mathbf{Q}_3$;
(5) on $\mathbf{S}_5$, the h-congruence θ collapsing only the least element and its cover on the left gives $\mathbf{S}_5/\theta \cong \mathbf{Q}_9$;
(6) on $\mathbf{S}_6$, the h-congruence θ collapsing only the two covers of the minimal element on the right produces $\mathbf{S}_6/\theta \cong \mathbf{Q}_3$;
(7) for $\mathbf{S}_7$, the subposet $(x]$ where x is the cover of the minimal element on the left is isomorphic to $\mathbf{Q}_3$;
(8) for $\mathbf{S}_8$, the subposet $(x]$ where x is either element covered by the maximal element is isomorphic to $\mathbf{Q}_3$;
(9) for $\mathbf{S}_9$, the subposet $(x]$ for either element x covering the minimal element on the left is isomorphic to $\mathbf{Q}_3$;
(10) on $\mathbf{S}_{10}$, the h-congruence θ collapsing only the least element and the element covering it on the left gives $\mathbf{S}_{10}/\theta \cong \mathbf{Q}_{10}$;
(11) on $\mathbf{S}_{11}$, the h-congruence θ collapsing only the least element and its cover on the left gives $\mathbf{S}_{11}/\theta \cong \mathbf{Q}_2$.

Theorem 1.3 completes the proof because if a variety $\mathbb{V}$ contains a $\mathbb{W}$-relatively alg-universal variety $\mathbb{V}'$ then $\mathbb{V}$ is also $\mathbb{W}$-relatively alg-universal, if $\mathbb{V}$ contains a variety $\mathbb{V}'$ having an A-D family then $\mathbb{V}$ also has an A-D family.

Proof of Corollary 1.5 Let $(X; \leq, \tau)$ be a finite h-space with the greatest element such that there exists an antichain A of $(X; \leq)$ with $|A| \geq 3$. Then we can assume that either

$A \subseteq \operatorname{Min}(X)$ or $A \cap \operatorname{Min}(X) = \emptyset$ or $|\operatorname{Min}(X)| = 2$ and $|A \cap \operatorname{Min}(X)| = 1$. Indeed, if $|\operatorname{Min}(X)| \geq 3$ or $\operatorname{Min}(X)$ is a singleton then we can choose an antichain $A \subseteq X$ such that $|A| \geq 3$ and either $A \cap \operatorname{Min}(X) = \emptyset$ or $A \subseteq \operatorname{Min}(X)$. If $|\operatorname{Min}(X)| = 2$ then either there exists an antichain $A \subseteq X$ with $|A| \geq 3$ and $A \cap \operatorname{Min}(X) = \emptyset$ or every antichain $A \subseteq X$ with $|A| \geq 3$ satisifies $A \cap \operatorname{Min}(X) \neq \emptyset$. In the second case necessarily $|A \cap \operatorname{Min}(X)| = 1$ for every antichain $A \subseteq X$ with $|A| \geq 3$.

Let x_0, x_1, x_2 be pairwise distinct elements of A such that if $A \cap \operatorname{Min}(X) \neq \emptyset$ then $x_0 \in \operatorname{Min}(X)$. Let θ be the least equivalence on X such that $u\theta v$ for all $u, v \in X$ with $(u] \cap \{x_0, x_1, x_2\} = (v] \cap \{x_0, x_1, x_2\}$. It is then straightforward to verify that θ is an h-congruence on $\mathcal{H}$ and $\mathcal{H}/\theta$ is isomorphic to one of the h-spaces $\mathbf{Q}_0$, $\mathbf{Q}_1$, $\mathbf{Q}_6$, $\mathbf{Q}_7$, $\mathbf{Q}_8$, $\mathbf{S}_0$, $\mathbf{S}_1$, $\mathbf{S}_2$, $\mathbf{S}_3$, $\mathbf{S}_4$, $\mathbf{S}_6$, $\mathbf{S}_7$, $\mathbf{S}_8$, and $\mathbf{S}_9$. By Theorem 1.3 and Corollary 1.4, if $(X; \leq, \tau) \in \mathbb{PHV}$ for a variety $\mathbb{V}$ of Heyting algebras then $\mathbb{V}$ is var-relatively alg-universal modulo a group and contains an A-D family.

M. E. Adams and W. Dziobiak [4] have improved Theorem 4.13 by showing that the variety $\operatorname{Var}\{\mathbf{DG}_0, \mathbf{DH}_0\}$ is a minimal Q-universal variety. It is an open question whether $\operatorname{Var}\{\mathbf{DG}_0, \mathbf{DH}_0\}$ is var-relatively alg-universal. Both questions appear to be open for the varieties $\operatorname{Var}\{\mathbf{DG}_1, \mathbf{DH}_1\}$ and $\operatorname{Var}\{\mathbf{DG}_2, \mathbf{DH}_2\}$: we do not know whether or not these varieties are Q-universal or var-relatively alg-universal.

Another interesting question is whether or not the varieties from Theorem 1.3 are minimal. Using the result of Adams and Dziobiak and the results from [11] we obtain that the variety $\operatorname{Var}(\mathbf{DQ}_i)$ is a minimal Q-universal variety for $i = 0, 1, \ldots, 9$. A more complicated is the question concerning minimal var-relatively alg-universal varieties. Here it is clear that the varieties $\operatorname{Var}(\mathbf{DQ}_i)$ for $i = 0, 1, 3, 6$ are minimal var-relatively alg-universal. For the other varieties this question is open. But we still conjecture that all varieties $\operatorname{Var}(\mathbf{DQ}_i)$ with $i = 0, 1, \ldots, 10$ will show to be both minimal Q-universal and minimal var-relatively alg-universal.

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communicated by *Klaus Denecke* ;

Department of Theoretical Computer Science and Mathematical Logic
and Institute of Theoretical Computer Science
The Faculty of Mathematics and Physics
Charles University
Malostranské nám. 25
118 00 Praha 1
Czech Republic
e-mail: koubekktiml.ms.mff.cuni.cz

Department of Mathematics
University of Manitoba
Winnipeg, Manitoba
Canada R3T2N2
e-mail: sichlercc.umanitoba.ca