A necessary and sufficient constraint qualification for DC programming problems with convex inequality constraints

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Abstract. The purpose of the paper is to consider a necessary and sufficient constraint qualification for local optimality conditions in DC programming problems with convex inequality constraints. Also, we consider necessary and sufficient constraint qualifications for local optimality conditions in fractional programming problems and weakly convex programming problems.

1 Introduction

We consider the following DC programming problem:

$$\begin{array}{ll} \text{minimize} & f(x) - g(x), \\ \text{subject to} & h_i(x) \leq 0, \ i \in I, \end{array}$$

where X is a real locally convex Hausdorff topological vector space, I is an arbitrary index set, f, $h_i : X \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, are lower semicontinuous (lsc) proper convex functions and $g : X \to \mathbb{R}$ is a lsc convex function. Hiriart-Urruty [6] established characterization theorems for local and global optimality in unconstrained DC programming problems by using subdifferential and ε-subdifferential. Jeyakumar and Glover [9] gave global optimality conditions for DC optimization problems with convex inequality constraints by using ε-subdifferential. In addition, they applied their results to weakly convex programming problems and fractional programming problems. Research of DC programming problems and fractional programming problems have been widely studied, for example, see [1, 2, 3, 4].

For convex programming problems, it is well known that the Slater condition is a constraint qualification for global optimality conditions. Li, Ng and Pong [10] studied constraint qualifications for global optimality conditions in convex programming problems and established that the basic constraint qualification (the BCQ) is a necessary and sufficient constraint qualification for global optimality conditions in convex programming problems. Similar research have been developed recently, see [5, 7, 8].

In this paper, we show that the BCQ is a necessary and sufficient constraint qualification for local optimality conditions in DC programming problems with convex inequality constraints. Also, we apply the result of DC programming problems to fractional programming problems and weakly convex programming problems.

The paper is organized as follows. In the next section, we introduce a theorem, the BCQ is a necessary and sufficient constraint qualification for global optimality conditions in convex programming problems, by Li, Ng and Pong [10], and also we introduce a characterization result for local optimality in unconstrained DC programming problems by Hiriart-Urruty [6]. We show that the BCQ is a necessary and sufficient constraint qualification for local optimality conditions in DC programming problems with convex inequality constraints. In section 4, we show that the BCQ is also necessary and sufficient constraint qualification for local optimality conditions in fractional programming problems and weakly convex programming problems. In the last section, we summarize our results.

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2 Preliminaries Let X be a real locally convex Hausdorff topological vector space and X^* denote the continuous dual space of X. Let $\langle x^*, x\rangle$ denote the value of a functional x^* in X^* at $x \in X$, that is, $\langle x^*, x\rangle = x^*(x)$. Let Z be a subset of X^*. The convex hull and conical hull of Z are denoted by co$Z$ and cone$Z$, respectively. Let A be a convex set in X. The normal cone of A at $z_0 \in A$, denoted by $N_A(z_0)$, is defined by

$$N_A(z_0) = \{x^* \in X^* \mid \langle x^*, z - z_0\rangle \leq 0 \text{ for each } z \in A\}.$$

The indicator function δ_A is defined by

$$\delta_A(x) = \begin{cases} 0 & x \in A, \\ +\infty & \text{otherwise.} \end{cases}$$

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. The effective domain and epigraph of f are defined by

$$\mathrm{dom} f = \{x \in X \mid f(x) < +\infty\},$$

and

$$\mathrm{epi} f = \{(x, r) \in X \times \mathbb{R} \mid f(x) \leq r\},$$

respectively. The conjugate function of f, $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) \mid x \in X\}$$

for each $x^* \in X^*$. The subdifferential of f at $x \in \mathrm{dom} f$, denoted by $\partial f(x)$, is defined by

$$\partial f(x) = \{x^* \in X^* \mid \langle x^*, y - x\rangle \leq f(y) - f(x) \text{ for each } y \in X\}.$$

In this paper, we consider mathematical programming problems under the following constraint set:

$$S = \{x \in X \mid h_i(x) \leq 0 \text{ for each } i \in I\},$$

where I is an arbitrary index set and $h_i : X \to \mathbb{R} \cup \{+\infty\}$, $i \in I$, are lsc proper convex functions.

First, we introduce the basic constraint qualification (the BCQ) that is a necessary and sufficient constraint qualification for global optimality conditions in convex programming problems by Li, Ng and Pong [10].

Definition 2.1 ([10]). *Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$. The family $\{h_i \mid i \in I\}$ is said to satisfy the BCQ at $\bar{x} \in S$ if*

$$N_S(\bar{x}) = \mathrm{cone}\ \mathrm{co} \bigcup_{i \in I(\bar{x})} \partial h_i(\bar{x}),$$

where $I(\bar{x}) = \{i \in I \mid h_i(\bar{x}) = 0\}$.

Theorem 2.1 ([10]). *Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Then the following statements are equivalent:*

(i) *The family $\{h_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$.*

(ii) *For each lsc proper convex function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *such that* $\mathrm{dom}f \cap S \neq \emptyset$ *and* $\mathrm{epi}f^* + \mathrm{epi}\delta_S^*$ *is weak***-closed,* $\bar{x}$ *is a minimizer of* f *in* S *if and only if there exists* $\lambda \in \mathbb{R}_+^{(I)}$ *such that* $\lambda_i h_i(\bar{x}) = 0$ *for each* $i \in I$*, and*

$$0 \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial h_i(\bar{x}),$$

where $\mathbb{R}_+^{(I)}$ *is the set of nonnegative real tuples* $\lambda = (\lambda_i)_{i\in I}$ *with only finitely many* $\lambda_i \neq 0$.

This theorem shows that the BCQ is a necessary and sufficient constraint qualification for global optimality conditions in convex programming problems.

Next, we introduce the following characterization result for local optimality in unconstrained DC programming problems by Hiriart-Urruty [6].

Theorem 2.2 ([6]). *Let* $f : X \to \mathbb{R}\cup\{+\infty\}$ *be a lsc proper convex function and* $g : X \to \mathbb{R}$ *be a lsc convex function. If* $\bar{x} \in X$ *is a local minimizer of* $f - g$ *in* X*, then*

$$\partial g(\bar{x}) \subset \partial f(\bar{x}).$$

By using this theorem, we show that the BCQ is a necessary and sufficient constraint qualification for local optimality conditions in DC programming problems with convex inequality constraints in the next section.

3 DC Programming

In this section, we consider again the DC programming problem:

$$\begin{array}{ll} \text{minimize} & f(x) - g(x), \\ \text{subject to} & h_i(x) \leq 0, \ i \in I, \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lsc proper convex function and $g : X \to \mathbb{R}$ is a lsc convex function.

Theorem 3.1. *Let* $\{h_i \mid i \in I\}$ *be a family of lsc proper convex functions from* X *to* $\mathbb{R} \cup \{+\infty\}$*, and* $\bar{x} \in S$*. Then the following statements are equivalent:*

(i) *The family* $\{h_i \mid i \in I\}$ *satisfies the BCQ at* $\bar{x}$.

(ii) *For each lsc proper convex function* $f : X \to \mathbb{R} \cup \{+\infty\}$ *such that* $\mathrm{dom}f \cap S \neq \emptyset$ *and* $\mathrm{epi}f^* + \mathrm{epi}\delta_S^*$ *is weak***-closed, and lsc convex function* $g : X \to \mathbb{R}$*, if* $\bar{x}$ *is a local minimizer of* $f - g$ *in* S*, then for each* $v \in \partial g(\bar{x})$*, there exists* $\lambda \in \mathbb{R}_+^{(I)}$ *such that* $\lambda_i h_i(\bar{x}) = 0$ *for each* $i \in I$*, and*

$$v \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial h_i(\bar{x}).$$

Proof. First, we prove (i) implies (ii). Assume that (i) holds. Let f be a lsc proper convex function from X to $\mathbb{R} \cup \{+\infty\}$ such that $\mathrm{dom}f \cap S \neq \emptyset$ and $\mathrm{epi}f^* + \mathrm{epi}\delta_S^*$ is weak*-closed, and g be a lsc convex function from X to $\mathbb{R}$. The point $\bar{x}$ is a local minimizer of $f - g$ in S if and only if $\bar{x}$ is a local minimizer of $(f + \delta_S) - g$ in X. We have from Theorem 2.2 that if $\bar{x}$ is a local minimizer of $(f + \delta_S) - g$ in X, then

$$\partial g(\bar{x}) \subset \partial(f + \delta_S)(\bar{x}).$$

By the assumption of f, the subdifferential sum formula holds, that is,

$$\partial(f+\delta_S)(\bar{x}) = \partial f(\bar{x}) + \partial\delta_S(\bar{x}).$$

Since $\partial\delta_S(\bar{x}) = N_S(\bar{x})$ and the assumption (i) holds,

$$\partial f(\bar{x}) + \partial\delta_S(\bar{x}) = \partial f(\bar{x}) + \text{cone co} \bigcup_{i\in I(\bar{x})} \partial h_i(\bar{x}).$$

Hence, if $\bar{x}$ is a local minimizer of $f-g$ in S, then

$$\partial g(\bar{x}) \subset \partial f(\bar{x}) + \text{cone co} \bigcup_{i\in I(\bar{x})} \partial h_i(\bar{x}).$$

This implies that (ii) holds.

Next, we prove (ii) implies (i). Assume that (ii) holds and let $x^* \in N_S(\bar{x})$. Then $\bar{x}$ is a minimizer of $-x^*$ in S. By setting $f=-x^*$ and $g=0$ in assumption (ii), there exist $\lambda \in \mathbb{R}_+^{(I)}$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and

$$0 \in -x^* + \sum_{i\in I} \lambda_i \partial h_i(\bar{x}).$$

Therefore, we have

$$x^* \in \sum_{i\in I} \lambda_i \partial h_i(\bar{x}) = \sum_{i\in I(\bar{x})} \lambda_i \partial h_i(\bar{x}) \subset \text{cone co} \bigcup_{i\in I(\bar{x})} \partial h_i(\bar{x}),$$

and hence $N_S(\bar{x}) \subset \text{cone co} \bigcup_{i\in I(\bar{x})} \partial h_i(\bar{x})$ holds. Since the converse inclusion is always satisfied, (i) holds. This completes the proof. □

This theorem shows that the BCQ is a necessary and sufficient constraint qualification for local optimality conditions in DC programming problems with convex inequality constraints.

4 Applications

In this section, we apply the result of previous section to fractional programming problems and weakly convex programming problems. In particular, we consider weakly convex programming problems in a smooth real Banach space.

4.1 Fractional Programming

We consider the following fractional programming problem:

$$\begin{array}{ll} \text{minimize} & f(x)/g(x), \\ \text{subject to} & h_i(x) \leq 0, \ i \in I, \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lsc proper convex function and $g : X \to \mathbb{R}$ is a lsc convex function such that f is nonnegative and g is positive on S.

Theorem 4.1.1. *Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Then the following statements are equivalent:*

(i) *The family $\{h_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$.*

(ii) *For each lsc proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that* $\mathrm{dom} f \cap S \neq \emptyset$, $\mathrm{epi} f^* + \mathrm{epi}\delta_S^*$ *is weak***-closed and f is nonnegative on S, and lsc convex function $g : X \to \mathbb{R}$ such that g is positive on S, if $\bar{x}$ is a local minimizer of f/g in S, then there exists $\lambda_0 \geq 0$ such that for each $v \in \lambda_0 \partial g(\bar{x})$, there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and*

$$v \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial h_i(\bar{x}).$$

Proof. We first prove (i) implies (ii). Let f be a lsc proper convex function from X to $\mathbb{R} \cup \{+\infty\}$ such that $\mathrm{dom} f \cap S \neq \emptyset$, $\mathrm{epi} f^* + \mathrm{epi}\delta_S^*$ is weak*-closed and f is nonnegative on S, and g be a lsc convex function from X to $\mathbb{R}$ such that g is positive on S. In addition, let $\bar{x}$ be a local minimizer of f/g in S. By putting $\lambda_0 = f(\bar{x})/g(\bar{x})$, $\bar{x}$ is a local minimizer of $f - \lambda_0 g$ in S. Because $f - \lambda_0 g$ is a DC function, we can prove (i) implies (ii) by using Theorem 3.1. Also, it is clear that (ii) implies (i) by taking $f = -x^* + \langle x^*, \bar{x} \rangle$ and $g = 1$. □

This theorem shows that the BCQ is also a necessary and sufficient constraint qualification for the fractional programming problems.

4.2 Weakly Convex Programming Let X be a real Banach space with norm $\|\cdot\|$. The norm of X^* is also denoted by $\|\cdot\|$ for convenience. The duality mapping of X, the multivalued operator $J : X \to X^*$, is defined by

$$J(x) = \{x^* \in X^* \mid \langle x^*, x \rangle = \|x\|^2 = \|x^*\|^2\}$$

for each $x \in X$. Let $S(X)$ denote the unit sphere of X, that is, $S(X) = \{x \in X \mid \|x\| = 1\}$. Then X is said to be smooth if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(X)$. In this case, because the duality mapping J of X is single valued, $J(x)$ is identified with the element of $J(x)$ for each $x \in X$; see [11].

Recall that a function p is weakly convex if it can be written as $p = q - \frac{\rho}{2}\|\cdot\|^2$ for some convex function q and $\rho \geq 0$. We consider the following weakly convex programming problem:

$$\begin{array}{ll} \text{minimize} & f(x) - \frac{\rho}{2}\|x\|^2, \\ \text{subject to} & h_i(x) \leq 0, \ i \in I, \end{array}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lsc proper convex function and $\rho \geq 0$.

We show the following theorem in a smooth real Banach space.

Theorem 4.2.1. *Let $\{h_i \mid i \in I\}$ be a family of lsc proper convex functions from X to $\mathbb{R} \cup \{+\infty\}$, and $\bar{x} \in S$. Assume that X is smooth. Then the following statements are equivalent:*

(i) *The family $\{h_i \mid i \in I\}$ satisfies the BCQ at $\bar{x}$.*

(ii) *For each lsc proper convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ such that* $\mathrm{dom} f \cap S \neq \emptyset$ *and* $\mathrm{epi} f^* + \mathrm{epi}\delta_S^*$ *is weak***-closed, and $\rho \geq 0$, if $\bar{x}$ is a local minimizer of $f - \frac{\rho}{2}\|\cdot\|^2$ in S, then there exists $\lambda \in \mathbb{R}_+^{(I)}$ such that $\lambda_i h_i(\bar{x}) = 0$ for each $i \in I$, and*

$$\rho J(\bar{x}) \in \partial f(\bar{x}) + \sum_{i \in I} \lambda_i \partial h_i(\bar{x}).$$

Proof. Since X is smooth, J is single valued. By taking g as $\frac{\rho}{2}\|\cdot\|^2$ in Theorem 3.1, we can prove (i) implies (ii) because $\partial g(\bar{x}) = \rho J(\bar{x})$. Also, it is clear that (ii) implies (i). $\square$

Example 4.2.1. *Consider the problem:*

$$\begin{array}{ll} \textit{minimize} & \frac{1}{4}x^4 + |x| - x^2, \\ \textit{subject to} & \max\{0, -x\} \le 0. \end{array}$$

Let $X = \mathbb{R}$, $I = \{1\}$, $f(x) = \frac{1}{4}x^4 + |x|$, $\rho = 2$, $h_1(x) = \max\{0, -x\}$ *and* $S = [0, +\infty)$. *Then* f *and* h_1 *are continuous convex functions and* $\{h_i \mid i \in I\}$ *satisfies the BCQ at each point of* S. *Let* $\bar{x}$ *be a local minimizer of* $f(x) - \frac{\rho}{2}x^2$ *in* S. *By Theorem 4.2.1, there exists* $\lambda_1 \ge 0$ *such that* $\rho\bar{x} \in \partial f(\bar{x}) + \lambda_1 \partial h_1(\bar{x})$ *and* $\lambda_1 h_1(\bar{x}) = 0$, *because* J *is an identity map for* X. *When* $\bar{x} > 0$, *since* $\partial f(\bar{x}) = \bar{x}^3 + \{1\}$ *and* $\partial h_1(\bar{x}) = \{0\}$, $\bar{x}$ *must be* 1 *or* $\frac{-1+\sqrt{5}}{2}$. *They also satisfy* $\lambda_1 h_1(\bar{x}) = 0$. *Otherwise, when* $\bar{x} = 0$, *since* $\partial f(\bar{x}) = \bar{x}^3 + [-1, 1]$ *and* $\partial h_1(\bar{x}) = [-1, 0]$, $\bar{x} \in [-\lambda_1 - 1, 1]$ *holds whenever* $\lambda_1 \ge 0$. *Also,* $\bar{x} = 0$ *satisfies* $\lambda_1 h_1(\bar{x}) = 0$. *Therefore* 0, 1 *and* $\frac{-1+\sqrt{5}}{2}$ *have possibilities for local minimizers, and actually,* 0 *is the global minimizer and* 1 *is a local minimizer. But* $\frac{-1+\sqrt{5}}{2}$ *is neither a minimizer nor a local minimizer.*

5 Conclusions In this paper we show that the BCQ is a necessary and sufficient constraint qualification for local optimality conditions in DC programming problems with convex inequality constraints. Also, we show that the BCQ is necessary and sufficient constraint qualification for local optimality conditions in fractional programming problems and weakly convex programming problems. In other words, we find the weakest condition under which local optimality conditions are valid for each mathematical programming problems appeared in this paper.

References

[1] R.I. Boţ, I.B. Hodrea, G. Wanka, Farkas-type results for fractional programming problems, Nonlinear Anal. 67 (2007) 1690–1703.

[2] R.I. Boţ, I.B. Hodrea, G. Wanka, Some new Farkas-type results for inequality systems with DC functions, J. Global Optim. 39 (2007) 595–608.

[3] N. Dinh, T.T.A. Nghia, G. Vallet, A closedness condition and its applications to DC programs with convex constraints, Optimization. 59 (2010) 541–560.

[4] N. Dinh, B. Mordukhovich, T.T.A. Nghia, Subdifferentials of value functions and optimality conditions for DC and bilevel infinite and semi-infinite programs, Math. Program. 123 (2010) 101–138.

[5] M.A. Goberna, V. Jeyakumar, M.A. López, Necessary and sufficient constraint qualifications for solvability of systems of infinite convex inequalities, Nonlinear Anal. 68 (2008) 1184–1194.

[6] J.-B. Hiriart-Urruty, From convex optimization to nonconvex optimization. Necessary and sufficient conditions for global optimality. in: F.H. Clarke, V.F. Dem'yanov, F. Giannessi (Eds.), Nonsmooth Optimization and Related Topics, Plenum, New York, 1989, pp. 219–239.

[7] V. Jeyakumar, Constraint qualifications characterizing Lagrangian duality in convex optimization, J. Optim. Theory Appl. 136 (2008) 31–41.

[8] V. Jeyakumar, N. Dinh, G.M. Lee, A new closed cone constraint qualification for convex optimization, Applied Mathematics Research Report AMR04/8, University of New South Wales, Sydney, Australia, 2004.

[9] V. Jeyakumar, B.M. Glover, Characterizing global optimality for DC optimization problems under convex inequality constraints, J. Global Optim. 8 (1996) 171–187.

[10] C. Li, K.F. Ng, T.K. Pong, Constraint qualifications for convex inequality systems with applications in constrained optimization, SIAM J. Optim. 19 (2008) 163–187.

[11] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.

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