# ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO STOCHASTIC PHASE TRANSITION MODEL 

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#### Abstract

In this paper, we consider a stochastic adsorbate-induced phase transition model. First, we prove the existence and uniqueness of global solution. Then, we estimate the long-time expectation of the adsorbate coverage rate of the surface and its average integral as well as the behaviour of solutions. Finally, numerical examples are presented to illustrate those results.


1 Introduction In 1990, Ertl [4] and Jakubith et al. [9] found out that, in the catalytic oxidation of CO molecules on a $\mathrm{Pt}(110)$ surface, adsorbed CO molecules and O atoms form various types of spatio-temporal patterns, such as propagating wave, spiral, target, stripe and chaotic patterns. To understand the mechanism of these phenomena from macroscopic point of view, Hildebrand et al. [6, 7] presented a simple kinematic model of the surface reaction coupled to a structural phase transition of the surface, that is, in $\Omega \times(0, \infty)$

$$
\left\{\begin{array}{l}
\frac{\partial x}{\partial t}=\mu_{1} \Delta x-\mu_{2} \nabla \cdot[x(1-x) \nabla \chi(y)]-a e^{-b \chi(y)} x_{t}-c x_{t}+h\left(1-x_{t}\right),  \tag{1.1}\\
\frac{\partial y}{\partial t}=\mu_{3} \Delta y+q y_{t}\left(x_{t}+y_{t}-1\right)\left(1-y_{t}\right)
\end{array}\right.
$$

The two-dimensional domain $\Omega$ denotes a Pt surface on which the patterns are performed. The unknown functions $x(z, t)$ and $y(z, t)$ denote the adsorbate coverage rate of the surface by CO molecules and the structural state of surface at a position $z \in \Omega$ and time $t$, respectively. The nonlinear advection $-\mu_{2} \nabla \cdot[x(1-x) \nabla \chi(y)]$ shows a flow of $u$ on $\Omega$ induced by the gradient of the local chemical potential $\chi(y)$ with mobility $1-x$. A typical form of $\chi(y)$ is such that $\chi(y)=y^{2}(3-2 y)$. In the growth function of $x$, $a e^{\chi(y)}$ denotes the desorption rate of the molecules depending on $\chi(y), c$ the desorption rate by the effect of a chemical reaction, and $h$ denotes the adsorption rate. We mention here some results related to this model:

- The square $\{(x, y): 0 \leqslant x, y \leqslant 1\}$ is an invariant and attractor set [18, p.381].
- $\left(\frac{h}{a+c+h}, 0\right),\left(\frac{h}{a e^{-b}+c+h}, 1\right)$ are homogeneous stationary solutions which are both stable.

However, given that models are often subject to environmental noise, it is important to investigate whether the presence of such noise affects this result or not. In this paper, we consider a stochastic version of model (1.1). The stochastic perturbation is as follows:

$$
\begin{array}{r}
-a e^{-b y_{t}^{2}\left(3-2 y_{t}\right)} x_{t}-c x_{t}+h\left(1-x_{t}\right) \rightsquigarrow-a e^{-b y_{t}^{2}\left(3-2 y_{t}\right)} x_{t}-c x_{t}+h\left(1-x_{t}\right) \\
+" \text { white noise" } \\
q y_{t}\left(x_{t}+y_{t}-1\right)\left(1-y_{t}\right) \rightsquigarrow q y_{t}\left(x_{t}+y_{t}-1\right)\left(1-y_{t}\right)+" \text { white noise". }
\end{array}
$$

[^0]The white noise here is effected by one-dimensional Brownian motion $\left\{w_{t}, t \geqslant 0\right\}$ defined on a complete probability space with filtration $\left(\Omega, \mathcal{F},\left\{\mathcal{F}_{t}\right\}_{t \geqslant 0}, \mathbb{P}\right)$ satisfying the usual conditions (see [10]). Since the unit square $\{(x, y): 0 \leqslant x, y \leqslant 1\}$ is an invariant and attractor set for model (1.1), we assume that the white noise vanishes on the boundary of this square. Thus, this environmentally perturbed system may be described by an Itô stochastic differential equation of the form

$$
\left\{\begin{align*}
d x_{t}= & {\left[-a e^{-b y_{t}^{2}\left(3-2 y_{t}\right)} x_{t}-c x_{t}+h\left(1-x_{t}\right)\right] d t }  \tag{1.2}\\
& +\sigma_{1}\left(x_{t}\right) x_{t}\left(1-x_{t}\right) d w_{t} \\
d y_{t}= & q y_{t}\left(x_{t}+y_{t}-1\right)\left(1-y_{t}\right) d t+\sigma_{2}\left(y_{t}\right) y_{t}\left(1-y_{t}\right) d w_{t},
\end{align*}\right.
$$

where $a, b, c, h$ and $q$ are positive constants, and $\sigma_{i}(\cdot)(i=1,2)$ are bounded functions. The generator of equation (1.2) is then given by

$$
\begin{align*}
& L f(t, x, y)=\frac{\partial f}{\partial t}+\frac{1}{2} \sigma_{1}^{2}(x) x^{2}(1-x)^{2} \frac{\partial^{2} f}{\partial x^{2}}+\frac{1}{2} \sigma_{2}^{2}(y) y^{2}(1-y)^{2} \frac{\partial^{2} f}{\partial y^{2}}  \tag{1.3}\\
& \quad \begin{aligned}
+\sigma_{1}(x) \sigma_{2}(y) x y(1-x)(1-y) \frac{\partial^{2} f}{\partial x \partial y}+\left[-a e^{-b y^{2}(3-2 y)} x\right. & -c x+h(1-x)] \frac{\partial f}{\partial x} \\
& +q y(x+y-1)(1-y) \frac{\partial f}{\partial y}
\end{aligned}
\end{align*}
$$

By some physical reasons, we only consider the solution $\left(x_{t}, y_{t}\right)$ of model (1.2) with an initial condition $\left(x_{0}, y_{0}\right) \in \mathbb{S} \equiv\{(x, y): 0<x, y<1\}$ and in order that the system (1.2) describes fitly the physical phenomena we expect that $\left(x_{t}, y_{t}\right)$ are in this square for all $t>0$.

The aim of this paper is to show that (1.2) has a global solution lying in $\mathbb{S}$. Next, we estimate the long-time expectation of $x_{t}$ and its average integral in order to investigate the behaviour of $\left(x_{t}, y_{t}\right)$.

The organization of paper is as follows. Section 2 provides the main results. First, we shall verify that the square $\mathbb{S}$ is an invariant set of model (1.2), i.e., the solution $\left(x_{t}, y_{t}\right)$ starting from $\mathbb{S}$ will remain in $\mathbb{S}$ with probability 1 . Next, using comparison theorems, we shall show that

$$
\frac{h}{a+c+h} \leqslant \liminf _{t \rightarrow \infty} \mathbb{E} x_{t} \leqslant \limsup _{t \rightarrow \infty} \mathbb{E} x_{t} \leqslant \frac{h}{a e^{-b}+c+h}
$$

and that there exist $\varrho_{i}, \bar{\varrho}_{i} \in(0,1), \bar{\varrho}_{i}>\varrho_{i}(i=1,2)$ such that as $t \rightarrow \infty$ it holds true that $\frac{1}{t} \int_{0}^{t} x_{s} d s \rightarrow \frac{h}{a e^{-b}+c+h}, \frac{1}{t} \int_{0}^{t} x_{s} d s \rightarrow \frac{h}{a+c+h}, y_{t} \rightarrow 1$ and $y_{t} \rightarrow 0$ with probability $\bar{\varrho}_{1}, \bar{\varrho}_{2}, \varrho_{1}$ and $\varrho_{2}$, respectively. Further, by calculating the Lie bracket and studying the controllability of differential equations, it is proved that the system is sweeping for every compact set of $\mathbb{S}$, i.e., for any $K$ such that $\bar{K} \subset \mathbb{S}$ we have

$$
\lim _{t \rightarrow \infty} \mathbb{P}\left\{\left(x_{t}, y_{t}\right) \in K\right\}=0
$$

The last section, Section 3, illustrates the above results by some numerical examples.
2 Main results First of all, we will prove the invariance of $\mathbb{S}$ and the global existence of solution to (1.2). Indeed, the following theorem is proved.

Theorem 2.1. For any $\left(x_{0}, y_{0}\right) \in \mathbb{S}$, there exists a unique solution $\left(x_{t}, y_{t}\right)$ of (1.2) for $t \geqslant 0$. Furthermore, with probability $1, \mathbb{S}$ is positively invariant for (1.2), i.e., if $\left(x_{0}, y_{0}\right) \in \mathbb{S}$ then $\left(x_{t}, y_{t}\right) \in \mathbb{S}$ for all $t \geqslant 0$ with probability 1 .

Proof. Since the coefficients of (1.2) are locally Lipschitz continuous, for any initial value $\left(x_{0}, y_{0}\right) \in \mathbb{S}$ there is a unique solution $\left(x_{t}, y_{t}\right)$ defined on an interval $[0, \tau)$, where $\tau$ is explosive time, i.e., $\lim _{t \rightarrow \tau}\left(\left|x_{t}\right|+\left|y_{t}\right|\right)=1$ (see $[1,5]$ ). Further, when either $y=0$ or $y=1$, the coefficients of the second equation in (1.2) equal to 0 . Therefore, by the local uniqueness of the solution, we see that if $y_{0} \in(0,1)$ then $y_{t} \in(0,1)$ a.s. for $t \in[0, \tau)$. Therefore, to show the existence of a global solution and also $\mathbb{S}$ to be positively invariant, we have only to prove that if $\left(x_{0}, y_{0}\right) \in \mathbb{S}$ then $0<x_{t}<1$ for all $t>0$ (which yields that $\tau=\infty$ ) a.s. We use the technique of localization dealt with in [10]. Let $\gamma_{0}>0$ be sufficiently large for $x_{0}$ lying within the interval $\left[\frac{1}{k_{0}}, k_{0}\right]$. Denote $H_{k}=\left[\frac{1}{k}, 1-\frac{1}{k}\right] \times(0,1)$. It is seen that $\cup_{k=1}^{\infty} H_{k}=\mathbb{S}$. Let us define a sequence of stopping times [10, Problem 1.2.7] for each integer $k \geqslant k_{0}$ by

$$
\tau_{k}=\inf \left\{t>0:\left(x_{t}, y_{t}\right) \notin H_{k}\right\}
$$

(with the convention $\inf \emptyset=\infty$ ). Since $\tau_{k}$ is nondecreasing as $k \rightarrow \infty$, there exists the limit $\tau_{\infty}=\lim _{k \rightarrow \infty} \tau_{k}$. It is clear $\tau_{\infty} \leqslant \tau$ a.s.

Consider a positive function $Q(x)=-\ln x-\ln (1-x)$ on $(0,1)$. By using (1.3), we have

$$
\begin{aligned}
L Q(x, y) & =-\frac{1}{x}\left[-a e^{-b y^{2}(3-2 y)} x-c x+h(1-x)\right]+h \\
& -\frac{1}{1-x}\left[a e^{-b y^{2}(3-2 y)} x+c x\right]+\frac{1}{2} \sigma_{1}^{2}(x)\left[x^{2}+(1-x)^{2}\right] .
\end{aligned}
$$

Since $\lim _{x \downarrow 0} L Q(x, y)=-\infty$ and $\lim _{x \uparrow 1} L Q(x, y)=-\infty$, there exists a positive number $M$ such that $L Q(x, y) \leqslant M$ for any $(x, y) \in \mathbb{S}$. Therefore, by Itô's formula, for any $t>0$ it holds that

$$
\begin{equation*}
\mathbb{E} Q\left(x_{t \wedge \tau_{k}}\right)=Q\left(x_{0}\right)+\int_{0}^{t} L Q\left(x_{s \wedge \tau_{k}}, y_{s \wedge \tau_{k}}\right) d s \leqslant Q\left(x_{0}\right)+M t \tag{2.1}
\end{equation*}
$$

Suppose on the contrary that $\mathbb{P}\left\{\tau_{\infty}<\infty\right\}>0$. Then, we can find a number $T$ such that if $A=\left\{\omega: \tau_{\infty}(\omega)<T\right\}$ then $\mathbb{P}(A)>0$. Therefore, from (2.1) we have

$$
\begin{aligned}
& \mathbb{E} Q\left(x_{T \wedge \tau_{k}}\right)=\mathbb{E}\left[1_{A} Q\left(x_{T \wedge \tau_{k}}\right)\right]+\mathbb{E}\left[1_{A^{c}} Q\left(x_{T \wedge \tau_{k}}\right)\right] \\
&=\mathbb{E}\left[1_{A} Q\left(x_{\tau_{k}}\right)\right]+\mathbb{E}\left[1_{A^{c}} Q\left(x_{T \wedge \tau_{k}}\right)\right] \leqslant Q\left(x_{0}\right)+M T .
\end{aligned}
$$

On the other hand, if $\tau_{k}<\infty$ then

$$
Q\left(x_{\tau_{k}}(\omega)\right)=-\ln \frac{1}{k}-\ln \left(1-\frac{1}{k}\right)
$$

Since $Q$ is positive, it follows that

$$
\left(-\ln \frac{1}{k}-\ln \left(1-\frac{1}{k}\right)\right) \mathbb{P}(A) \leqslant Q\left(x_{0}\right)+M T
$$

Letting $k \rightarrow \infty$ we get a contradiction. Thus $\mathbb{P}\left\{\tau_{\infty}=\infty\right\}=1$ and this implies that $\mathbb{P}\{\tau=\infty\}=1$. Therefore, if $\left(x_{0}, y_{0}\right) \in \mathbb{S}$, then we have $\left(x_{t}, y_{t}\right) \in \mathbb{S}$ for any $t>0$. This means that $\left(x_{t}, y_{t}\right)$ is defined on $[0, \infty)$ and $\mathbb{S}$ is an invariant set. The proof is complete.

To study the dynamics of solutions of the system (1.2) we suppose that:
Hypothesis 2.2. $\sigma_{i}(\cdot) \in C^{1}\left(\mathbb{R}_{+}\right)(i=1,2)$ and there exist two positive constants $\varepsilon_{1}, \varepsilon_{2}$ such that $\varepsilon_{1}<\sigma_{i}(x)<\varepsilon_{2}$ and $\left|\sigma_{i}^{\prime}(x)\right|<\varepsilon_{2}$ for all $x \geqslant 0$.

Theorem 2.3. Under Hypothesis 2.2, it holds true that
(i) there exist $\varrho_{i} \in(0,1)(i=1,2)$ such that $\mathbb{P}\left[\lim _{t \rightarrow \infty} y_{t}=1\right]=\varrho_{1}$ and $\mathbb{P}\left[\lim _{t \rightarrow \infty} y_{t}=\right.$ $0]=\varrho_{2} ;$
(ii) $\frac{h}{a+c+h} \leqslant \liminf _{t \rightarrow \infty} \mathbb{E} x_{t} \leqslant \limsup _{t \rightarrow \infty} \mathbb{E} x_{t} \leqslant \frac{h}{a e^{-b}+c+h}$;
(iii) there exist $\bar{\varrho}_{i} \in\left[\varrho_{i}, 1\right)(i=1,2)$ for which it holds that $\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{s} d s=\frac{h}{a e^{-b}+c+h}$ and $\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{s} d s=\frac{h}{a+c+h}$ with probability $\bar{\varrho}_{1}, \bar{\varrho}_{2}$, respectively.

Proof. To prove (i), we define the following two stochastic differential equations:

$$
\left\{\begin{array}{l}
d \bar{y}_{t}=-q \bar{y}_{t}\left(\bar{y}_{t}-1\right)^{2} d t+\sigma_{2}\left(\bar{y}_{t}\right) \bar{y}_{t}\left(1-\bar{y}_{t}\right) d w_{t},  \tag{2.2}\\
d \overline{\bar{y}}_{t}=q \overline{\bar{y}}_{t}^{2}\left(1-\overline{\bar{y}}_{t}\right) d t+\sigma_{2}\left(\overline{\bar{y}}_{t}\right) \overline{\bar{y}}_{t}\left(1-\overline{\bar{y}}_{t}\right) d w_{t}, \\
\bar{y}_{0}=\overline{\bar{y}}_{0}=y_{0} \in(0,1)
\end{array}\right.
$$

From Theorem 2.1 and the comparison theorem (see [8, Theorem 1.1, p.352]), we see that $\bar{y}_{t} \leqslant y_{t} \leqslant \overline{\bar{y}}_{t}$ for all $t \geqslant 0$ a.s. The scale functions $s_{1}(y)$ and $s_{2}(y)$ of the stochastic differential equations in (2.2) are given, respectively, by

$$
s_{1}(y)=\int_{\frac{1}{2}}^{y} \exp \left\{-2 \int_{\frac{1}{2}}^{u} \frac{-q v(v-1)^{2}}{v^{2}(v-1)^{2} \sigma_{2}^{2}(v)} d v\right\} d u=\int_{\frac{1}{2}}^{y} \exp \left\{\int_{\frac{1}{2}}^{u} \frac{2 q}{v \sigma_{2}^{2}(v)} d v\right\} d u
$$

and

$$
s_{2}(y)=\int_{\frac{1}{2}}^{y} \exp \left\{-2 \int_{\frac{1}{2}}^{u} \frac{q v^{2}(1-v)}{v^{2}(v-1)^{2} \sigma_{2}^{2}(v)} d v\right\} d u=\int_{\frac{1}{2}}^{y} \exp \left\{\int_{\frac{1}{2}}^{u} \frac{-2 q}{(1-v) \sigma_{2}^{2}(v)} d v\right\} d u
$$

Since $\sigma_{2}(\cdot)$ is bounded above and below by positive constants, it is easily seen that

$$
A_{i}:=\lim _{y \rightarrow 1} s_{i}(y)<\infty, \quad B_{i}:=\lim _{y \rightarrow 0} s_{i}(y)>-\infty \quad(i=1,2)
$$

Therefore, by using [10, Proposition 5.22, p.345] we have

$$
\begin{aligned}
& \mathbb{P}\left[\lim _{t \rightarrow \infty} \bar{y}_{t}=0\right]=1-\mathbb{P}\left[\lim _{t \rightarrow \infty} \bar{y}_{t}=1\right]=\frac{A_{1}-s_{1}\left(y_{0}\right)}{A_{1}-B_{1}}, \\
& \mathbb{P}\left[\lim _{t \rightarrow \infty} \overline{\bar{y}}_{t}=0\right]=1-\mathbb{P}\left[\lim _{t \rightarrow \infty} \overline{\bar{y}}_{t}=1\right]=\frac{A_{2}-s_{2}\left(y_{0}\right)}{A_{2}-B_{2}} .
\end{aligned}
$$

This implies existence of $\varrho_{1} \in\left[1-\frac{A_{1}-s_{1}\left(y_{0}\right)}{A_{1}-B_{1}}, 1\right)$ and $\varrho_{2} \in\left[\frac{A_{2}-s_{2}\left(y_{0}\right)}{A_{2}-B_{2}}, 1\right)$ such that

$$
\mathbb{P}\left[\lim _{t \rightarrow \infty} y_{t}=1\right]=\varrho_{1}, \quad \mathbb{P}\left[\lim _{t \rightarrow \infty} y_{t}=0\right]=\varrho_{2}
$$

Next, we prove (ii). We consider two one-dimensional stochastic processes $\bar{x}_{t}$ and $\overline{\bar{x}}_{t}$ given by

$$
\left\{\begin{array}{l}
d \bar{x}_{t}=\left[h-(a+c+h) \bar{x}_{t}\right] d t+\sigma_{1}\left(\bar{x}_{t}\right) \bar{x}_{t}\left(1-\bar{x}_{t}\right) d w_{t}  \tag{2.3}\\
d \overline{\bar{x}}_{t}=\left[h-\left(a e^{-b}+c+h\right) \overline{\bar{x}}_{t}\right] d t+\sigma_{1}\left(\bar{x}_{t}\right) \overline{\bar{x}}_{t}\left(1-\overline{\bar{x}}_{t}\right) d w_{t} \\
\bar{x}_{0}=\overline{\bar{x}}_{0}=x_{0} \in(0,1)
\end{array}\right.
$$

Using again Theorem 2.1 and the comparison theorem as before, we get

$$
\begin{equation*}
\bar{x}_{t} \leqslant x_{t} \leqslant \overline{\bar{x}}_{t} \quad \text { for all } t \geqslant 0 \quad \text { a.s. } \tag{2.4}
\end{equation*}
$$

It follows from the first equation of (2.3) and Fubini's theorem that

$$
\left\{\begin{array}{l}
d \mathbb{E} \bar{x}_{t}=\left[h-(a+c+h) \mathbb{E} \bar{x}_{t}\right] d t \\
\mathbb{E} \bar{x}_{0}=x_{0}
\end{array}\right.
$$

It is easy to see that

$$
\mathbb{E} \bar{x}_{t}=\left(x_{0}-\frac{h}{a+c+h}\right) e^{-(a+c+h) t}+\frac{h}{a+c+h} .
$$

Therefore, $\lim _{t \rightarrow \infty} \mathbb{E} \bar{x}_{t}=\frac{h}{a+c+h}$. Similarly, we have $\lim _{t \rightarrow \infty} \mathbb{E} \overline{\bar{x}}_{t}=\frac{h}{a e^{-b}+c+h}$. Hence, by using (2.4) we obtain

$$
\frac{h}{a+c+h} \leqslant \liminf _{t \rightarrow \infty} \mathbb{E} x_{t} \leqslant \limsup _{t \rightarrow \infty} \mathbb{E} x_{t} \leqslant \frac{h}{a e^{-b}+c+h}
$$

We now prove (iii). It follows from the first equation of (1.2) that

$$
\begin{align*}
\frac{x_{t}}{t}=\frac{x_{0}}{t}+\frac{1}{t} \int_{0}^{t}\left[-a e^{-b y_{s}^{2}\left(3-2 y_{s}\right)} x_{s}-c x_{s}+h\left(1-x_{s}\right)\right] d s &  \tag{2.5}\\
& +\frac{1}{t} \int_{0}^{t} \sigma_{1}\left(x_{s}\right) x_{s}\left(1-x_{s}\right) d w_{s}
\end{align*}
$$

Since $0<x_{t}<1$, then $\lim _{t \rightarrow \infty} \frac{x_{t}}{t}=0$ a.s. Furthermore, by putting $M_{t}=\int_{0}^{t} \sigma_{1}\left(x_{s}\right) x_{s}(1-$ $\left.x_{s}\right) d w_{s}$ and by Hypothesis $2.2,\left\{M_{t}, \mathcal{F}_{t}, t \geqslant 0\right\}$ is a continuous martingale with the variation $<M>_{t}=\int_{0}^{t} \sigma_{1}^{2}\left(x_{s}\right) x_{s}^{2}\left(1-x_{s}\right)^{2} d s$ satisfying

$$
<M>_{t}<\epsilon_{2}^{2} t / 16, \quad \text { a.s. }
$$

Using the corollary to the theorem of time-change for martingales [10, Theorem 4.6, p.174], there exists a Brownian motion $\left(B_{t}, \mathcal{G}_{t}\right)$ on $(\Omega, \mathcal{F}, P)$ such that $M_{t}=B_{<M>_{t}}, t \geqslant 0$. For $P$-a.s. $\omega \in \Omega$,

- If $<M>_{\infty}(\omega):=\lim _{t \rightarrow \infty}<M>_{t}(\omega)<\infty$, then

$$
\lim _{t \rightarrow \infty} \frac{M_{t}(\omega)}{t}=\lim _{t \rightarrow \infty} \frac{B_{<M>_{t}(\omega)}(\omega)}{t}=\lim _{t \rightarrow \infty} \frac{B_{<M>_{\infty}(\omega)}(\omega)}{t}=0
$$

- If $<M>_{\infty}(\omega)=\infty$, then from

$$
\left|\frac{M_{t}(\omega)}{t}\right| \leqslant\left|\frac{\epsilon_{2}^{2} M_{t}(\omega)}{16<M>_{t}(\omega)}\right|
$$

and from the strong law of large numbers for Brownian motion (see [10, p.104]) we have

$$
\lim _{t \rightarrow \infty}\left|\frac{M_{t}(\omega)}{t}\right| \leqslant \lim _{t \rightarrow \infty}\left|\frac{\epsilon_{2}^{2} B_{<M>_{t}(\omega)}(\omega)}{16<M>_{t}(\omega)}\right|=\frac{\epsilon_{2}^{2}}{16} \lim _{s \rightarrow \infty}\left|\frac{B_{s}(\omega)}{s}\right|=0
$$

Therefore,

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \sigma_{1}\left(x_{s}\right) x_{s}\left(1-x_{s}\right) d w_{s}=0 \tag{2.6}
\end{equation*}
$$

Combining (2.5) and (2.6) gives

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t}\left[a e^{-b y_{s}^{2}\left(3-2 y_{s}\right)}+c+h\right] x_{s} d s=h \tag{2.7}
\end{equation*}
$$

Denote $A=\left\{\omega: \lim _{t \rightarrow \infty} y_{t}(\omega)=1\right\}$, we then have from (i) that $\mathbb{P}(A)=\varrho_{1}$. For every $\omega \in A$ and $\varepsilon \in(0,1)$, there exists $T=T(\varepsilon, \omega)>0$ such that $y_{t}(\omega)>1-\varepsilon$. Then, for all $s \geqslant T$ we have

$$
\begin{equation*}
a e^{-b} \leqslant a e^{-b y_{s}^{2}(\omega)\left(3-2 y_{s}(\omega)\right)} \leqslant a e^{-b(1-\varepsilon)^{2}(1+2 \varepsilon)} \tag{2.8}
\end{equation*}
$$

Because of $\left(x_{t}, y_{t}\right) \in \mathbb{S}$, we see from (2.7) and (2.8) that on the set $A$,

$$
\left(a e^{-b}+c+h\right) \limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{s} d s \leqslant h \leqslant\left(a e^{-b(1-\varepsilon)^{2}(1+2 \varepsilon)}+c+h\right) \liminf _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{s} d s
$$

Letting $\varepsilon \rightarrow 0$ gives $\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{s} d s=\frac{h}{a e^{-b}+c+h}$ on $A$. It means that there exists $\bar{\varrho}_{1} \in$ $\left[\varrho_{1}, 1\right)$ such that $\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{s} d s=\frac{h}{a e^{-b}+c+h}$ with probability $\bar{\varrho}_{1}$.

Similarly, there exists $\bar{\varrho}_{2} \in\left[\varrho_{2}, 1\right)$ such that

$$
\lim _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} x_{s} d s=\frac{h}{a+c+h}
$$

with probability $\bar{\varrho}_{2}$. Hence, the proof is complete.
Since Theorem 2.1 implies $\left(x_{t}, y_{t}\right) \in \mathbb{S}$, we can use the transformation $x_{t}=\frac{e^{\xi_{t}}}{1+e^{\xi_{t}}}, y_{t}=$ $\frac{e^{\eta_{t}}}{1+e^{\eta_{t}}}(t \geqslant 0)$. Then, by Itô's formula, the system (1.2) is rewritten into

$$
\left\{\begin{array}{l}
d \xi_{t}=P_{1}\left(\xi_{t}, \eta_{t}\right) d t+\sigma_{1}\left(\frac{e^{\xi_{t}}}{1+\varepsilon_{t}}\right) d w_{t}  \tag{2.9}\\
d \eta_{t}=P_{2}\left(\xi_{t}, \eta_{t}\right) d t+\sigma_{2}\left(\frac{e^{\eta t}}{1+e^{\eta_{t}}}\right) d w_{t}
\end{array}\right.
$$

where

$$
\begin{aligned}
P_{1}(\xi, \eta)= & \left(1+e^{\xi}\right)\left(-a \exp \left\{\frac{-b e^{2 \eta}\left(e^{\eta}+3\right)}{\left(e^{\eta}+1\right)^{3}}\right\}-c+\frac{h}{e^{\xi}}\right) \\
& +\frac{e^{\xi}-1}{2\left(1+e^{\xi}\right)} \sigma_{1}^{2}\left(\frac{e^{\xi}}{1+e^{\xi}}\right), \\
P_{2}(\xi, \eta)= & q\left(\frac{e^{\xi}}{1+e^{\xi}}+\frac{e^{\eta}}{1+e^{\eta}}-1\right)+\frac{e^{\eta}-1}{2\left(1+e^{\eta}\right)} \sigma_{2}^{2}\left(\frac{e^{\eta}}{1+e^{\eta}}\right) .
\end{aligned}
$$

It is seen that $x_{t}$ and $y_{t}$ tend to 0 or 1 if and only if $\xi_{t}$ and $\eta_{t}$ tend to $-\infty$ or $+\infty$, respectively.

To study the positivity of density of transition probability function to (2.9) we put

$$
\rho_{i}(x)=\int_{0}^{x} \frac{d u}{\sigma_{i}\left(\frac{e^{u}}{1+e^{u}}\right)}(i=1,2), \alpha_{t}=\rho_{1}\left(\xi_{t}\right), \beta_{t}=\rho_{2}\left(\eta_{t}\right), \gamma_{t}=\alpha_{t}-\beta_{t}
$$

By using Itô's formula, we obtain from (2.9) that

$$
\left\{\begin{array}{l}
d \beta_{t}=g_{1}\left(\beta_{t}, \gamma_{t}\right) d t+d w_{t}  \tag{2.10}\\
d \gamma_{t}=g_{2}\left(\beta_{t}, \gamma_{t}\right) d t
\end{array}\right.
$$

where

$$
\begin{equation*}
g_{1}(\beta, \gamma)=\frac{P_{2}\left(\rho_{1}^{-1}(\beta+\gamma), \rho_{2}^{-1}(\beta)\right)}{\sigma_{2}\left(\frac{e^{\rho_{2}^{-1}(\beta)}}{1+e^{\rho_{2}^{-1}(\beta)}}\right)}-\frac{1}{2} \frac{e^{\rho_{2}^{-1}(\beta)}}{\left[1+e^{\rho_{2}^{-1}(\beta)}\right]^{2}} \sigma_{2}^{\prime}\left(\frac{e^{\rho_{2}^{-1}(\beta)}}{1+e^{\rho_{2}^{-1}(\beta)}}\right) \tag{2.11}
\end{equation*}
$$

and

$$
\begin{align*}
& g_{2}(\beta, \gamma)=\frac{P_{1}\left(\rho_{1}^{-1}(\beta+\gamma), \rho_{2}^{-1}(\beta)\right)}{\sigma_{1}\left(\frac{e^{\rho_{1}^{-1}(\beta+\gamma)}}{1+e^{\rho_{1}^{-1}(\beta+\gamma)}}\right)}  \tag{2.12}\\
&-\frac{1}{2} \frac{e^{\rho_{1}^{-1}(\beta+\gamma)}}{\left[1+e^{\left.\rho_{1}^{-1}(\beta+\gamma)\right]^{2}}\right.} \sigma_{1}^{\prime}\left(\frac{e^{\rho_{1}^{-1}(\beta+\gamma)}}{1+e^{\rho_{1}^{-1}(\beta+\gamma)}}\right)-g_{1}(\beta, \gamma)
\end{align*}
$$

Here $\rho_{i}^{-1}(\cdot)$ denotes the inverse of $\rho_{i}(\cdot)(i=1,2)$. It is known that the density $u(t, \xi, \eta)$ of the random variable $\left(\xi_{t}, \eta_{t}\right)$, if it exists and is smooth, can be found from the following Fokker-Planck equation

$$
\begin{align*}
\frac{\partial u(t, \xi, \eta)}{\partial t}=\frac{1}{2} \frac{\partial^{2}}{\partial \xi^{2}}\left[\sigma_{1}^{2}\left(\frac{e^{\xi}}{1+e^{\xi}}\right) u\right] & +\frac{1}{2} \frac{\partial^{2}}{\partial \eta^{2}}\left[\sigma_{2}^{2}\left(\frac{e^{\eta}}{1+e^{\eta}}\right) u\right]  \tag{2.13}\\
& \quad+\frac{\partial^{2}}{\partial \xi \partial \eta}\left[\sigma_{1}\left(\frac{e^{\xi}}{1+e^{\xi}}\right) \sigma_{2}\left(\frac{e^{\eta}}{1+e^{\eta}}\right) u\right]-\frac{\partial\left(P_{1} u\right)}{\partial \xi}-\frac{\partial\left(P_{2} u\right)}{\partial \eta}
\end{align*}
$$

For convenience of arguments, we will study the long-time behaviour of the density of solution $\left(\xi_{t}, \eta_{t}\right)$ of the system (2.9) or of the system (2.10) instead of considering the system (1.2). Let us first survey some basic results on Markov semigroups.

Let the triple $(X, \Sigma, \mathbf{m})$ be a $\sigma$-finite measure space. Denote by $\mathcal{D}$ a subset of $L^{1}=$ $L^{1}(X, \Sigma, \mathbf{m})$, consisting of the densities, i.e., $\mathcal{D}=\left\{f \in L^{1}: f \geqslant 0,\|f\|=1\right\}$.
Definition 2.4. Let $P$ be a linear mapping from $L^{1}$ to itself and let $\{P(t)\}_{t \geqslant 0}$ denote a semigroup of linear operators on $L^{1}$.
(i) $P$ is called a Markov operator if $P(\mathcal{D}) \subset \mathcal{D}$;
(ii) A Markov operator $P$ is called a kernel operator if there exists a measurable function $k: X \times X \rightarrow[0, \infty)$ such that $P f(x)=\int_{X} k(x, y) f(y) \mathbf{m}(d y)$ for every density $f \in \mathcal{D}$.
(iii) Semigroup $\{P(t)\}_{t \geqslant 0}$ is a Markov semigroup if $P(t)$ is a Markov operator for every $t \geqslant 0$;
(iv) Semigroup $\{P(t)\}_{t \geqslant 0}$ is integral if $P(t)$ is an integral Markov operator for every $t \geqslant 0$.
(v) Semigroup $\{P(t)\}_{t \geqslant 0}$ is asymptotically stable if there is an invariant density $f_{*}$ such that $\lim _{t \rightarrow \infty}\left\|P(t) f-f_{*}\right\|=0$ for $f \in \mathcal{D}$ (a density $f_{*}$ is said to be invariant under the semigroup $\{P(t)\}_{t \geqslant 0}$ if $P(t) f_{*}=f_{*}$ for each $\left.t \geqslant 0\right)$.
(vi) Makov semigroup $\{P(t)\}_{t \geqslant 0}$ is said to be sweeping with respect to a set $K \in \Sigma$ if for all $f \in \mathcal{D}$,

$$
\lim _{t \rightarrow \infty} \int_{K} P(t) f(x) \mathbf{m}(d x)=0
$$

If a semigroup is either asymptotically stable or sweeping with respect to compact sets then we say that the semigroup has the Foguel alternative. Then, the following results are known. For the proof, see [14].

Theorem 2.5. Let $X$ be a metric space and let $\Sigma$ be the its Borel $\sigma$-field. Let $\{P(t)\}_{t \geqslant 0}$ be an integral Markov semigroup with a continuous kernel $k(t, x, y)$. Assume that for every $f \in \mathcal{D}$,

$$
\int_{0}^{\infty} P(t) f d t>0, \quad \text { a.e. }
$$

Then, the semigroup enjoys the Foguel alternative.
We now consider the space $\left(\mathbb{R}^{2}, \mathcal{B}\left(\mathbb{R}^{2}\right), \mathbf{m}\right)$, where $\mathcal{B}\left(\mathbb{R}^{2}\right)$ is the Borel $\sigma$ - field of $\mathbb{R}^{2}$ and $\mathbf{m}$ is the Lebesgue measure on $\left(\mathbb{R}^{2}, \mathcal{B}\left(\mathbb{R}^{2}\right)\right)$. We can then construct a semigroup $\{P(t)\}_{t \geqslant 0}$ generated by the system (2.9) and can prove that it has the Foguel alternative. To do this, we add a condition to Hypothesis 2.2 and assume that the following hypothesis holds throughout the remaining of the paper.

Hypothesis 2.6. The functions $\sigma_{i}(\cdot)(i=1,2)$ satisfy Hypothesis 2.2 and are analytic.
Lemma 2.7. Let $\left(\xi_{t}, \eta_{t}\right)$ be a solution of (2.9) with initial condition $\left(\xi_{0}, \eta_{0}\right) \in \mathbb{R}^{2}$. Then, the transition probability function $\mathcal{P}\left(t, \xi_{0}, \eta_{0}, \cdot\right)$ of the Markov diffusion process $\left(\xi_{t}, \eta_{t}\right)$, i.e., $\mathcal{P}\left(t, \xi_{0}, \eta_{0}, K\right)=\mathbb{P}\left\{\left(\xi_{t}, \eta_{t}\right) \in K\right\}$ for $K \in \mathcal{B}\left(\mathbb{R}^{2}\right)$, has a density $k\left(t, \xi, \eta, \xi_{0}, \eta_{0}\right)$ with respect to $\mathbf{m}$ with the regularity $k \in C^{\infty}\left([0, \infty) \times \mathbb{R}^{2} \times \mathbb{R}^{2}\right)$.

Proof. Since $\rho_{i}(x)$ is a strictly increasing function, it suffices to show that the transition probability function of solution $\left(\beta_{t}, \gamma_{t}\right)$ of (2.10) has a density, say by $\bar{k}\left(t, \beta, \gamma, \beta^{0}, \gamma^{0}\right)$ satisfying $\bar{k} \in C^{\infty}\left((0, \infty) \times \mathbb{R}^{2} \times \mathbb{R}^{2}\right)$. We here apply the Hörmander theorem on existence of smooth densities of the transition probability function for the degenerate diffusion processes described by (2.10) (see Nualart [13, Theorem 7.4], Malliavin [11] and Norris [12]).

When $X(x)=\left(X_{1}, \cdots, X_{d}\right)$ and $Y(x)=\left(Y_{1}, \cdots, Y_{d}\right)$ are a vector field on $\mathbb{R}^{d}$, the Lie bracket $[X, Y]$ is defined by a vector field

$$
[X, Y]_{j}(x)=\sum_{k=1}^{d}\left[X_{k} \frac{\partial Y_{j}}{\partial x_{k}}(x)-Y_{k} \frac{\partial X_{j}}{\partial x_{k}}(x)\right] \quad(j=1, \cdots, d)
$$

Put

$$
\begin{aligned}
& a_{0}(\beta, \gamma)=\left[\begin{array}{l}
g_{1}(\beta, \gamma) \\
g_{2}(\beta, \gamma)
\end{array}\right], \quad a_{1}=\left[\begin{array}{l}
1 \\
0
\end{array}\right], \\
& a_{2}(\beta, \gamma)=\left[a_{0}, a_{1}\right], \quad a_{i+1}(\beta, \gamma)=\left[a_{i}, a_{1}\right] \quad(i \geqslant 2) .
\end{aligned}
$$

Then,

$$
a_{i}=\left[\begin{array}{l}
(-1)^{i-1} \frac{\partial^{i-1} g_{1}}{\partial \beta^{i-1}} \\
(-1)^{i-1} \frac{\partial^{i-1} g_{2}}{\partial \beta^{i-1}}
\end{array}\right] \quad(i \geqslant 2) .
$$

Denote by $\mathbb{R}^{2}(\beta, \gamma)$ the vector space generated by vectors $\left\{a_{i}(\beta, \gamma)\right\}_{i \geqslant 1}$. If there exists a point $\left(\beta^{0}, \gamma^{0}\right) \in \mathbb{R}^{2}$ such that $\mathbb{R}^{2}\left(\beta^{0}, \gamma^{0}\right) \neq \mathbb{R}^{2}$, then the vectors $a_{i}(i \geqslant 1)$ and $a_{1}$ are parallel. Hence,

$$
\frac{\partial^{i-1} g_{2}}{\partial \beta^{i-1}}\left(\beta^{0}, \gamma^{0}\right)=0, \quad \text { for all } i \geqslant 2
$$

It then follows that $g_{2}\left(\beta, \gamma^{0}\right)$ is constant for $\beta \in \mathbb{R}$.

On the other hand, from the fact that $\lim _{\beta \rightarrow \infty} \rho_{1}^{-1}\left(\beta+\gamma_{0}\right)=\lim _{\beta \rightarrow \infty} \rho_{2}^{-1}(\beta)=\infty$ we have $\lim _{\beta \rightarrow \infty} g_{2}\left(\beta, \gamma_{0}\right)=-\infty$. This is contradiction to the above result. Therefore, we obtain the Hörmander condition:
(H) For every $(\beta, \gamma) \in \mathbb{R}^{2}$, the vectors $a_{i}(\beta, \gamma)(i \geqslant 1)$ span the space $\mathbb{R}^{2}$.

Under the condition (H), the transition probability function $\mathcal{P}\left(t, \beta^{0}, \gamma^{0}, \cdot\right)$ of system (2.10) has a density $\bar{k}\left(t, \beta, \gamma, \beta^{0}, \gamma^{0}\right)$ with regularity $\bar{k} \in C^{\infty}\left((0, \infty) \times \mathbb{R}^{2} \times \mathbb{R}^{2}\right)$. Thus, the lemma is proved.

We now turn to study the solution $\left(\xi_{t}, \eta_{t}\right)$ of $(2.9)$ with an initial random variable $\left(\xi_{0}, \eta_{0}\right)$ whose distribution is absolutely continuous with the density $v(\xi, \eta)$. From Lemma 2.7, for any $t>0,\left(\xi_{t}, \eta_{t}\right)$ has a density $u(t, \xi, \eta)$ satisfying the Fokker-Planck equation (2.13). Further,

$$
u(t, \xi, \eta)=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k\left(t, \xi, \eta, \varsigma_{1}, \varsigma_{2}\right) v\left(\varsigma_{1}, \varsigma_{2}\right) d \varsigma_{1} d \varsigma_{2}
$$

As in [14] we see that system (2.9) generates a Markov semigroup. For any $t \geqslant 0$, we define the operator $P(t): \mathcal{D} \rightarrow \mathcal{D}$ by

$$
P(t) v(\xi, \eta)=u(t, \xi, \eta)=\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k\left(t, \xi, \eta, \varsigma_{1}, \varsigma_{2}\right) v\left(\varsigma_{1}, \varsigma_{2}\right) d \varsigma_{1} d \varsigma_{2}
$$

Using Lemma 2.7 and an analytic prolongation, we know that the family of operators $\{P(t)\}_{t \geqslant 0}$ on $L^{1}\left(\mathbb{R}^{2}, \mathcal{B}\left(\mathbb{R}^{2}\right), \mathbf{m}\right)$ becomes an integral Markov semigroup with a continuous kernel $k$. The positivity of $k$ is equivalent to that of $\bar{k}$ which is defined in the proof of Lemma 2.7. To verify this positivity we use the method based on support theorems (see Aida et al. [2], Ben Arous and Leandre [3], Stroock and Varadhan [15]). Fix a point $\left(\beta_{0}, \gamma_{0}\right) \in \mathbb{R}^{2}$ and a function $\phi \in L^{2}([0, T] ; \mathbb{R}), T>0$. Consider the following system of differential equations:

$$
\left\{\begin{array}{l}
\beta_{\phi}(t)=\beta_{0}+\int_{0}^{t}\left[\phi(s)+g_{1}\left(\beta_{\phi}(s), \gamma_{\phi}(s)\right)\right] d s  \tag{2.14}\\
\gamma_{\phi}(t)=\gamma_{0}+\int_{0}^{t} g_{2}\left(\beta_{\phi}(s), \gamma_{\phi}(s)\right) d s
\end{array}\right.
$$

Let $\left(\beta_{\phi}, \gamma_{\phi}\right)$ be a solution of (2.14) with the initial condition $\beta_{\phi}(0)=\beta_{0}, \gamma_{\phi}(0)=\gamma_{0}$ and let $F: L^{2}([0, T], \mathbb{R}) \rightarrow \mathbb{R}^{2}$ be a mapping defined by $F(\psi)=\left(\beta_{\phi+\psi}(T), \gamma_{\phi+\psi}(T)\right)$. Denote by $D_{\beta_{0}, \gamma_{0}, \phi}$ the Fréchet derivative of $F$. It is known that if $\operatorname{Rank}\left(D_{\beta_{0}, \gamma_{0}, \phi}\right)=2$ then $\bar{k}\left(T, \bar{\beta}, \bar{\gamma}, \beta_{0}, \gamma_{0}\right)>0$ where $(\bar{\beta}, \bar{\gamma})=\left(\beta_{\phi}(T), \gamma_{\phi}(T)\right)$. On the space $C([0, T], \mathbb{R})$ of all continuous functions from $[0, T]$ to $\mathbb{R}, D_{\beta_{0}, \gamma_{0}, \phi}$ can be calculated by means of the perturbation method for ordinary differential equations as follows. Let $\mathbf{f}=\left(g_{1}, g_{2}\right)$ and $\Lambda(t)=\mathbf{f}^{\prime}\left(\beta_{\phi}(t), \gamma_{\phi}(t)\right)$. Denote by $R(t, s), 0 \leqslant s \leqslant t \leqslant T$, the fundamental matrix of solutions of the equation

$$
\dot{Z}=\Lambda(t) Z
$$

i.e., $\frac{\partial R(t, s)}{\partial t}=\Lambda(t) R(t, s)$ and $R(s, s)=I$. Then,

$$
D_{\beta_{0}, \gamma_{0}, \phi} \psi=\int_{0}^{T} R(T, s) a_{1} \psi(s) d s
$$

where $a_{1}=\left[\begin{array}{l}1 \\ 0\end{array}\right]$. Let $\epsilon \in(0, T)$ and put $\psi(t)=0$ if $0 \leqslant t \leqslant T-\epsilon$ and $\psi(t)=\frac{1}{\epsilon}(t-T+\epsilon)$ if $T-\epsilon \leqslant t \leqslant T$. By the Taylor formula we have $R(T, s)=I-\Lambda(T)(T-s)+o(T-s)$ as
$s \rightarrow T$. Thus,

$$
\begin{aligned}
D_{\beta_{0}, \gamma_{0}, \phi} \psi & =\int_{T-\epsilon}^{T} \frac{s-T+\epsilon}{\epsilon} a_{1} d s+\Lambda(T) \int_{T-\epsilon}^{T} \frac{s-T+\epsilon}{\epsilon}(s-T) a_{1} d s+o\left(\epsilon^{2}\right) \\
& =\frac{\epsilon}{2} a_{1}-\frac{\epsilon^{2}}{6} \Lambda(T) a_{1}+o\left(\epsilon^{2}\right)
\end{aligned}
$$

Since $\Lambda(t)=\left[\begin{array}{ll}\frac{\partial g_{1}}{\partial \beta} & \frac{\partial g_{1}}{\partial \gamma} \\ \frac{\partial g_{2}}{\partial \beta} & \frac{\partial g_{2}}{\partial \gamma}\end{array}\right]$, it follows that $\Lambda(T) a_{1}=\left[\begin{array}{c}\frac{\partial g_{1}(\bar{\beta}, \bar{\gamma})}{\partial \beta} \\ \frac{\partial g_{2}(\bar{\beta}, \bar{\gamma})}{\partial \beta}\end{array}\right]$ where $\bar{\beta}=\beta_{\phi}(T), \bar{\gamma}=\gamma_{\phi}(T)$.
If the vectors $a_{1}$ and $\Lambda(T) a_{1}$ are linearly independent, then $\operatorname{Rank}\left(D_{x_{0}, y_{0}, \phi}\right)=2$. In the case where $a_{1}$ and $\Lambda(T) a_{1}$ are linearly dependent, $(\bar{\beta}, \bar{\gamma})$ is a solution to the following equation:

$$
\begin{equation*}
\frac{\partial g_{2}(\bar{\beta}, \bar{\gamma})}{\partial \beta}=0 \tag{2.15}
\end{equation*}
$$

Denote by $G$ the set of all solutions of (2.15), then $\mathbf{m}(G)=0$. Indeed, put $g(\beta, \gamma)=\frac{\partial g_{2}}{\partial \beta}$, $g_{i}(\beta, \gamma)=\frac{\partial^{i} g(\beta, \gamma)}{\partial \beta^{i} \partial \gamma^{i}{ }^{2}}$, where $i=\left(i_{1}, i_{2}\right)$ is a two-dimensional multi-index and denote $S_{i}=$ $\left\{(\beta, \gamma): g_{i} \neq 0\right\}$, then $S_{i}$ is open set. Given a two-dimensional multi-index $i$, we assume without loss of generality that

$$
g_{i}(\beta, \gamma)=\frac{\partial g_{i^{*}}(\beta, \gamma)}{\partial \beta}
$$

where $i^{*}$ is other two-dimensional multi-index with $\left|i^{*}\right|<|i|$. Since $\frac{\partial g_{i^{*}}(\beta, \gamma)}{\partial \beta} \neq 0$ on $S_{i}$, it follows from the implicit function theorem that

$$
g_{i^{*}}(\beta, \gamma) \neq 0, \quad \mathbf{m} \text { - a.e. }(\beta, \gamma) \text { on } S_{i} .
$$

Denote by $S_{i^{*}}^{1}$ the zero set of $g_{i^{*}}$ in $S_{i}$, then

- $\mathbf{m}\left(S_{i^{*}}^{1}\right)=0$;
- $S_{i} \backslash S_{i^{*}}^{1}$ is open and on which $g_{i^{*}} \neq 0$.

We now consider the function $g_{i^{*}}$ on $S_{i} \backslash S_{i^{*}}^{1}$. Iteration of the above process leads to $g(\beta, \gamma) \neq 0, \mathbf{m}$ - a.e. $(\beta, \gamma)$ on $S_{i}$. Thus,

$$
g(\beta, \gamma) \neq 0, \quad \mathbf{m} \text {-a.e. }(\beta, \gamma) \text { on } \cup_{i} S_{i} .
$$

If there exists $\left(\beta_{0}, \gamma_{0}\right) \in \mathbb{R}^{2} \backslash \cup_{i} S_{i}$, then $g_{i}\left(\beta_{0}, \gamma_{0}\right)=0$ for all two-dimensional multi-index $i$. It then follows that $g(\beta, \gamma)$ is constant on $\mathbb{R}^{2}$. Hence, $g_{2}(\beta, \gamma)$ has the following form on $\mathbb{R}^{2}$ :

$$
g_{2}(\beta, \gamma)=c \beta+\bar{g}(\gamma), \quad c \in \mathbb{R}
$$

This leads to a contradiction, because $g_{2}(\beta, \gamma)$ tends to $-\infty$ for every $\gamma$ when $\beta$ tends to $\infty$ or $-\infty$. Therefore, $\mathbb{R}^{2} \backslash \cup_{i} S_{i}=\emptyset$ and then $\mathbf{m}(G)=0$.

Summing up these results yields
Lemma 2.8. $\bar{k}\left(T, \bar{\beta}, \bar{\gamma}, \beta_{0}, \gamma_{0}\right)>0$ if $(\bar{\beta}, \bar{\gamma})=\left(\beta_{\phi}(T), \gamma_{\phi}(T)\right) \notin G$.

We are now in a position to consider the controllability of system (2.14) in the space $C([0, T], \mathbb{R})$. System (2.14) becomes

$$
\left\{\begin{array}{l}
\beta_{\phi}^{\prime}(t)=\phi(t)+g_{1}\left(\beta_{\phi}(t), \gamma_{\phi}(t)\right)  \tag{2.16}\\
\gamma_{\phi}^{\prime}(t)=g_{2}\left(\beta_{\phi}(t), \gamma_{\phi}(t)\right)
\end{array}\right.
$$

We then have the following claims.
Claim 1. Let $\gamma_{0}>\gamma_{1}$. Since $\lim _{\beta \rightarrow \infty} \rho_{1}^{-1}\left(\beta+\gamma_{0}\right)=\lim _{\beta \rightarrow \infty} \rho_{2}^{-1}(\beta)=\infty$, it is easy to see that $\lim _{\beta \rightarrow \infty} g_{2}(\beta, \gamma)=-\infty$ uniformly in $\gamma \in\left[\gamma_{1}, \gamma_{0}\right]$. Then, there exists $\beta_{0}$ such that $g_{2}\left(\beta_{0}, \gamma\right) \leqslant-1$ for any $\gamma \in\left[\gamma_{1}, \gamma_{0}\right]$. Choose $\phi(t)=-g_{1}\left(\beta_{0}, \gamma(t)\right)$, where $\gamma(t)$ is a solution of the following equation

$$
\left\{\begin{array}{l}
\gamma^{\prime}(t)=g_{2}\left(\beta_{0}, \gamma(t)\right) \\
\gamma(0)=\gamma_{0}
\end{array}\right.
$$

Then, system (2.16) has a solution $\left(\beta_{\phi}(t), \gamma_{\phi}(t)\right)=\left(\beta_{0}, \gamma(t)\right)$ and $\gamma_{\phi}(0)=\gamma_{0}$. Since $\gamma_{\phi}^{\prime}=$ $g_{2}\left(\beta_{0}, \gamma_{\phi}\right) \leqslant-1$ whenever $\gamma_{\phi} \in\left[\gamma_{1}, \gamma_{0}\right]$, we can find $T>0$ such that $\gamma_{\phi}(T)=\gamma_{1}$.
Claim 2. Let $\gamma_{0}<\gamma_{1}$. Since $\lim _{\beta \rightarrow-\infty} \rho_{1}^{-1}\left(\beta+\gamma_{0}\right)=\lim _{\beta \rightarrow-\infty} \rho_{2}^{-1}(\beta)=-\infty$, it is easy to see that $\lim _{\beta \rightarrow-\infty} g_{2}(\beta, \gamma)=\infty$ uniformly in $\gamma \in\left[\gamma_{0}, \gamma_{1}\right]$. Similarly to Claim 1, there exist $\beta_{0}$, a function $\phi(t)$ and $T>0$ such that $(2.16)$ has a solution $\left(\beta_{\phi}(t), \gamma_{\phi}(t)\right)=$ $\left(\beta_{0}, \gamma(t)\right), \gamma_{\phi}(0)=\gamma_{0}$ and $\gamma_{\phi}(T)=\gamma_{1}$.
Claim 3. Fix $y_{0} \in \mathbb{R}, L>0, A_{1}>A_{0}$ and $\epsilon>0$ such that $\epsilon<\min \left\{\frac{L}{4}, \frac{A_{1}-A_{0}}{4}\right\}$. Let

$$
m^{*}=\max \left\{\left|g_{1}(\beta, \gamma)\right|+\left|g_{2}(\beta, \gamma)\right|:(\beta, \gamma) \in\left[\beta_{0}, \beta_{0}+L\right] \times\left[A_{0}, A_{1}\right]\right\}
$$

$t_{0}=\frac{\epsilon}{m^{*}}$ and $\phi=\frac{3 m^{*} L}{4 \epsilon}$. For every $\gamma_{0} \in\left[A_{0}+\epsilon, A_{1}-\epsilon\right]$ the solution of system (2.16) with $\left(\beta_{\phi}(0), \gamma_{\phi}(0)\right)=\left(\beta_{0}, \gamma_{0}\right)$ satisfies

$$
\left\{\begin{array}{l}
\beta_{\phi}\left(t_{0}\right) \in\left(\beta_{0}+\frac{L}{2}, \beta_{0}+L\right)  \tag{2.17}\\
\gamma_{\phi}(t) \in\left[\gamma_{0}-\epsilon, \gamma_{0}+\epsilon\right], t \in\left[0, t_{0}\right]
\end{array}\right.
$$

Indeed, it follows from system (2.16) that

$$
\left\{\begin{array}{l}
\beta_{\phi}^{\prime}(t) \geqslant \phi-m^{*}>\frac{m^{*} L}{2 \epsilon}>0 \\
\beta_{\phi}^{\prime}(t) \leqslant \phi+m^{*}=m^{*}\left(\frac{3 L}{4 \epsilon}+1\right)
\end{array}\right.
$$

Therefore, for every $0 \leqslant s \leqslant t \leqslant t_{0}$, we have

$$
\left\{\begin{array}{l}
\beta_{\phi}\left(t_{0}\right)>\beta_{0}+\frac{m^{*} L}{2 \epsilon} t_{0}=\beta_{0}+\frac{L}{2} \\
\beta_{\phi}(t)>\beta_{\phi}(s), \\
\beta_{\phi}\left(t_{0}\right) \leqslant m^{*} t_{0}\left(\frac{3 L}{4 \epsilon}+1\right)+\beta_{0}<\beta_{0}+L
\end{array}\right.
$$

and

$$
\left|\gamma_{\phi}(t)-\gamma_{\phi}(0)\right|=\left|\int_{0}^{t} g_{2}\left(\beta_{\phi}(s), \gamma_{\phi}(s)\right) d s\right| \leqslant \int_{0}^{t_{0}} m^{*} d s=m^{*} t_{0}=\epsilon
$$

It now follows from (2.17) that for $\left(\beta_{1}, \gamma_{1}\right) \in\left(\beta_{0}, \beta_{0}+\frac{L}{2}\right] \times\left[A_{0}+2 \epsilon, A_{1}-2 \epsilon\right]$ there exists $\gamma_{0} \in\left[\gamma_{1}-\epsilon, \gamma_{1}+\epsilon\right]$ and $T \in\left(0, t_{0}\right)$ such that $\beta_{\phi}(T)=\beta_{1}$ and $\gamma_{\phi}(T)=\gamma_{1}$. The same proof works for $\beta_{1} \in\left(\beta_{0}-\frac{L}{2}, \beta_{0}\right]$ and gives that: for all $\left(\beta_{1}, \gamma_{1}\right) \in\left(\beta_{0}-\frac{L}{2}, \beta_{0}+\frac{L}{2}\right) \times\left[A_{0}+2 \epsilon, A_{1}-2 \epsilon\right]$, there exist $\gamma_{0} \in\left[\gamma_{1}-\epsilon, \gamma_{1}+\epsilon\right]$ and $T \in\left(0, t_{0}\right)$ such that $\beta_{\phi}(T)=\beta_{1}, \gamma_{\phi}(T)=\gamma_{1}$.

Summing up these claims, we get

Lemma 2.9. System (2.16) is controllable in $\mathbb{R}^{2}$ by piecewise continuous controls, i.e., for any $\left(\beta_{0}, \gamma_{0}\right),\left(\beta_{1}, \gamma_{1}\right) \in \mathbb{R}^{2}$ there exist a piecewise continuous control function $\phi$ and $T>0$ such that $\left(\beta_{\phi}(0), \gamma_{\phi}(0)\right)=\left(\beta_{0}, \gamma_{0}\right)$ and $\left(\beta_{\phi}(T), \gamma_{\phi}(T)\right)=\left(\beta_{1}, \gamma_{1}\right)$.

Combining two Lemmas 2.8-2.9 yields
Lemma 2.10. Let $\left(\xi^{0}, \eta^{0}\right),\left(\beta^{0}, \gamma^{0}\right) \in \mathbb{R}^{2}$ be given.

- For almost every $(\beta, \gamma) \in \mathbb{R}^{2}$, there exists $\bar{T}>0$ such that $\bar{k}\left(\bar{T}, \beta, \gamma, \beta^{0}, \gamma^{0}\right)>0$.
- For almost every $(\xi, \eta) \in \mathbb{R}^{2}$, there exists $T>0$ such that $k\left(T, \xi, \eta, \xi^{0}, \eta^{0}\right)>0$.

We have the following theorem.
Theorem 2.11. Under Hypothesis 2.6, let $\left(x_{t}, y_{t}\right)$ be the solution of (1.2) with initial value $\left(x_{0}, y_{0}\right) \in \mathbb{S}$. Then for every $t>0,\left(x_{t}, y_{t}\right)$ has a density $\varphi(t, x, y)$ satisfying the FokkerPlanck equation $L^{*} \varphi(t, x, y)=0$, where $L^{*}$ is the adjoint operator of $L$ in (1.3) which is given by

$$
\begin{aligned}
& L^{*} \varphi(t, x, y)=-\frac{\partial \varphi}{\partial t}+\frac{1}{2} \frac{\partial^{2}}{\partial x^{2}}\left[\sigma_{1}^{2}(x) x^{2}(1-x)^{2} \varphi\right]+\frac{1}{2} \frac{\partial^{2}}{\partial y^{2}}\left[\sigma_{2}^{2}(y) y^{2}(1-y)^{2} \varphi\right] \\
& +\frac{\partial^{2}}{\partial x \partial y}\left[\sigma_{1}(x) \sigma_{2}(y) x y(1-x)(1-y) \varphi\right]-\frac{\partial}{\partial x}\left[\left\{-a e^{-b y^{2}(3-2 y)} x-c x+h(1-x)\right\} \varphi\right] \\
& \\
& -\frac{\partial}{\partial y}[q y(x+y-1)(1-y) \varphi]
\end{aligned}
$$

Furthermore, for any compact set $K \subset \mathbb{S}, \lim _{t \rightarrow \infty} \mathbb{P}\left\{\left(x_{t}, y_{t}\right) \in K\right\}=0$.
Proof. By virtue of Lemma 2.7, $\{P(t)\}_{t \geqslant 0}$ is an integral Markov semigroup with a continuous kernel $k\left(t, \xi, \eta, \xi_{0}, \eta_{0}\right)$ for $t>0$. Then, the distribution of $\left(\xi_{t}, \eta_{t}\right)$ has a density $u(t, \xi, \eta)$ satisfying the Fokker-Planck equation (2.13). This implies the first statement of theorem. Next, using Lemma 2.10, we have for $v \in \mathcal{D}$

$$
\int_{0}^{\infty} P(t) v d t>0, \quad \text { a.e. on } \mathbb{R}^{2}
$$

Therefore, from Theorem 2.5, $\{P(t)\}_{t \geqslant 0}$ has the Foguel alternative, i.e., one of two following assertions holds true:
(A) $\{P(t)\}_{t \geqslant 0}$ is asymptotically stable on $\mathbb{R}^{2}$.
(B) $\{P(t)\}_{t \geqslant 0}$ is sweeping with respect to compact sets.

We shall show that the assertion (A) is not the case. Indeed, if (A) takes place then there exists a unique stationary density $u_{*}(\xi, \eta)$ of the Fokker-Planck equation (2.13) such that

$$
\lim _{t \rightarrow \infty} \iint_{\mathbb{R}^{2}}\left|u(t, \xi, \eta)-u_{*}(\xi, \eta)\right| d \xi d \eta=0
$$

As a consequence, $\lim _{t \rightarrow \infty} \mathbb{P}\left\{\left(\xi_{t}, \eta_{t}\right) \in K_{1}\right\}=\iint_{K_{1}} u_{*}(\xi, \eta) d \xi d \eta$ for any $K_{1} \in \mathcal{B}\left(\mathbb{R}^{2}\right)$. We choose a compact set $K_{1}$ satisfying $\iint_{K_{1}} u_{*}(\xi, \eta) d \xi d \eta>1-\varrho_{1}$. On the other hand, from Theorem 2.3 we have $\mathbb{P}\left[\lim _{t \rightarrow \infty} \eta_{t}=\infty\right]=\varrho_{1}$. Therefore, $1 \geqslant \mathbb{P}\left[\lim _{t \rightarrow \infty} \eta_{t}=\infty\right]+$ $\lim _{t \rightarrow \infty} \mathbb{P}\left\{\left(\xi_{t}, \eta_{t}\right) \in K_{1}\right\}>\varrho_{1}+\left(1-\varrho_{1}\right)=1$. This contradiction implies that (B) must hold. It means that $\lim _{t \rightarrow \infty} \mathbb{P}\left\{\left(\xi_{t}, \eta_{t}\right) \in K_{1}\right\}=0$ for any compact set $K_{1} \subset \mathbb{R}$. Therefore, $\lim _{t \rightarrow \infty} \mathbb{P}\left\{\left(x_{t}, y_{t}\right) \in K\right\}=0$ for any compact set $K \subset \mathbb{S}$. Thus, the proof is complete.

3 Numerical examples and conclusion In this section, we present numerical examples. Set $a=4, b=5, c=0.2, h=1, q=1,\left(x_{0}, y_{0}\right)=(0.8,0.45)$, and $\left(\sigma_{1}, \sigma_{2}\right)=(0.5,0.2)$. In the following figures, $d_{1}$ and $d_{2}$ denote, respectively, the lines $x=\frac{h}{a+c+h}$ and $x=\frac{h}{a e^{-b}+c+h}$.
In Figure 1(a), $y_{t} \rightarrow 1$ as $t \rightarrow \infty$. It follows that $\frac{1}{t} \int_{0}^{t} x_{s} d s \rightarrow \frac{h}{a e^{-b}+c+h}$ as $t \rightarrow \infty$, see Figure 1(b). In this case, the trajectory of $x_{t}$ oscillates around $d_{2}$. Similarly, in Figure 2(a), the trajectory of $y_{t}$ approaches to 0 and in Figure 2(b) one of $x_{t}$ oscillates around $d_{1}$ as $t \rightarrow \infty$. The model (1.2) is sweeping with respect to compact sets.

In conclusion, we recall that this work provides some results about the asymptotic behavior of a stochastic adsorbate-induced phase transition model (1.2). The mathematical analysis shows that the square $(0,1) \times(0,1)$ is an invariant set of this model and the model is sweeping with respect to compact sets of $(0,1) \times(0,1)$. Furthermore, we show that $y_{t} \rightarrow 0, y_{t} \rightarrow 1, \frac{1}{t} \int_{0}^{t} x_{s} d s \rightarrow \frac{h}{a e^{-b}+c+h}$, and $\frac{1}{t} \int_{0}^{t} x_{s} d s \rightarrow \frac{h}{a+c+h}$ as $t \rightarrow \infty$ with positive probability. These results agree to those of Hildebrand et al. [6, 7] where they confirm that $\left(\frac{h}{a+c+h}, 0\right)$ and $\left(\frac{h}{a e^{-b}+c+h}, 1\right)$ are stable homogeneous stationary solutions.


Figure 1: Case when $y_{t} \rightarrow 1$ and $\frac{1}{t} \int_{0}^{t} x_{s} d s \rightarrow \frac{h}{a e^{-b}+c+h}$


Figure 2: Case when $y_{t} \rightarrow 0$ and $\frac{1}{t} \int_{0}^{t} x_{s} d s \rightarrow \frac{h}{a+c+h}$

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