GLOBAL EXISTENCE AND DECAY RATE FOR A COUPLED DEGENERATE HYPERBOLIC SYSTEM WITH DISSIPATION

Kosuke Ono

Received February 10, 2011 ; revised May 13, 2011

Abstract.We study on the initial-boundary value problem for the coupled degenerate hyperbolic system with dissipation :

$$\begin{cases} \rho\dfrac{\partial^2 u}{\partial t^2} - \left(\displaystyle\int_\Omega |\nabla u(x,t)|^2\,dx + \int_\Omega |\nabla v(x,t)|^2\,dx\right)\Delta u + \delta\dfrac{\partial u}{\partial t} = 0\,, \\ \rho\dfrac{\partial^2 v}{\partial t^2} - \left(\displaystyle\int_\Omega |\nabla u(x,t)|^2\,dx + \int_\Omega |\nabla v(x,t)|^2\,dx\right)\Delta v + \delta\dfrac{\partial v}{\partial t} = 0 \end{cases}$$

with $\rho > 0$ and $\delta > 0$ and a homogeneous Dirichlet boundary condition. When either the coefficient ρ or the initial data are appropriately smaller than the coefficient δ, we show the global-in-time solvability for the system and the optimal decay rate for the H^2-norm for the solutions. Moreover, we derive the sharp decay estimates of their derivatives.

1 Introduction. In this paper we consider the initial-boundary value problem for the following coupled degenerate hyperbolic system with dissipation :

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\int_\Omega |\nabla u(x,t)|^2\,dx + \int_\Omega |\nabla v(x,t)|^2\,dx\right)\Delta u + \delta\frac{\partial u}{\partial t} = 0 \quad \text{in } \Omega\times[0,\infty)\,, \tag{1.1}$$

$$\rho\frac{\partial^2 v}{\partial t^2} - \left(\int_\Omega |\nabla u(x,t)|^2\,dx + \int_\Omega |\nabla v(x,t)|^2\,dx\right)\Delta v + \delta\frac{\partial v}{\partial t} = 0 \quad \text{in } \Omega\times[0,\infty) \tag{1.2}$$

with the initial and boundary conditions

$$u(x,0) = u_0(x)\,, \quad \frac{\partial u}{\partial t}(x,0) = u_1(x)\,, \quad v(x,0) = v_0(x)\,, \quad \frac{\partial v}{\partial t}(x,0) = v_1(x) \quad \text{in } \Omega$$

and

$$u(x,t) = v(x,t) = 0 \quad \text{on } \partial\Omega\times[0,\infty)\,.$$

Here $u = u(x,t)$ and $v = v(x,t)$ are unknown real functions, Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\Delta = \nabla\cdot\nabla = \sum_{j=1}^N \partial^2/\partial x_j^2$ is the Laplace operator with the domain $H^2(\Omega)\cap H_0^1(\Omega)$, the coefficients $\rho > 0$ and $\delta > 0$ are positive constants.

The coupled degenerate hyperbolic system (1.1)–(1.2) comes from the single hyperbolic equation :

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\mu + \int_\Omega |\nabla u(x,t)|^2\,dx\right)\Delta u + \delta\frac{\partial u}{\partial t} = 0 \quad \text{in } \Omega\times[0,\infty) \tag{1.3}$$

2000 *Mathematics Subject Classification.* Primary 35L15, 35B40; Secondary 35L80, 35L20.
Key words and phrases. Degenerate hyperbolic system, dissipation, decay rate.

with $u(x,0) = u_0(x)$ and $u_t(x,t) = u_1(x)$, which is called a non-degenerate equation when $\mu > 0$ and a degenerate one when $\mu = 0$. When the dimension N is one, it is well-known that (1.3) describes small amplitude vibrations of an elastic stretched string, and (1.3) with $\delta = 0$ was introduced by Kirchhoff [7] (also see [3], [5]). The coupled hyperbolic system (1.1)–(1.2) will be useful for the research of amplitude vibrations of two kinds of elastic stretched strings.

When $\mu > 0$ and $\delta > 0$, it is easy to see that the energy of the non-degenerate equation (1.3) has an exponential decay rate.

On the other hand, when $\mu = 0$ and $\delta > 0$, Nishihara and Yamada [14] have shown the global existence theorem under the assumption that the initial data $\{u_0, u_1\}$ belonging to $H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ are sufficiently small and $\|\nabla u_0\| \neq 0$, and they have derived some decay properties of the solution by using the energy method, e.g., $\|\nabla u(t)\|^2 + \|\nabla u_t(t)\|^2 \leq C(1+t)^{-1}$ and $\|u_t(t)\|^2 + \|u_{tt}(t)\|^2 \leq C(t+1)^{-2}$ for $t \geq 0$ (also see [10], [21]), where $\|\cdot\|$ is the norm of $L^2(\Omega)$. In the previous paper [15], we have improved some decay rates of the solution as in [14], e.g., $\|\Delta u(t)\|^2 \leq C(1+t)^{-1}$ and $\|\nabla u_t(t)\|^2 \leq C(1+t)^{-3}$ and $\|u_{tt}(t)\|^2 \leq C(1+t)^{-4}$ for $t \geq 0$. Moreover, Mizumachi [9] has derived the lower decay estimate $\|u(t)\|^2 \geq C(1+t)^{-1}$ for $t \geq 0$, when the initial data are sufficiently small (cf. [13], [16] for equations with strong dissipation). And, Ghisi and Gobbino [6] have given the lower decay estimate $\|\nabla u(t)\|^2 \geq C(1+t)^{-1}$ for $t \geq 0$, too, when the coefficient ρ is sufficiently small by using another technique as [9] (also see [17]).

The purpose of this paper is to derive the optimal decay rate for the H^2-norm of the solutions $\{u(t), v(t)\}$ of the system (1.1)–(1.2) including the lower decay estimate under weaker conditions. Moreover, when either the coefficient ρ or the initial data $\{u_0, u_1, v_0, v_1\}$ are appropriately smaller than the coefficient δ, we prove the global existence theorem.

We will use the following function through this paper.

$$
\begin{aligned}
&K(t) \equiv \|u(t)\|^2 + \|v(t)\|^2\,, \quad L(t) \equiv \|u_t(t)\|^2 + \|v_t(t)\|^2\,,\\
&M(t) \equiv \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\,, \quad X(t) \equiv \|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2\,,\\
&Y(t) \equiv \|\nabla u_t(t)\|^2 + \|\nabla v_t(t)\|^2\,, \quad Z(t) \equiv \|\Delta u(t)\|^2 + \|\Delta v(t)\|^2\,,
\end{aligned}
$$

and

$$
H(t) \equiv \rho \frac{Y(t)}{M(t)} + Z(t)\,. \tag{1.4}
$$

In particular, when $t = 0$, it means that

$$
H(0) \equiv \rho \frac{\|\nabla u_1\|^2 + \|\nabla v_1\|^2}{\|\nabla u_0\|^2 + \|\nabla v_0\|^2} + \|\Delta u_0\|^2 + \|\Delta v_0\|^2\,. \tag{1.5}
$$

Our main result is as follows.

Theorem 1.1 *Let initial data* $\{u_0, v_0\} \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^2$ *and* $\{u_1, v_1\} \in \left(H_0^1(\Omega)\right)^2$ *satisfy* $\|\nabla u_0\|^2 + \|\nabla v_0\|^2 \neq 0$. *Suppose that* ρ *and* $\{u_0, u_1, v_0, v_1\}$ *satisfy*

$$
\rho\, H(0) < \delta^2\,. \tag{1.6}
$$

Then, the problem (1.1)–(1.2) *admits unique global solutions* $\{u(t), v(t)\}$ *in the class*

$$
\left(C^0([0,\infty); H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,\infty); H_0^1(\Omega)) \cap C^2([0,\infty); L^2(\Omega))\right)^2\,, \tag{1.7}
$$

and $\{u(t), v(t)\}$ *satisfy that*

$$C'(1+t)^{-1} \leq \|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 \leq C(1+t)^{-1} \tag{1.8}$$

for $t \geq 0$*. Moreover, when* $4\rho H(0) < \delta^2$ *instead of* (1.6), $\{u(t), v(t)\}$ *satisfy that*

$$\|u_t(t)\|_{H^1}^2 + \|v_t(t)\|_{H^1}^2 \leq C(1+t)^{-3}\,, \tag{1.9}$$

$$\|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 \leq C(1+t)^{-4} \tag{1.10}$$

for $t \geq 0$*, where* C *and* C' *are certain positive constants depending on the initial data and the coefficient* $\rho > 0$*.*

The above theorem is obtained by gathering Theorems 2.2–4.3 in the following sections.

The notations we use in this paper are standard. The symbol $(\cdot, \cdot)$ means the inner product in $L^2(\Omega)$ or sometimes duality between the space X and its dual X', and the norm of $L^2(\Omega)$ is often written as $\|\cdot\| = \|\cdot\|_{L^2}$ for simplicity. Positive constants will be denoted by C and will change from line to line.

2 Global Existence.

By applying the Banach contraction mapping theorem to the problem (1.1)–(1.2), we obtain the following local existence theorem. The proof is standard and we omit it here (see [1], [2], [4], [15], [18], [19], [20]).

Proposition 2.1 *If the initial data* $\{u_0, v_0\} \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^2$ *and* $\{u_1, v_1\} \in \left(H_0^1(\Omega)\right)^2$ *satisfy* $\|\nabla u_0\|^2 + \|\nabla v_0\|^2 \neq 0$*, then the problem* (1.1)–(1.2) *admits unique local solutions* $\{u(t), v(t)\}$ *in the class*

$$\left(C^0([0,T]; H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1([0,T]; H_0^1(\Omega)) \cap C^2([0,T]; L^2(\Omega))\right)^2$$

for some $T = T(\|u_0\|_{H^2}, \|v_0\|_{H^2}, \|u_1\|_{H^1}, \|v_1\|_{H^1}) > 0$*. Moreover, if* $\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 > 0$ *and* $\|u(t)\|_{H^2}^2 + \|v(t)\|_{H^2}^2 + \|u(t)\|_{H^1}^2 + \|v(t)\|_{H^1}^2 < \infty$ *for* $t \geq 0$*, then we can take* $T = \infty$*.*

By deriving a-priori estimates $Y(t) + Z(t) < \infty$ and $M(t) > 0$ for $t \geq 0$, we will show the global-in-time solvability for the system (1.1)–(1.2).

Theorem 2.2 *Let the initial data* $\{u_0, u_1\} \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right)^2$ *and* $\{u_1, v_1\} \in \left(H_0^1(\Omega)\right)^2$ *satisfy* $M(0) > 0$*. Suppose that* ρ *and* $\{u_0, u_1, v_0, v_1\}$ *satisfy* (1.6)*. Then, the problem* (1.1)–(1.2) *admits unique global solutions* $\{u(t), v(t)\}$ *in the class* (1.7) *and it holds that*

$$M(t) \equiv \|\nabla u(t)\|^2 + \|\nabla v(t)\|^2 > 0 \tag{2.1}$$

and

$$H(t) \equiv \rho \frac{Y(t)}{M(t)} + Z(t) \leq H(0) \tag{2.2}$$

for $t \geq 0$*.*

Proof. Multiplying (1.1) and (1.2) by $-2\Delta u_t$ and $-2\Delta v_t$, respectively, and integrating them over Ω, we have

$$\frac{d}{dt}\rho\|\nabla u_t(t)\|^2 + M(t)\frac{d}{dt}\|\Delta u(t)\|^2 + 2\delta\|\nabla u_t(t)\|^2 = 0$$

and

$$\frac{d}{dt}\rho\|\nabla v_t(t)\|^2 + M(t)\frac{d}{dt}\|\Delta v(t)\|^2 + 2\delta\|\nabla v_t(t)\|^2 = 0\,.$$

Adding these two equations, we obtain

$$\frac{d}{dt}\rho Y(t) + M(t)\frac{d}{dt}Z(t) + 2\delta Y(t) = 0\,. \tag{2.3}$$

Since $M(0) > 0$, putting

$$T \equiv \sup\left\{t \in [0,\infty) \;\middle|\; M(s) > 0 \text{ for } 0 \le s < t\right\}\,,$$

we see that $T > 0$ and $M(t) > 0$ for $0 \le t < T$.

Then, multiplying (2.3) by $M(t)^{-1}$, we have

$$\frac{d}{dt}H(t) + 2\left(\delta + \frac{\rho}{2}\frac{M'(t)}{M(t)}\right)\frac{Y(t)}{M(t)} = 0 \tag{2.4}$$

for $0 \le t < T$, where $H(t)$ is defined by (1.4). Moreover, since

$$\frac{\rho}{2}\frac{|M'(t)|}{M(t)} \le \rho\left(\frac{Y(t)}{M(t)}\right)^{1/2} \le \left(\rho H(t)\right)^{1/2}\,, \tag{2.5}$$

we obtain

$$\frac{d}{dt}H(t) + 2\left(\delta - \left(\rho H(t)\right)^{1/2}\right)\frac{Y(t)}{M(t)} \le 0 \tag{2.6}$$

for $0 \le t < T$.

If $(\rho H(0))^{1/2} < \delta$, then there exists $0 < T_1 \le T$ such that

$$(\rho H(t))^{1/2} \le \delta \qquad \text{for } 0 \le t \le T_1$$

and we see from (2.6) that $H(t) \le H(0)$ for $0 \le t \le T_1$, and hence,

$$H(t) \le H(0) \qquad \text{for } 0 \le t < T\,. \tag{2.7}$$

Next, we will show that $T = \infty$. If $M(T) = 0$, then $\lim_{t\to T} Y(t) = 0$, and we see

$$\lim_{t\to T} E(u(t), v(t)) = 0 \tag{2.8}$$

where

$$E(u(t), v(t)) \equiv \rho\left(\|u_t(t)\|^2 + \|v_t(t)\|^2\right) + \frac{1}{2}\left(\|\nabla u(t)\|^2 + \|\nabla v(t)\|^2\right)^2\,. \tag{2.9}$$

On the other hand, we perform the change of variable $s = T - t$ or $t = T - s$, then the functions $U(s) = u(T - t)$ and $V(s) = v(T - t)$ on $[0, T]$ satisfy that

$$\rho U_{ss} - \left(\|\nabla U\|^2 + \|\nabla V\|^2\right)\Delta U - \delta U_s = 0\,, \tag{2.10}$$

$$\rho V_{ss} - \left(\|\nabla U\|^2 + \|\nabla V\|^2\right)\Delta V - \delta V_s = 0\,. \tag{2.11}$$

Multiplying (2.10) and (2.11) by $2U_s$ and $2V_s$, respectively, and integrating them over Ω, and adding the resulting equations, we have

$$\frac{d}{ds}E(U(s),V(s)) = 2\delta\left(\|U_s(s)\|^2 + \|V_s(s)\|^2\right) \le \frac{2\delta}{\rho}E(U(s),V(s))$$

where $E(U(s),V(s))$ is defined by (2.9). Thus, since $E(U(0),V(0)) = \lim_{t\to T} E(u(t),v(t)) = 0$ by (2.8), we obtain

$$E(U(s),V(s)) \le \frac{2\delta}{\rho}\int_0^s E(U(\tau),V(\tau))\,d\tau\,,$$

and from the Gronwall inequality that

$$E(U(s),V(s)) = 0 \text{ on } [0,T] \quad \text{or} \quad E(u(t),v(t)) = 0 \text{ on } [0,T]$$

which contradicts $M(0) \equiv \|\nabla u_0\|^2 + \|\nabla v_0\|^2 \ne 0$. Then, we see that $T = \infty$, and $M(t) > 0$ and (2.4)–(2.7) hold true for $t \ge 0$, that is, we have that $Y(t) + Z(t) < \infty$ for $t \ge 0$. Thus, by the second statement of Proposition 2.1, we see that the problem (1.1)–(1.2) admits unique global solutions $\{u(t), v(t)\}$. □

3 Optimal Decay Estimates for $\{u, v\}$.

First, we will derive the upper decay estimate for $\|u(t)\|^2_{H^2} + \|v(t)\|^2_{H^2}$.

Theorem 3.1 *Under the assumption of Theorem 2.2, it holds that*

$$H(t) \equiv \rho\frac{Y(t)}{M(t)} + Z(t) \le C(1+t)^{-1}\,, \tag{3.1}$$

$$M(t) \le C(1+t)^{-1} \quad \textit{and} \quad Y(t) \le C(1+t)^{-2} \tag{3.2}$$

for $t \ge 0$.

Proof. From (2.2) and (2.4) (or (2.6)) we have

$$\frac{d}{dt}H(t) + b\frac{Y(t)}{M(t)} \le 0 \qquad \text{for } t \ge 0 \tag{3.3}$$

with $b = 2(\delta - (\rho H(0))^{1/2}) > 0$. For any $t \ge 0$, integrating (3.3) over $[t, t+1]$, we obtain

$$b\int_t^{t+1}\frac{Y(s)}{M(s)}\,ds \le H(t) - H(t+1) \qquad (\equiv bD_1(t)^2\,)\,. \tag{3.4}$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\frac{Y(t_j)}{M(t_j)} \le 4\int_t^{t+1}\frac{Y(s)}{M(s)}\,ds = 4D_1(t)^2 \qquad \text{for } j = 1,2\,. \tag{3.5}$$

On the other hand, multiplying (1.1) and (1.2) by $-\Delta u$ and $-\Delta v$, respectively, and integrating them over Ω, we have

$$M(t)\|\Delta u(t)\|^2 + \frac{d}{dt}\rho(\nabla u(t), \nabla u_t(t)) - \rho\|\nabla u_t(t)\|^2 + \delta(\nabla u(t), \nabla u_t(t)) = 0\,,$$
$$M(t)\|\Delta v(t)\|^2 + \frac{d}{dt}\rho(\nabla v(t), \nabla v_t(t)) - \rho\|\nabla v_t(t)\|^2 + \delta(\nabla v(t), \nabla v_t(t)) = 0\,.$$

Adding these two equations and multiplying the resulting equation by $1/M(t)$, we observe

$$Z(t)+\frac{\rho}{2}\frac{|M'(t)|^2}{M(t)^2}=\rho\frac{Y(t)}{M(t)}-\frac{\rho}{2}\frac{d}{dt}\frac{M'(t)}{M(t)}-\frac{\delta}{2}\frac{M'(t)}{M(t)}\,,$$

and integrating it over $[t_1,t_2]$, we obtain from (2.5), (3.4) and (3.5) that

$$\begin{aligned}
&\int_{t_1}^{t_2}\left(Z(s)+\frac{\rho}{2}\frac{|M'(s)|^2}{M(s)^2}\right)ds\\
&\leq\int_{t_1}^{t_2}\rho\frac{Y(s)}{M(s)}\,ds+\frac{\rho}{2}\sum_{j=1}^{2}\frac{|M'(t_j)|}{M(t_j)}+\frac{\delta}{2}\int_{t_1}^{t_2}\frac{|M'(s)|}{M(s)}\,ds\\
&\leq\int_{t}^{t+1}\rho\frac{Y(s)}{M(s)}\,ds+\frac{\rho}{2}\sum_{j=1}^{2}\left(\frac{Y(t_j)}{M(t_j)}\right)^{1/2}+\frac{\delta}{2}\int_{t}^{t+1}\left(\frac{Y(s)}{M(s)}\right)^{1/2}ds\\
&\leq\rho D_1(t)^2+CD_1(t)\,, \qquad (3.6)
\end{aligned}$$

and moreover, from (3.4) and (3.6) that

$$\begin{aligned}
\int_{t_1}^{t_2}H(s)\,ds&=\int_{t_1}^{t_2}\rho\frac{Y(s)}{M(s)}\,ds+\int_{t_1}^{t_2}Z(s)\,ds\\
&\leq 2\rho D_1(t)^2+CD_1(t)\,. \qquad (3.7)
\end{aligned}$$

Integrating (2.4) over $[t,t_2]$, we have from (2.2) and (2.5) that

$$\begin{aligned}
H(t)&=H(t_2)+2\int_{t}^{t_2}\left(\delta+\frac{\rho}{2}\frac{M'(s)}{M(s)}\right)\frac{Y(s)}{M(s)}\,ds\\
&\leq 2\int_{t_1}^{t_2}H(s)\,ds+C\int_{t}^{t+1}\frac{Y(s)}{M(s)}\,ds\\
&\leq CD_1(t)^2+CD_1(t)
\end{aligned}$$

and since $bD_1(t)^2\leq H(t)-H(t+1)\leq H(0)$,

$$H(t)^2\leq CD_1(t)^2\leq C(H(t)-H(t+1))\,. \qquad (3.8)$$

Thus, applying Lemma 3.2 to (3.8) we obtain the desired estimates (3.1) and (3.2). □

In order to derive the decay estimate of the function $H(t)$, we used the following Nakao inequality in the proof of Theorem 3.1 (see [10], [11], [12] for the proof).

Lemma 3.2 *Let $\phi(t)$ be a non-increasing non-negative function on $[0,\infty)$ and satisfy*

$$\phi(t)^{1+\alpha}\leq k_0\left(\phi(t)-\phi(t+1)\right)$$

with certain constants $k_0\geq 0$ and $\alpha>0$. Then, the function $\phi(t)$ satisfies

$$\phi(t)\leq\left(\phi(0)^{-\alpha}+\alpha k_0^{-1}[t-1]^{+}\right)^{-1/\alpha}$$

for $t\geq 0$, where $[t-1]^{+}=\max\{t-1,0\}$.

Next, we will derive the lower decay estimate for $\|u(t)\|_{H_2}+\|v(t)\|_{H^2}$.

Theorem 3.3 *Under the assumption of Theorem 2.2, it holds that*

$$K(t) \equiv \|u(t)\|^2 + \|v(t)\|^2 \geq C'(1+t)^{-1} \tag{3.9}$$

for $t \geq 0$ *with a positive constant* $C' > 0$.

Proof. Multiplying (1.1) and (1.2) by $2u_t$ and $2v_t$, respectively, and integrating them over Ω, and adding the resulting equations we have

$$\frac{d}{dt}E(t) + 2\delta L(t) = 0 \tag{3.10}$$

where we put

$$E(t) \equiv \rho L(t) + \frac{1}{2}M(t)^2 \,.$$

Multiplying (1.1) and (1.2) by u and v, respectively, and integrating them over Ω, and adding the resulting equation, we have

$$\frac{d}{dt}\frac{1}{2}\left(\delta K(t) + \rho K'(t)\right) + \rho L(t) + M(t)^2 = 0 \,. \tag{3.11}$$

Multiplying (3.10) by ρ/δ and adding (3.11), we obtain

$$\frac{d}{dt}E^*(t) + \rho L(t) + M(t)^2 = 0 \tag{3.12}$$

where we put

$$E^*(t) \equiv \frac{\rho}{\delta}E(t) + \frac{1}{2}\left(\delta K(t) + \rho K'(t)\right) \,. \tag{3.13}$$

Since $L(t) \leq CM(t)^2$, $M(t) \leq CK(t)$, $|K'(t)| \leq C(L(t)+K(t))$, and $K(t) \leq C$, we observe

$$E^*(t) \leq CK(t) \tag{3.14}$$

and

$$\rho L(t) + M(t)^2 \leq CM(t)^2 \leq CK(t)^2 \leq C_1 K(t) \,. \tag{3.15}$$

On the other hand, since

$$|K'(t)| \leq 2(L(t)K(t))^{1/2} \leq \frac{2\rho}{\delta}L(t) + \frac{\delta}{2\rho}K(t)$$

by the Young inequality, we have

$$E^*(t) \geq \frac{\delta}{4}K(t) \,. \tag{3.16}$$

Thus, we obtain from (3.12)–(3.16) that

$$\frac{d}{dt}E^*(t) + \frac{4}{\delta}C_1 E^*(t)^2 \geq \frac{d}{dt}E^*(t) + \left(\rho L(t) + M(t)^2\right) \geq 0$$

and hence, $E^*(t) \geq C(1+t)^{-1}$ for $t \geq 0$ with $C > 0$, which implies the desired estimate (3.9). □

4 Sharp Decay Estimates for $\{u_t, v_t, u_{tt}, v_{tt}\}$. First we will derive the decay estimate for $L(t)$.

Theorem 4.1 *Under the assumption of Theorem 2.2, if $4\rho H(0) < \delta^2$, then it holds that*

$$L(t) \equiv \|u_t(t)\|^2 + \|v_t(t)\|^2 \le C(1+t)^{-3} \tag{4.1}$$

for $t \ge 0$.

Proof. The proof is divided in three steps.
Step 1. We will derive the boundedness of $\int_0^t X(s)/M(s)\,ds$.

Differentiating (1.1) and (1.2) once with respect to t and multiplying the resulting equations by $2u_{tt}$ and $2v_{tt}$, respectively, and integrating them over Ω, and adding the resulting equations, we have

$$\frac{d}{dt}\rho X(t) + M(t)\frac{d}{dt}Y(t) + \frac{d}{dt}\frac{1}{2}|M'(t)|^2 - 2M'(t)Y(t) + 2\delta X(t) = 0\,. \tag{4.2}$$

Moreover, multiplying (4.2) by $M(t)^{-2}$, we observe

$$\begin{aligned}\frac{d}{dt}&\left(\rho\frac{X(t)}{M(t)^2} + \frac{Y(t)}{M(t)} + \frac{1}{2}\frac{|M'(t)|^2}{M(t)^2}\right) + 2\left(\delta + \rho\frac{M'(s)}{M(s)}\right)\frac{X(t)}{M(t)^2}\\ &= \frac{M'(t)Y(t)}{M(t)^2} - \frac{|M'(t)|^2 M'(t)}{M(t)^3} \le C(1+t)^{-3/2}\end{aligned}$$

where we used the facts that

$$\frac{|M'(t)|}{M(t)} \le 2\left(\frac{Y(t)}{M(t)}\right)^{1/2} \quad \text{and} \quad \frac{Y(t)}{M(t)} \le C(1+t)^{-1}\,.$$

Thus, if $4\rho H(0) < 1$, then since it follows from (2.2) and (2.5) that

$$\delta + \rho\frac{M'(s)}{M(s)} \ge \delta - 2(\rho H(0))^{1/2} > 0$$

we have

$$\int_0^t \frac{X(s)}{M(s)^2}\,ds \le C + C\int_0^\infty (1+t)^{-3/2}\,dt \le C\,. \tag{4.3}$$

Step 2. We will derive the boundedness of $M(t)/K(t)$.

From the equations (1.1) and (1.2), it follows that

$$\begin{aligned}\frac{d}{dt}\delta\frac{M(t)}{K(t)} &= \frac{\delta}{K(t)}\left(M'(t) - \frac{M(t)}{K(t)}K'(t)\right)\\ &= \frac{-2}{K(t)}\left(\left(\Delta u + \frac{M(t)}{K(t)}\,,\,\delta u_t\right) + \left(\Delta v + \frac{M(t)}{K(t)}v\,,\,\delta v_t\right)\right)\\ &= \frac{2\rho}{K(t)}\left(\left(\Delta u + \frac{M(t)}{K(t)}u\,,\,u_{tt}\right) + \left(\Delta v + \frac{M(t)}{K(t)}v\,,\,v_{tt}\right)\right)\\ &\quad - \frac{2M(t)}{K(t)}\left(\left(\Delta u + \frac{M(t)}{K(t)}u\,,\,\Delta u\right) + \left(\Delta v + \frac{M(t)}{K(t)}v\,,\,\Delta v\right)\right)\,.\end{aligned} \tag{4.4}$$

Since we observe

$$\left(\Delta u+\frac{M(t)}{K(t)}u\,,\,\Delta u\right)=\|\Delta u+\frac{M(t)}{K(t)}u\|^2+\frac{M(t)}{K(t)}\left(\|\nabla u\|^2-\frac{M(t)}{K(t)}\|u\|^2\right),$$
$$\left(\Delta v+\frac{M(t)}{K(t)}v\,,\,\Delta v\right)=\|\Delta v+\frac{M(t)}{K(t)}v\|^2+\frac{M(t)}{K(t)}\left(\|\nabla v\|^2-\frac{M(t)}{K(t)}\|v\|^2\right),$$

and hence,

(4.5)
$$\left(\Delta u+\frac{M(t)}{K(t)}u\,,\,\Delta u\right)+\left(\Delta v+\frac{M(t)}{K(t)}v\,,\,\Delta v\right)=\|\Delta u+\frac{M(t)}{K(t)}u\|^2+\|\Delta v+\frac{M(t)}{K(t)}v\|^2,$$

we have from (4.4) and (4.5) that

$$\begin{aligned}&\frac{d}{dt}\delta\frac{M(t)}{K(t)}+2\frac{M(t)}{K(t)}\left(\|\Delta u+\frac{M(t)}{K(t)}u\|^2+\|\Delta v+\frac{M(t)}{K(t)}v\|^2\right)\\&=\frac{2\rho}{K(t)}\left(\left(\Delta u+\frac{M(t)}{K(t)}u\,,\,u_{tt}\right)+\left(\Delta v+\frac{M(t)}{K(t)}v\,,\,v_{tt}\right)\right)\\&\le\frac{2\rho}{K(t)}\left(\|\Delta u+\frac{M(t)}{K(t)}u\|^2+\|\Delta v+\frac{M(t)}{K(t)}v\|^2\right)^{1/2}X(t)^{1/2}.\end{aligned}$$

Thus, from the Young inequality we obtain

$$\frac{d}{dt}\delta\frac{M(t)}{K(t)}\le\rho^2\frac{M(t)}{K(t)}\frac{X(t)}{M(t)^2},$$

and hence, from (4.3) that

$$\frac{M(t)}{K(t)}\le\frac{M(0)}{K(0)}\exp\left(\frac{\rho^2}{\delta}\int_0^\infty\frac{X(t)}{M(t)^2}\,dt\right)\le C. \tag{4.6}$$

Step 3. We will derive the decay estimate (4.1).

From (3.10) it follows that

$$\frac{d}{dt}\rho L(t)+M(t)M'(t)+2\delta L(t)=0. \tag{4.7}$$

Multiplying (4.7) by $M(t)^{-2}$, we have

$$\frac{d}{dt}\rho\frac{L(t)}{M(t)^2}+2\left(\delta+\rho\frac{M'(s)}{M(s)}\right)\frac{L(t)}{M(t)^2}=-\frac{M'(t)}{M(t)}$$

and from (2.2) and (2.4) (or (2.6)) that

$$\frac{d}{dt}\rho\frac{L(t)}{M(t)^2}+b\frac{L(t)}{M(t)^2}\le 2\frac{L(t)^{1/2}}{M(t)}Z(t)^{1/2}$$

with $b=2(\delta-(\rho H(0))^{1/2})>0$, and from the Young inequality and (3.1) that

$$\frac{d}{dt}\rho\frac{L(t)}{M(t)^2}+\frac{b}{2}\frac{L(t)}{M(t)^2}\le CZ(t)\le C(1+t)^{-1},$$

and hence, we obtain

$$\frac{L(t)}{M(t)^2} \le C(1+t)^{-1} \tag{4.8}$$

which gives the desired estimate (4.1). □

The following generalized Nakao type inequality is useful to derive decay estimates of the solutions (see [8] for the proof).

Lemma 4.2 *Let $\phi(t)$ be a non-negative function on $[0,\infty)$ satisfying*

$$\sup_{t\le s\le t+1} \phi(s)^{1+\alpha} \le k_1(1+t)^{\beta}(\phi(t)-\phi(t+1)) + k_2(1+t)^{-\gamma}$$

with certain constants $k_1>0$, $k_2\ge 0$, $\alpha>0$, $\beta<1$, and $\gamma>0$. Then, it holds that

$$\phi(t) \le C_0(1+t)^{-\theta}\,, \qquad \theta = \min\left\{\frac{1-\beta}{\alpha}, \frac{\gamma}{1+\alpha}\right\}$$

for $t\ge 0$, where C_0 is a positive constant depending on $\phi(0)$ and other known quantities.

Next, we will derive the decay estimates of $X(t)$ and $Y(t)$.

Theorem 4.3 *Under the assumption of Theorem 4.1, it holds that*

$$X(t) \equiv \|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 \le C(1+t)^{-4}\,, \tag{4.9}$$

$$Y(t) \equiv \|\nabla u_t(t)\|^2 + \|\nabla v_t(t)\|^2 \le C(1+t)^{-3} \tag{4.10}$$

for $t\ge 0$.

Proof. Multiplying (4.2) by $M(t)^{-1}$, we have

$$\frac{d}{dt}G(t) + 2\left(\delta + \frac{\rho}{2}\frac{M'(t)}{M(t)}\right)\frac{X(t)}{M(t)} = \frac{M'(t)}{M(t)}\left(2Y(t) - \frac{1}{2}\frac{|M'(t)|^2}{M(t)}\right), \tag{4.11}$$

where we put

$$G(t) \equiv \rho\frac{X(t)}{M(t)} + Y(t) + \frac{1}{2}\frac{|M'(t)|^2}{M(t)}\,. \tag{4.12}$$

Since $|M'(t)|^2 \le 2L(t)Z(t)$ and $|M'(t)|^2 \le 2Y(t)M(t)$, we observe from (3.1) and (4.8) that

$$\text{(R.H.S) of (4.11)} \le C\left(\frac{L(t)}{M(t)^2}Z(t)\right)^{1/2} Y(t) \le C(1+t)^{-1}Y(t)\,, \tag{4.13}$$

and from (2.2) and (2.5) that

$$\frac{d}{dt}G(t) + b\frac{X(t)}{M(t)} \le C(1+t)^{-1}Y(t) \tag{4.14}$$

with $b = 2(\delta - (\rho H(0))^{1/2}) > 0$. Moreover, since $Y(t)\le C(1+t)^{-3}$, we see

$$G(t) \le C \qquad \text{for } t\ge 0\,. \tag{4.15}$$

For any $t \geq 0$, integrating (4.14) over $[t, t+1]$, we obtain

$$b \int_t^{t+1} \frac{X(s)}{M(s)}\, ds \leq G(t) - G(t+1) + C(1+t)^{-1} \sup_{t \leq s \leq t+1} Y(s) \qquad \left(\equiv bD_2(t)^2\right) . \tag{4.16}$$

Then, there exist two numbers $t_1 \in [t, t+1/4]$ and $t_2 \in [t+3/4, t+1]$ such that

$$\frac{X(t_j)}{M(t_j)} \leq 4 \int_t^{t+1} \frac{X(s)}{M(s)}\, ds \leq 4D_2(t)^2 \qquad \text{for } j = 1, 2\, . \tag{4.17}$$

Moreover, there exists $t_* \in [t_1, t_2]$ such that

$$G(t_*) \leq 2 \int_{t_1}^{t_2} G(s)\, ds\, . \tag{4.18}$$

On the other hand, differentiating (1.1) and (1.2) once with respect to t and multiplying the resulting equations by u_t and v_t, respectively, and integrating them over Ω, and adding the resulting equations, we have

$$\frac{d}{dt}\frac{\rho}{2}L'(t) - \rho X(t) + M(t)Y(t) + \frac{1}{2}|M'(t)|^2 + \frac{\delta}{2}L'(t) = 0\, . \tag{4.19}$$

Moreover, multiplying (4.19) by $M(t)^{-1}$, we observe that

$$Y(t) + \frac{1}{2}\frac{|M'(t)|^2}{M(t)} = \rho\frac{X(t)}{M(t)} - \frac{d}{dt}\frac{\rho}{2}\frac{L'(t)}{M(t)} - \frac{1}{2}\left(\delta + \rho\frac{M'(s)}{M(s)}\right)\frac{L'(t)}{M(t)}\, . \tag{4.20}$$

And integrating (4.20) over $[t_1, t_2]$, we have from (3.1), (3.2), (4.8) and (4.15)–(4.17) that

$$\begin{aligned}
&\int_{t_1}^{t_2} \left(Y(s) + \frac{1}{2}\frac{|M'(s)|^2}{M(s)}\right) ds \\
&\leq \int_{t_1}^{t_2} \rho\frac{X(s)}{M(s)}\, ds + \frac{\rho}{2}\sum_{j=1}^{2}\frac{|L'(t_j)|}{M(t_j)} + \frac{1}{2}\int_{t_1}^{t_2}\left(\delta + \rho\frac{|M'(s)|}{M(s)}\right)\frac{|L'(s)|}{M(s)}\, ds \\
&\leq \int_t^{t+1} \rho\frac{X(s)}{M(s)}\, ds + C(1+t)^{-1}\sum_{j=1}^{2}\left(\frac{X(t_j)}{M(t_j)}\right)^{1/2} + C(1+t)^{-1}\int_t^{t+1}\left(\frac{X(s)}{M(s)}\right)^{1/2} ds \\
&\leq \rho D_2(t)^2 + C(1+t)^{-1}D_2(t)\, ,
\end{aligned} \tag{4.21}$$

where we used the fact that

$$\frac{|L'(t)|}{M(t)} \leq 2\frac{(L(t)M(t))^{1/2}}{M(t)} \leq C(1+t)^{-1}\left(\frac{X(t)}{M(t)}\right)^{1/2}$$

(see (3.2) and (4.8)). Then we have from (4.12), (4.16) and (4.21) that

$$\begin{aligned}
\int_{t_1}^{t_2} G(s)\, ds &= \int_{t_1}^{t_2} \rho\frac{X(s)}{M(s)}\, ds + \int_{t_1}^{t_2}\left(Y(s) + \frac{1}{2}\frac{|M'(s)|^2}{M(s)}\right) ds \\
&\leq 2\rho D_2(t)^2 + C(1+t)^{-1}D_2(t)\, .
\end{aligned} \tag{4.22}$$

For $\tau \in [t, t+1]$, integrating (4.11) over $[\tau, t_*]$ (or $[t_*, \tau]$), we have from (4.13), (4.16), (4.18) and (4.22) that

$$\begin{aligned} G(\tau) &= G(t_*) + \int_\tau^{t_*} \left(2\left(\delta + \frac{\rho}{2}\frac{M'(s)}{M(s)}\right)\frac{X(s)}{M(s)} - \frac{M'(s)}{M(s)}\left(2Y(s) - \frac{1}{2}\frac{|M'(s)|^2}{M(s)}\right)\right) ds \\ &\le 2\int_{t_1}^{t_2} G(s)\,ds + C\int_t^{t+1} \frac{X(s)}{M(s)}\,ds + C\int_t^{t+1}(1+s)^{-1}Y(s)\,ds \\ &\le CD_2(t)^2 + C(1+t)^{-1}D_2(t) + C(1+t)^{-1}\sup_{t\le s\le t+1} Y(s)\,. \end{aligned}$$

Moreover, since $Y(t) \le G(t)$ and $Y(t) \le C(1+t)^{-2}$, it follows from (4.16) and the Young inequality that

$$\sup_{t\le s\le t+1} G(s)^2 \le C\left(G(t) + (1+t)^{-2}\right)(G(t) - G(t+1)) + C(1+t)^{-6}\,. \tag{4.23}$$

Applying Lemma 4.2 to (4.23) together with (4.15), we have

$$G(t) \le C(1+t)^{-1} \qquad \text{for } t \ge 0\,, \tag{4.24}$$

and again, applying Lemma 4.2 to (4.23) together with (4.24), we have

$$G(t) \le C(1+t)^{-2} \qquad \text{for } t \ge 0\,, \tag{4.25}$$

and hence, applying Lemma 4.2 to (4.23) together with (4.25), we obtain

$$G(t) \le C(1+t)^{-3} \qquad \text{for } t \ge 0\,,$$

which implies the desired estimates (4.9) and (4.10). □

Acknowledgment. This work was in part supported by Grant-in-Aid for Science Research (C) 21540186 of JSPS.

References

[1] Arosio, A.; Garavaldi, S. *On the mildly degenerate Kirchhoff string.* Math. Methods Appl. Sci. **14** (1991) 177–195.

[2] Arosio, A.; Panizzi, S. *On the well-posedness of the Kirchhoff string.* Trans. Amer. Math. Soc. **348** (1996) 305–330.

[3] Carrier, G.F. *On the non-linear vibration problem of the elastic string.* Quart. Appl. Math. **3** (1945) 157–165.

[4] Crippa, H.R. *On local solutions of some mildly degenerate hyperbolic equations.* Nonlinear Anal. **21** (1993) 565–574.

[5] Dickey, R.W. *Infinite systems of nonlinear oscillation equations with linear damping.* SIAM J. Appl. Math. **19** (1970) 208–214.

[6] Ghisi, M.; Gobbino, M. *Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: time-decay estimates.* J. Differential Equations **245** (2008) 2979–3007.

[7] Kirchhoff G. *Vorlesungen über Mechanik.* Teubner, Leipzig, 1883.

[8] Kawashima, S.; Nakao, M.; Ono, K. *On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term.* J. Math. Soc. Japan **47** (1995) 617–653.

[9] Mizumachi T. *Decay properties of solutions to degenerate wave equations with dissipative terms.* Adv. Differential Equations **2** (1997) 573–592.

[10] Nakao, M. *Decay of solutions of some nonlinear evolution equations.* J. Math. Anal. Appl. **60** (1977) 542–549.

[11] Nakao, M. *A difference inequality and its application to nonlinear evolution equations.* J. Math. Soc. Japan **30** (1978) 747–762.

[12] Nakao, M.; Ono, K. *Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations.* Math. Z. **214** (1993) 325–342.

[13] Nishihara, K. *Decay properties of solutions of some quasilinear hyperbolic equations with strong damping.* Nonlinear Anal. **21** (1993) 17–21.

[14] Nishihara, K.; Yamada, Y. *On global solutions of some degenerate quasilinear hyperbolic equations with dissipative terms.* Funkcial. Ekvac. **33** (1990) 151–159.

[15] Ono, K. *Global existence and decay properties of solutions for some mildly degenerate nonlinear dissipative Kirchhoff strings.* Funkcial. Ekvac. **40** (1997) 255–270.

[16] Ono, K. *Asymptotic behavior for degenerate nonlinear Kirchhoff type equations with damping.* Funkcial. Ekvac. **43** (2000) 381–393.

[17] Ono, K. *On sharp decay estimates of solutions for mildly degenerate dissipative wave equations of Kirchhoff type*. Math. Methods Appl. Sci. (in press).

[18] Strauss, W.A. *On continuity of functions with values in various Banach spaces.* Pacific J. Math. **19** (1966) 543–551.

[19] Strauss, W.A. *Nonlinear wave equations*. CBMS Regional Conf. Ser. in Math., Amer. Math. Soc., Providence, RI, 1989.

[20] Temam, R. *Infinite Dimensional Dynamical Systems in Mechanics and Physics*. Springer-Verlag, (Applied Mathematical Sciences), Vol.68, New York, 1988.

[21] Yamada, Y. *On the decay of solutions for some nonlinear evolution equations of second order.* Nagoya Math. J. **73** (1979) 67–98.

communicated by *Atsushi Yagi* ;

Department of Mathematical Sciences, The University of Tokushima,
Tokushima 770-8502, JAPAN
E-mail : ono@ias.tokushima-u.ac.jp