## ASYMPTOTIC BEHAVIOUR OF SOLUTIONS TO STOCHASTIC PHASE TRANSITION MODEL

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ABSTRACT. In this paper, we consider a stochastic adsorbate-induced phase transition model. First, we prove the existence and uniqueness of global solution. Then, we estimate the long-time expectation of the adsorbate coverage rate of the surface and its average integral as well as the behaviour of solutions. Finally, numerical examples are presented to illustrate those results.

1 Introduction In 1990, Ertl [4] and Jakubith et al. [9] found out that, in the catalytic oxidation of CO molecules on a Pt(110) surface, adsorbed CO molecules and O atoms form various types of spatio-temporal patterns, such as propagating wave, spiral, target, stripe and chaotic patterns. To understand the mechanism of these phenomena from macroscopic point of view, Hildebrand et al. [6, 7] presented a simple kinematic model of the surface reaction coupled to a structural phase transition of the surface, that is, in  $\Omega \times (0, \infty)$ 

(1.1) 
$$\begin{cases} \frac{\partial x}{\partial t} = \mu_1 \Delta x - \mu_2 \bigtriangledown \cdot [x(1-x) \bigtriangledown \chi(y)] - ae^{-b\chi(y)} x_t - cx_t + h(1-x_t), \\ \frac{\partial y}{\partial t} = \mu_3 \Delta y + qy_t (x_t + y_t - 1)(1-y_t). \end{cases}$$

The two-dimensional domain  $\Omega$  denotes a Pt surface on which the patterns are performed. The unknown functions x(z,t) and y(z,t) denote the adsorbate coverage rate of the surface by CO molecules and the structural state of surface at a position  $z \in \Omega$  and time t, respectively. The nonlinear advection  $-\mu_2 \bigtriangledown [x(1-x) \bigtriangledown \chi(y)]$  shows a flow of u on  $\Omega$  induced by the gradient of the local chemical potential  $\chi(y)$  with mobility 1-x. A typical form of  $\chi(y)$ is such that  $\chi(y) = y^2(3-2y)$ . In the growth function of x,  $ae^{\chi(y)}$  denotes the desorption rate of the molecules depending on  $\chi(y)$ , c the desorption rate by the effect of a chemical reaction, and h denotes the adsorption rate. We mention here some results related to this model:

- The square  $\{(x, y) : 0 \leq x, y \leq 1\}$  is an invariant and attractor set [18, p.381].
- $(\frac{h}{a+c+h}, 0), (\frac{h}{ae^{-b}+c+h}, 1)$  are homogeneous stationary solutions which are both stable.

However, given that models are often subject to environmental noise, it is important to investigate whether the presence of such noise affects this result or not. In this paper, we consider a stochastic version of model (1.1). The stochastic perturbation is as follows:

$$-ae^{-by_t^2(3-2y_t)}x_t - cx_t + h(1-x_t) \rightsquigarrow -ae^{-by_t^2(3-2y_t)}x_t - cx_t + h(1-x_t) + \text{"white noise"},$$
$$qy_t(x_t + y_t - 1)(1-y_t) \rightsquigarrow qy_t(x_t + y_t - 1)(1-y_t) + \text{"white noise"}.$$

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The white noise here is effected by one-dimensional Brownian motion  $\{w_t, t \ge 0\}$  defined on a complete probability space with filtration  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$  satisfying the usual conditions (see [10]). Since the unit square  $\{(x, y) : 0 \le x, y \le 1\}$  is an invariant and attractor set for model (1.1), we assume that the white noise vanishes on the boundary of this square. Thus, this environmentally perturbed system may be described by an Itô stochastic differential equation of the form

(1.2) 
$$\begin{cases} dx_t = \left[ -ae^{-by_t^2(3-2y_t)}x_t - cx_t + h(1-x_t) \right] dt \\ +\sigma_1(x_t)x_t(1-x_t)dw_t, \\ dy_t = qy_t(x_t+y_t-1)(1-y_t)dt + \sigma_2(y_t)y_t(1-y_t)dw_t, \end{cases}$$

where a, b, c, h and q are positive constants, and  $\sigma_i(\cdot)$  (i = 1, 2) are bounded functions. The generator of equation (1.2) is then given by

$$(1.3) \quad Lf(t,x,y) = \frac{\partial f}{\partial t} + \frac{1}{2}\sigma_1^2(x)x^2(1-x)^2\frac{\partial^2 f}{\partial x^2} + \frac{1}{2}\sigma_2^2(y)y^2(1-y)^2\frac{\partial^2 f}{\partial y^2} \\ + \sigma_1(x)\sigma_2(y)xy(1-x)(1-y)\frac{\partial^2 f}{\partial x\partial y} + [-ae^{-by^2(3-2y)}x - cx + h(1-x)]\frac{\partial f}{\partial x} \\ + qy(x+y-1)(1-y)\frac{\partial f}{\partial y}.$$

By some physical reasons, we only consider the solution  $(x_t, y_t)$  of model (1.2) with an initial condition  $(x_0, y_0) \in \mathbb{S} \equiv \{(x, y) : 0 < x, y < 1\}$  and in order that the system (1.2) describes fitly the physical phenomena we expect that  $(x_t, y_t)$  are in this square for all t > 0.

The aim of this paper is to show that (1.2) has a global solution lying in S. Next, we estimate the long-time expectation of  $x_t$  and its average integral in order to investigate the behaviour of  $(x_t, y_t)$ .

The organization of paper is as follows. Section 2 provides the main results. First, we shall verify that the square S is an invariant set of model (1.2), i.e., the solution  $(x_t, y_t)$  starting from S will remain in S with probability 1. Next, using comparison theorems, we shall show that

$$\frac{h}{a+c+h} \leqslant \liminf_{t \to \infty} \mathbb{E}x_t \leqslant \limsup_{t \to \infty} \mathbb{E}x_t \leqslant \frac{h}{ae^{-b}+c+h}$$

and that there exist  $\varrho_i, \overline{\varrho}_i \in (0, 1), \overline{\varrho}_i > \varrho_i(i = 1, 2)$  such that as  $t \to \infty$  it holds true that  $\frac{1}{t} \int_0^t x_s ds \to \frac{h}{ae^{-b}+c+h}, \frac{1}{t} \int_0^t x_s ds \to \frac{h}{a+c+h}, y_t \to 1$  and  $y_t \to 0$  with probability  $\overline{\varrho}_1, \overline{\varrho}_2, \varrho_1$  and  $\varrho_2$ , respectively. Further, by calculating the Lie bracket and studying the controllability of differential equations, it is proved that the system is sweeping for every compact set of  $\mathbb{S}$ , i.e., for any K such that  $\overline{K} \subset \mathbb{S}$  we have

$$\lim \mathbb{P}\{(x_t, y_t) \in K\} = 0.$$

The last section, Section 3, illustrates the above results by some numerical examples.

2 Main results First of all, we will prove the invariance of S and the global existence of solution to (1.2). Indeed, the following theorem is proved.

**Theorem 2.1.** For any  $(x_0, y_0) \in \mathbb{S}$ , there exists a unique solution  $(x_t, y_t)$  of (1.2) for  $t \ge 0$ . Furthermore, with probability 1,  $\mathbb{S}$  is positively invariant for (1.2), i.e., if  $(x_0, y_0) \in \mathbb{S}$  then  $(x_t, y_t) \in \mathbb{S}$  for all  $t \ge 0$  with probability 1.

Proof. Since the coefficients of (1.2) are locally Lipschitz continuous, for any initial value  $(x_0, y_0) \in \mathbb{S}$  there is a unique solution  $(x_t, y_t)$  defined on an interval  $[0, \tau)$ , where  $\tau$  is explosive time, i.e.,  $\lim_{t\to\tau} (|x_t| + |y_t|) = 1$  (see [1, 5]). Further, when either y = 0 or y = 1, the coefficients of the second equation in (1.2) equal to 0. Therefore, by the local uniqueness of the solution, we see that if  $y_0 \in (0, 1)$  then  $y_t \in (0, 1)$  a.s. for  $t \in [0, \tau)$ . Therefore, to show the existence of a global solution and also S to be positively invariant, we have only to prove that if  $(x_0, y_0) \in \mathbb{S}$  then  $0 < x_t < 1$  for all t > 0 (which yields that  $\tau = \infty$ ) a.s. We use the technique of localization dealt with in [10]. Let  $\gamma_0 > 0$  be sufficiently large for  $x_0$  lying within the interval  $[\frac{1}{k_0}, k_0]$ . Denote  $H_k = [\frac{1}{k}, 1 - \frac{1}{k}] \times (0, 1)$ . It is seen that  $\cup_{k=1}^{\infty} H_k = \mathbb{S}$ . Let us define a sequence of stopping times [10, Problem 1.2.7] for each integer  $k \ge k_0$  by

$$\tau_k = \inf \left\{ t > 0 : (x_t, y_t) \notin H_k \right\}$$

(with the convention  $\inf \emptyset = \infty$ ). Since  $\tau_k$  is nondecreasing as  $k \to \infty$ , there exists the limit  $\tau_{\infty} = \lim_{k \to \infty} \tau_k$ . It is clear  $\tau_{\infty} \leq \tau$  a.s.

Consider a positive function  $Q(x) = -\ln x - \ln(1-x)$  on (0,1). By using (1.3), we have

$$LQ(x,y) = -\frac{1}{x} \left[ -ae^{-by^2(3-2y)}x - cx + h(1-x) \right] + h$$
  
$$-\frac{1}{1-x} \left[ ae^{-by^2(3-2y)}x + cx \right] + \frac{1}{2}\sigma_1^2(x)[x^2 + (1-x)^2].$$

Since  $\lim_{x\downarrow 0} LQ(x,y) = -\infty$  and  $\lim_{x\uparrow 1} LQ(x,y) = -\infty$ , there exists a positive number M such that  $LQ(x,y) \leq M$  for any  $(x,y) \in \mathbb{S}$ . Therefore, by Itô's formula, for any t > 0 it holds that

(2.1) 
$$\mathbb{E}Q(x_{t\wedge\tau_k}) = Q(x_0) + \int_0^t LQ(x_{s\wedge\tau_k}, y_{s\wedge\tau_k}) ds \leq Q(x_0) + Mt.$$

Suppose on the contrary that  $\mathbb{P}\{\tau_{\infty} < \infty\} > 0$ . Then, we can find a number T such that if  $A = \{\omega : \tau_{\infty}(\omega) < T\}$  then  $\mathbb{P}(A) > 0$ . Therefore, from (2.1) we have

$$\mathbb{E}Q(x_{T\wedge\tau_k}) = \mathbb{E}[1_A Q(x_{T\wedge\tau_k})] + \mathbb{E}[1_{A^c} Q(x_{T\wedge\tau_k})]$$
$$= \mathbb{E}[1_A Q(x_{\tau_k})] + \mathbb{E}[1_{A^c} Q(x_{T\wedge\tau_k})] \leqslant Q(x_0) + MT.$$

On the other hand, if  $\tau_k < \infty$  then

$$Q(x_{\tau_k}(\omega)) = -\ln\frac{1}{k} - \ln(1 - \frac{1}{k}).$$

Since Q is positive, it follows that

$$\left(-\ln\frac{1}{k}-\ln(1-\frac{1}{k})\right)\mathbb{P}(A) \leq Q(x_0) + MT.$$

Letting  $k \to \infty$  we get a contradiction. Thus  $\mathbb{P}\{\tau_{\infty} = \infty\} = 1$  and this implies that  $\mathbb{P}\{\tau = \infty\} = 1$ . Therefore, if  $(x_0, y_0) \in \mathbb{S}$ , then we have  $(x_t, y_t) \in \mathbb{S}$  for any t > 0. This means that  $(x_t, y_t)$  is defined on  $[0, \infty)$  and  $\mathbb{S}$  is an invariant set. The proof is complete.  $\Box$ 

To study the dynamics of solutions of the system (1.2) we suppose that:

**Hypothesis 2.2.**  $\sigma_i(\cdot) \in C^1(\mathbb{R}_+)$  (i = 1, 2) and there exist two positive constants  $\varepsilon_1, \varepsilon_2$  such that  $\varepsilon_1 < \sigma_i(x) < \varepsilon_2$  and  $|\sigma'_i(x)| < \varepsilon_2$  for all  $x \ge 0$ .

Theorem 2.3. Under Hypothesis 2.2, it holds true that

- (i) there exist  $\varrho_i \in (0,1)(i=1,2)$  such that  $\mathbb{P}[\lim_{t\to\infty} y_t = 1] = \varrho_1$  and  $\mathbb{P}[\lim_{t\to\infty} y_t = 0] = \varrho_2;$
- (ii)  $\frac{h}{a+c+h} \leq \liminf_{t \to \infty} \mathbb{E}x_t \leq \limsup_{t \to \infty} \mathbb{E}x_t \leq \frac{h}{ae^{-b}+c+h};$
- (iii) there exist  $\overline{\varrho}_i \in [\varrho_i, 1)(i = 1, 2)$  for which it holds that  $\lim_{t \to \infty} \frac{1}{t} \int_0^t x_s ds = \frac{h}{ae^{-b} + c + h}$ and  $\lim_{t \to \infty} \frac{1}{t} \int_0^t x_s ds = \frac{h}{a + c + h}$  with probability  $\overline{\varrho}_1, \overline{\varrho}_2$ , respectively.

*Proof.* To prove (i), we define the following two stochastic differential equations:

(2.2) 
$$\begin{cases} d\overline{y}_t = -q\overline{y}_t(\overline{y}_t - 1)^2 dt + \sigma_2(\overline{y}_t)\overline{y}_t(1 - \overline{y}_t) dw_t, \\ d\overline{y}_t = q\overline{y}_t^2(1 - \overline{y}_t) dt + \sigma_2(\overline{y}_t)\overline{y}_t(1 - \overline{y}_t) dw_t, \\ \overline{y}_0 = \overline{y}_0 = y_0 \in (0, 1). \end{cases}$$

From Theorem 2.1 and the comparison theorem (see [8, Theorem 1.1, p.352]), we see that  $\overline{y}_t \leq y_t \leq \overline{\overline{y}}_t$  for all  $t \geq 0$  a.s. The scale functions  $s_1(y)$  and  $s_2(y)$  of the stochastic differential equations in (2.2) are given, respectively, by

$$s_1(y) = \int_{\frac{1}{2}}^{y} \exp\left\{-2\int_{\frac{1}{2}}^{u} \frac{-qv(v-1)^2}{v^2(v-1)^2\sigma_2^2(v)} dv\right\} du = \int_{\frac{1}{2}}^{y} \exp\left\{\int_{\frac{1}{2}}^{u} \frac{2q}{v\sigma_2^2(v)} dv\right\} du$$

and

$$s_2(y) = \int_{\frac{1}{2}}^{y} \exp\left\{-2\int_{\frac{1}{2}}^{u} \frac{qv^2(1-v)}{v^2(v-1)^2\sigma_2^2(v)}dv\right\} du = \int_{\frac{1}{2}}^{y} \exp\left\{\int_{\frac{1}{2}}^{u} \frac{-2q}{(1-v)\sigma_2^2(v)}dv\right\} du.$$

Since  $\sigma_2(\cdot)$  is bounded above and below by positive constants, it is easily seen that

$$A_i := \lim_{y \to 1} s_i(y) < \infty, \ B_i := \lim_{y \to 0} s_i(y) > -\infty \quad (i = 1, 2).$$

Therefore, by using [10, Proposition 5.22, p.345] we have

$$\begin{split} \mathbb{P}[\lim_{t \to \infty} \overline{y}_t = 0] &= 1 - \mathbb{P}[\lim_{t \to \infty} \overline{y}_t = 1] = \frac{A_1 - s_1(y_0)}{A_1 - B_1}, \\ \mathbb{P}[\lim_{t \to \infty} \overline{\overline{y}}_t = 0] &= 1 - \mathbb{P}[\lim_{t \to \infty} \overline{\overline{y}}_t = 1] = \frac{A_2 - s_2(y_0)}{A_2 - B_2}. \end{split}$$

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This implies existence of  $\varrho_1 \in [1 - \frac{A_1 - s_1(y_0)}{A_1 - B_1}, 1)$  and  $\varrho_2 \in [\frac{A_2 - s_2(y_0)}{A_2 - B_2}, 1)$  such that

$$\mathbb{P}[\lim_{t \to \infty} y_t = 1] = \varrho_1, \quad \mathbb{P}[\lim_{t \to \infty} y_t = 0] = \varrho_2.$$

Next, we prove (ii). We consider two one-dimensional stochastic processes  $\overline{x}_t$  and  $\overline{\overline{x}}_t$  given by

(2.3) 
$$\begin{cases} d\overline{x}_t = [h - (a+c+h)\overline{x}_t]dt + \sigma_1(\overline{x}_t)\overline{x}_t(1-\overline{x}_t)dw_t, \\ d\overline{x}_t = [h - (ae^{-b}+c+h)\overline{x}_t]dt + \sigma_1(\overline{x}_t)\overline{x}_t(1-\overline{x}_t)dw_t, \\ \overline{x}_0 = \overline{x}_0 = x_0 \in (0,1). \end{cases}$$

Using again Theorem 2.1 and the comparison theorem as before, we get

(2.4) 
$$\overline{x}_t \leqslant x_t \leqslant \overline{x}_t$$
 for all  $t \ge 0$  a.s

It follows from the first equation of (2.3) and Fubini's theorem that

$$\begin{cases} d\mathbb{E}\overline{x}_t = [h - (a + c + h)\mathbb{E}\overline{x}_t]dt, \\ \mathbb{E}\overline{x}_0 = x_0. \end{cases}$$

It is easy to see that

$$\mathbb{E}\overline{x}_t = (x_0 - \frac{h}{a+c+h})e^{-(a+c+h)t} + \frac{h}{a+c+h}$$

Therefore,  $\lim_{t\to\infty} \mathbb{E}\overline{x}_t = \frac{h}{a+c+h}$ . Similarly, we have  $\lim_{t\to\infty} \mathbb{E}\overline{\overline{x}}_t = \frac{h}{ae^{-b}+c+h}$ . Hence, by using (2.4) we obtain

$$\frac{h}{a+c+h} \leqslant \liminf_{t \to \infty} \mathbb{E}x_t \leqslant \limsup_{t \to \infty} \mathbb{E}x_t \leqslant \frac{h}{ae^{-b}+c+h}$$

We now prove (iii). It follows from the first equation of (1.2) that

(2.5) 
$$\frac{x_t}{t} = \frac{x_0}{t} + \frac{1}{t} \int_0^t \left[ -ae^{-by_s^2(3-2y_s)} x_s - cx_s + h(1-x_s) \right] ds + \frac{1}{t} \int_0^t \sigma_1(x_s) x_s(1-x_s) dw_s.$$

Since  $0 < x_t < 1$ , then  $\lim_{t\to\infty} \frac{x_t}{t} = 0$  a.s. Furthermore, by putting  $M_t = \int_0^t \sigma_1(x_s) x_s(1 - x_s) dw_s$  and by Hypothesis 2.2,  $\{M_t, \mathcal{F}_t, t \ge 0\}$  is a continuous martingale with the variation  $< M >_t = \int_0^t \sigma_1^2(x_s) x_s^2(1 - x_s)^2 ds$  satisfying

$$< M >_t < \epsilon_2^2 t/16,$$
 a.s.

Using the corollary to the theorem of time-change for martingales [10, Theorem 4.6, p.174], there exists a Brownian motion  $(B_t, \mathcal{G}_t)$  on  $(\Omega, \mathcal{F}, P)$  such that  $M_t = B_{\langle M \rangle_t}, t \ge 0$ . For P-a.s.  $\omega \in \Omega$ ,

• If  $\langle M \rangle_{\infty} (\omega) := \lim_{t \to \infty} \langle M \rangle_t (\omega) \langle \infty$ , then

$$\lim_{t \to \infty} \frac{M_t(\omega)}{t} = \lim_{t \to \infty} \frac{B_{_t(\omega)}(\omega)}{t} = \lim_{t \to \infty} \frac{B_{_\infty(\omega)}(\omega)}{t} = 0.$$

• If  $\langle M \rangle_{\infty} (\omega) = \infty$ , then from

$$\left|\frac{M_t(\omega)}{t}\right| \leqslant \left|\frac{\epsilon_2^2 M_t(\omega)}{16 < M >_t(\omega)}\right|$$

and from the strong law of large numbers for Brownian motion (see [10, p.104]) we have

$$\lim_{t \to \infty} \left| \frac{M_t(\omega)}{t} \right| \leq \lim_{t \to \infty} \left| \frac{\epsilon_2^2 B_{\leq M > t(\omega)}(\omega)}{16 < M >_t(\omega)} \right| = \frac{\epsilon_2^2}{16} \lim_{s \to \infty} \left| \frac{B_s(\omega)}{s} \right| = 0.$$

Therefore,

(2.6) 
$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \sigma_1(x_s) x_s (1 - x_s) dw_s = 0, \qquad \text{a.s.}$$

Combining (2.5) and (2.6) gives

(2.7) 
$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \left[ a e^{-by_s^2(3-2y_s)} + c + h \right] x_s ds = h, \qquad \text{a.s}$$

Denote  $A = \{\omega : \lim_{t\to\infty} y_t(\omega) = 1\}$ , we then have from (i) that  $\mathbb{P}(A) = \varrho_1$ . For every  $\omega \in A$  and  $\varepsilon \in (0, 1)$ , there exists  $T = T(\varepsilon, \omega) > 0$  such that  $y_t(\omega) > 1 - \varepsilon$ . Then, for all  $s \ge T$  we have

(2.8) 
$$ae^{-b} \leqslant ae^{-by_s^2(\omega)(3-2y_s(\omega))} \leqslant ae^{-b(1-\varepsilon)^2(1+2\varepsilon)}.$$

Because of  $(x_t, y_t) \in \mathbb{S}$ , we see from (2.7) and (2.8) that on the set A,

$$(ae^{-b}+c+h)\limsup_{t\to\infty}\frac{1}{t}\int_0^t x_s ds \leqslant h \leqslant (ae^{-b(1-\varepsilon)^2(1+2\varepsilon)}+c+h)\liminf_{t\to\infty}\frac{1}{t}\int_0^t x_s ds.$$

Letting  $\varepsilon \to 0$  gives  $\lim_{t\to\infty} \frac{1}{t} \int_0^t x_s ds = \frac{h}{ae^{-b}+c+h}$  on A. It means that there exists  $\overline{\varrho}_1 \in [\varrho_1, 1)$  such that  $\lim_{t\to\infty} \frac{1}{t} \int_0^t x_s ds = \frac{h}{ae^{-b}+c+h}$  with probability  $\overline{\varrho}_1$ .

Similarly, there exists  $\overline{\varrho}_2 \in [\varrho_2, 1)$  such that

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t x_s ds = \frac{h}{a+c+h}$$

with probability  $\overline{\varrho}_2$ . Hence, the proof is complete.

Since Theorem 2.1 implies  $(x_t, y_t) \in \mathbb{S}$ , we can use the transformation  $x_t = \frac{e^{\xi_t}}{1+e^{\xi_t}}, y_t = \frac{e^{\eta_t}}{1+e^{\eta_t}}$   $(t \ge 0)$ . Then, by Itô's formula, the system (1.2) is rewritten into

(2.9) 
$$\begin{cases} d\xi_t = P_1(\xi_t, \eta_t) dt + \sigma_1(\frac{e^{\xi_t}}{1+e^{\xi_t}}) dw_t, \\ d\eta_t = P_2(\xi_t, \eta_t) dt + \sigma_2(\frac{e^{\eta_t}}{1+e^{\eta_t}}) dw_t, \end{cases}$$

where

$$P_{1}(\xi,\eta) = (1+e^{\xi}) \left( -a \exp\left\{\frac{-be^{2\eta}(e^{\eta}+3)}{(e^{\eta}+1)^{3}}\right\} - c + \frac{h}{e^{\xi}}\right) \\ + \frac{e^{\xi}-1}{2(1+e^{\xi})} \sigma_{1}^{2} \left(\frac{e^{\xi}}{1+e^{\xi}}\right), \\ P_{2}(\xi,\eta) = q\left(\frac{e^{\xi}}{1+e^{\xi}} + \frac{e^{\eta}}{1+e^{\eta}} - 1\right) + \frac{e^{\eta}-1}{2(1+e^{\eta})} \sigma_{2}^{2} \left(\frac{e^{\eta}}{1+e^{\eta}}\right).$$

It is seen that  $x_t$  and  $y_t$  tend to 0 or 1 if and only if  $\xi_t$  and  $\eta_t$  tend to  $-\infty$  or  $+\infty$ , respectively.

To study the positivity of density of transition probability function to (2.9) we put

$$\rho_i(x) = \int_0^x \frac{du}{\sigma_i(\frac{e^u}{1+e^u})} \ (i=1,2), \alpha_t = \rho_1(\xi_t), \beta_t = \rho_2(\eta_t), \gamma_t = \alpha_t - \beta_t.$$

By using Itô's formula, we obtain from (2.9) that

(2.10) 
$$\begin{cases} d\beta_t = g_1(\beta_t, \gamma_t)dt + dw_t, \\ d\gamma_t = g_2(\beta_t, \gamma_t)dt, \end{cases}$$

where

(2.11) 
$$g_1(\beta,\gamma) = \frac{P_2(\rho_1^{-1}(\beta+\gamma),\rho_2^{-1}(\beta))}{\sigma_2(\frac{e^{\rho_2^{-1}(\beta)}}{1+e^{\rho_2^{-1}(\beta)}})} - \frac{1}{2}\frac{e^{\rho_2^{-1}(\beta)}}{[1+e^{\rho_2^{-1}(\beta)}]^2}\sigma_2'(\frac{e^{\rho_2^{-1}(\beta)}}{1+e^{\rho_2^{-1}(\beta)}})$$

and

$$(2.12) \quad g_2(\beta,\gamma) = \frac{P_1(\rho_1^{-1}(\beta+\gamma),\rho_2^{-1}(\beta))}{\sigma_1(\frac{e^{\rho_1^{-1}(\beta+\gamma)}}{1+e^{\rho_1^{-1}(\beta+\gamma)}})} \\ \quad -\frac{1}{2}\frac{e^{\rho_1^{-1}(\beta+\gamma)}}{[1+e^{\rho_1^{-1}(\beta+\gamma)}]^2}\sigma_1'(\frac{e^{\rho_1^{-1}(\beta+\gamma)}}{1+e^{\rho_1^{-1}(\beta+\gamma)}}) - g_1(\beta,\gamma).$$

Here  $\rho_i^{-1}(\cdot)$  denotes the inverse of  $\rho_i(\cdot)$  (i = 1, 2). It is known that the density  $u(t, \xi, \eta)$  of the random variable  $(\xi_t, \eta_t)$ , if it exists and is smooth, can be found from the following Fokker-Planck equation

$$(2.13) \quad \frac{\partial u(t,\xi,\eta)}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial \xi^2} \left[ \sigma_1^2 \left(\frac{e^{\xi}}{1+e^{\xi}}\right) u \right] + \frac{1}{2} \frac{\partial^2}{\partial \eta^2} \left[ \sigma_2^2 \left(\frac{e^{\eta}}{1+e^{\eta}}\right) u \right] \\ + \frac{\partial^2}{\partial \xi \partial \eta} \left[ \sigma_1 \left(\frac{e^{\xi}}{1+e^{\xi}}\right) \sigma_2 \left(\frac{e^{\eta}}{1+e^{\eta}}\right) u \right] - \frac{\partial (P_1 u)}{\partial \xi} - \frac{\partial (P_2 u)}{\partial \eta} \cdot$$

For convenience of arguments, we will study the long-time behaviour of the density of solution  $(\xi_t, \eta_t)$  of the system (2.9) or of the system (2.10) instead of considering the system (1.2). Let us first survey some basic results on Markov semigroups.

Let the triple  $(X, \Sigma, \mathbf{m})$  be a  $\sigma$ -finite measure space. Denote by  $\mathcal{D}$  a subset of  $L^1 = L^1(X, \Sigma, \mathbf{m})$ , consisting of the densities, i.e.,  $\mathcal{D} = \{f \in L^1 : f \ge 0, \|f\| = 1\}$ .

**Definition 2.4.** Let P be a linear mapping from  $L^1$  to itself and let  $\{P(t)\}_{t\geq 0}$  denote a semigroup of linear operators on  $L^1$ .

(i) P is called a Markov operator if  $P(\mathcal{D}) \subset \mathcal{D}$ ;

(ii) A Markov operator P is called a kernel operator if there exists a measurable function  $k: X \times X \to [0, \infty)$  such that  $Pf(x) = \int_X k(x, y) f(y) \mathbf{m}(dy)$  for every density  $f \in \mathcal{D}$ .

(iii) Semigroup  $\{P(t)\}_{t\geq 0}$  is a Markov semigroup if P(t) is a Markov operator for every  $t\geq 0$ ;

(iv) Semigroup  $\{P(t)\}_{t\geq 0}$  is integral if P(t) is an integral Markov operator for every  $t\geq 0$ .

(v) Semigroup  $\{P(t)\}_{t\geq 0}$  is asymptotically stable if there is an invariant density  $f_*$  such that  $\lim_{t\to\infty} ||P(t)f - f_*|| = 0$  for  $f \in \mathcal{D}$  (a density  $f_*$  is said to be invariant under the semigroup  $\{P(t)\}_{t\geq 0}$  if  $P(t)f_* = f_*$  for each  $t \geq 0$ ).

(vi) Makov semigroup  $\{P(t)\}_{t\geq 0}$  is said to be sweeping with respect to a set  $K \in \Sigma$  if for all  $f \in D$ ,

$$\lim_{t \to \infty} \int_K P(t) f(x) \mathbf{m}(dx) = 0.$$

If a semigroup is either asymptotically stable or sweeping with respect to compact sets then we say that the semigroup has the Foguel alternative. Then, the following results are known. For the proof, see [14].

**Theorem 2.5.** Let X be a metric space and let  $\Sigma$  be the its Borel  $\sigma$  -field. Let  $\{P(t)\}_{t\geq 0}$  be an integral Markov semigroup with a continuous kernel k(t, x, y). Assume that for every  $f \in \mathcal{D}$ ,

$$\int_0^\infty P(t)fdt > 0, \qquad \qquad a.e$$

Then, the semigroup enjoys the Foguel alternative.

We now consider the space  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathbf{m})$ , where  $\mathcal{B}(\mathbb{R}^2)$  is the Borel  $\sigma$ -field of  $\mathbb{R}^2$  and **m** is the Lebesgue measure on  $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ . We can then construct a semigroup  $\{P(t)\}_{t\geq 0}$  generated by the system (2.9) and can prove that it has the Foguel alternative. To do this, we add a condition to Hypothesis 2.2 and assume that the following hypothesis holds throughout the remaining of the paper.

**Hypothesis 2.6.** The functions  $\sigma_i(\cdot)$  (i = 1, 2) satisfy Hypothesis 2.2 and are analytic.

**Lemma 2.7.** Let  $(\xi_t, \eta_t)$  be a solution of (2.9) with initial condition  $(\xi_0, \eta_0) \in \mathbb{R}^2$ . Then, the transition probability function  $\mathcal{P}(t, \xi_0, \eta_0, \cdot)$  of the Markov diffusion process  $(\xi_t, \eta_t)$ , i.e.,  $\mathcal{P}(t, \xi_0, \eta_0, K) = \mathbb{P}\{(\xi_t, \eta_t) \in K\}$  for  $K \in \mathcal{B}(\mathbb{R}^2)$ , has a density  $k(t, \xi, \eta, \xi_0, \eta_0)$  with respect to **m** with the regularity  $k \in C^{\infty}([0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2)$ .

*Proof.* Since  $\rho_i(x)$  is a strictly increasing function, it suffices to show that the transition probability function of solution  $(\beta_t, \gamma_t)$  of (2.10) has a density, say by  $\overline{k}(t, \beta, \gamma, \beta^0, \gamma^0)$  satisfying  $\overline{k} \in C^{\infty}((0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2)$ . We here apply the Hörmander theorem on existence of smooth densities of the transition probability function for the degenerate diffusion processes described by (2.10) (see Nualart [13, Theorem 7.4], Malliavin [11] and Norris [12]).

When  $X(x) = (X_1, \dots, X_d)$  and  $Y(x) = (Y_1, \dots, Y_d)$  are a vector field on  $\mathbb{R}^d$ , the Lie bracket [X, Y] is defined by a vector field

$$[X,Y]_j(x) = \sum_{k=1}^d \left[ X_k \frac{\partial Y_j}{\partial x_k}(x) - Y_k \frac{\partial X_j}{\partial x_k}(x) \right] \quad (j = 1, \cdots, d).$$

Put

$$a_0(\beta,\gamma) = \begin{bmatrix} g_1(\beta,\gamma) \\ g_2(\beta,\gamma) \end{bmatrix}, \quad a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ a_2(\beta,\gamma) = [a_0,a_1], \quad a_{i+1}(\beta,\gamma) = [a_i,a_1] \quad (i \ge 2).$$

Then,

$$a_i = \begin{bmatrix} (-1)^{i-1} \frac{\partial^{i-1}g_1}{\partial\beta^{i-1}}\\ (-1)^{i-1} \frac{\partial^{i-1}g_2}{\partial\beta^{i-1}} \end{bmatrix} \qquad (i \ge 2).$$

Denote by  $\mathbb{R}^2(\beta, \gamma)$  the vector space generated by vectors  $\{a_i(\beta, \gamma)\}_{i \ge 1}$ . If there exists a point  $(\beta^0, \gamma^0) \in \mathbb{R}^2$  such that  $\mathbb{R}^2(\beta^0, \gamma^0) \neq \mathbb{R}^2$ , then the vectors  $a_i \ (i \ge 1)$  and  $a_1$  are parallel. Hence,

$$\frac{\partial^{i-1}g_2}{\partial\beta^{i-1}}(\beta^0,\gamma^0) = 0, \quad \text{for all } i \ge 2.$$

It then follows that  $g_2(\beta, \gamma^0)$  is constant for  $\beta \in \mathbb{R}$ .

On the other hand, from the fact that  $\lim_{\beta\to\infty} \rho_1^{-1}(\beta + \gamma_0) = \lim_{\beta\to\infty} \rho_2^{-1}(\beta) = \infty$ we have  $\lim_{\beta\to\infty} g_2(\beta, \gamma_0) = -\infty$ . This is contradiction to the above result. Therefore, we obtain the Hörmander condition:

(H) For every  $(\beta, \gamma) \in \mathbb{R}^2$ , the vectors  $a_i(\beta, \gamma)$   $(i \ge 1)$  span the space  $\mathbb{R}^2$ .

Under the condition (H), the transition probability function  $\mathcal{P}(t, \beta^0, \gamma^0, \cdot)$  of system (2.10) has a density  $\overline{k}(t, \beta, \gamma, \beta^0, \gamma^0)$  with regularity  $\overline{k} \in C^{\infty}((0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2)$ . Thus, the lemma is proved.

We now turn to study the solution  $(\xi_t, \eta_t)$  of (2.9) with an initial random variable  $(\xi_0, \eta_0)$ whose distribution is absolutely continuous with the density  $v(\xi, \eta)$ . From Lemma 2.7, for any t > 0,  $(\xi_t, \eta_t)$  has a density  $u(t, \xi, \eta)$  satisfying the Fokker-Planck equation (2.13). Further,

$$u(t,\xi,\eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(t,\xi,\eta,\varsigma_1,\varsigma_2) v(\varsigma_1,\varsigma_2) d\varsigma_1 d\varsigma_2.$$

As in [14] we see that system (2.9) generates a Markov semigroup. For any  $t \ge 0$ , we define the operator  $P(t): \mathcal{D} \to \mathcal{D}$  by

$$P(t)v(\xi,\eta) = u(t,\xi,\eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k(t,\xi,\eta,\varsigma_1,\varsigma_2)v(\varsigma_1,\varsigma_2)d\varsigma_1d\varsigma_2.$$

Using Lemma 2.7 and an analytic prolongation, we know that the family of operators  $\{P(t)\}_{t\geq 0}$  on  $L^1(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mathbf{m})$  becomes an integral Markov semigroup with a continuous kernel k. The positivity of k is equivalent to that of  $\overline{k}$  which is defined in the proof of Lemma 2.7. To verify this positivity we use the method based on support theorems (see Aida et al. [2], Ben Arous and Leandre [3], Stroock and Varadhan [15]). Fix a point  $(\beta_0, \gamma_0) \in \mathbb{R}^2$  and a function  $\phi \in L^2([0, T]; \mathbb{R}), T > 0$ . Consider the following system of differential equations:

(2.14) 
$$\begin{cases} \beta_{\phi}(t) = \beta_0 + \int_0^t [\phi(s) + g_1(\beta_{\phi}(s), \gamma_{\phi}(s))] ds, \\ \gamma_{\phi}(t) = \gamma_0 + \int_0^t g_2(\beta_{\phi}(s), \gamma_{\phi}(s)) ds. \end{cases}$$

Let  $(\beta_{\phi}, \gamma_{\phi})$  be a solution of (2.14) with the initial condition  $\beta_{\phi}(0) = \beta_0, \gamma_{\phi}(0) = \gamma_0$  and let  $F: L^2([0, T], \mathbb{R}) \to \mathbb{R}^2$  be a mapping defined by  $F(\psi) = (\beta_{\phi+\psi}(T), \gamma_{\phi+\psi}(T))$ . Denote by  $D_{\beta_0,\gamma_0,\phi}$  the Fréchet derivative of F. It is known that if Rank  $(D_{\beta_0,\gamma_0,\phi}) = 2$ then  $\overline{k}(T, \overline{\beta}, \overline{\gamma}, \beta_0, \gamma_0) > 0$  where  $(\overline{\beta}, \overline{\gamma}) = (\beta_{\phi}(T), \gamma_{\phi}(T))$ . On the space  $C([0, T], \mathbb{R})$  of all continuous functions from [0, T] to  $\mathbb{R}$ ,  $D_{\beta_0,\gamma_0,\phi}$  can be calculated by means of the perturbation method for ordinary differential equations as follows. Let  $\mathbf{f} = (g_1, g_2)$  and  $\Lambda(t) = \mathbf{f}'(\beta_{\phi}(t), \gamma_{\phi}(t))$ . Denote by  $R(t, s), 0 \leq s \leq t \leq T$ , the fundamental matrix of solutions of the equation

$$\dot{Z} = \Lambda(t)Z,$$

i.e.,  $\frac{\partial R(t,s)}{\partial t} = \Lambda(t)R(t,s)$  and R(s,s) = I. Then,

$$D_{\beta_0,\gamma_0,\phi}\psi = \int_0^T R(T,s)a_1\psi(s)ds,$$

where  $a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ . Let  $\epsilon \in (0,T)$  and put  $\psi(t) = 0$  if  $0 \leq t \leq T - \epsilon$  and  $\psi(t) = \frac{1}{\epsilon}(t - T + \epsilon)$  if  $T - \epsilon \leq t \leq T$ . By the Taylor formula we have  $R(T,s) = I - \Lambda(T)(T-s) + o(T-s)$  as

 $s \to T$ . Thus,

$$D_{\beta_0,\gamma_0,\phi}\psi = \int_{T-\epsilon}^T \frac{s-T+\epsilon}{\epsilon} a_1 ds + \Lambda(T) \int_{T-\epsilon}^T \frac{s-T+\epsilon}{\epsilon} (s-T) a_1 ds + o(\epsilon^2)$$
$$= \frac{\epsilon}{2} a_1 - \frac{\epsilon^2}{6} \Lambda(T) a_1 + o(\epsilon^2).$$

Since  $\Lambda(t) = \begin{bmatrix} \frac{\partial g_1}{\partial \beta} & \frac{\partial g_1}{\partial \gamma} \\ \frac{\partial g_2}{\partial \beta} & \frac{\partial g_2}{\partial \gamma} \end{bmatrix}$ , it follows that  $\Lambda(T)a_1 = \begin{bmatrix} \frac{\partial g_1(\overline{\beta},\overline{\gamma})}{\partial \beta} \\ \frac{\partial g_2(\overline{\beta},\overline{\gamma})}{\partial \beta} \end{bmatrix}$  where  $\overline{\beta} = \beta_{\phi}(T), \overline{\gamma} = \gamma_{\phi}(T)$ . If the vectors  $a_1$  and  $\Lambda(T)a_1$  are linearly independent, then Rank  $(D_{x_0,y_0,\phi}) = 2$ . In the case

where  $a_1$  and  $\Lambda(T)a_1$  are linearly independent, then  $\operatorname{Kan}(D_{x_0,y_0,\phi}) = 2$ . In the case

(2.15) 
$$\frac{\partial g_2(\overline{\beta},\overline{\gamma})}{\partial \beta} = 0$$

Denote by G the set of all solutions of (2.15), then  $\mathbf{m}(G) = 0$ . Indeed, put  $g(\beta, \gamma) = \frac{\partial g_2}{\partial \beta}$ ,  $g_i(\beta, \gamma) = \frac{\partial^i g(\beta, \gamma)}{\partial \beta^{i_1} \partial \gamma^{i_2}}$ , where  $i = (i_1, i_2)$  is a two-dimensional multi-index and denote  $S_i = \{(\beta, \gamma) : g_i \neq 0\}$ , then  $S_i$  is open set. Given a two-dimensional multi-index i, we assume without loss of generality that

$$g_i(\beta,\gamma) = rac{\partial g_{i^*}(\beta,\gamma)}{\partial eta},$$

where  $i^*$  is other two-dimensional multi-index with  $|i^*| < |i|$ . Since  $\frac{\partial g_{i^*}(\beta,\gamma)}{\partial \beta} \neq 0$  on  $S_i$ , it follows from the implicit function theorem that

$$g_{i^*}(\beta, \gamma) \neq 0,$$
 m-a.e.  $(\beta, \gamma)$  on  $S_i$ .

Denote by  $S_{i^*}^1$  the zero set of  $g_{i^*}$  in  $S_i$ , then

- $\mathbf{m}(S_{i^*}^1) = 0;$
- $S_i \setminus S_{i^*}^1$  is open and on which  $g_{i^*} \neq 0$ .

We now consider the function  $g_{i^*}$  on  $S_i \setminus S_{i^*}^1$ . Iteration of the above process leads to  $g(\beta, \gamma) \neq 0$ , **m**-a.e.  $(\beta, \gamma)$  on  $S_i$ . Thus,

$$g(\beta, \gamma) \neq 0$$
, **m**-a.e.  $(\beta, \gamma)$  on  $\cup_i S_i$ .

If there exists  $(\beta_0, \gamma_0) \in \mathbb{R}^2 \setminus \bigcup_i S_i$ , then  $g_i(\beta_0, \gamma_0) = 0$  for all two-dimensional multi-index *i*. It then follows that  $g(\beta, \gamma)$  is constant on  $\mathbb{R}^2$ . Hence,  $g_2(\beta, \gamma)$  has the following form on  $\mathbb{R}^2$ :

$$g_2(\beta, \gamma) = c\beta + \overline{g}(\gamma), \qquad c \in \mathbb{R}.$$

This leads to a contradiction, because  $g_2(\beta, \gamma)$  tends to  $-\infty$  for every  $\gamma$  when  $\beta$  tends to  $\infty$  or  $-\infty$ . Therefore,  $\mathbb{R}^2 \setminus \bigcup_i S_i = \emptyset$  and then  $\mathbf{m}(G) = 0$ .

Summing up these results yields

**Lemma 2.8.**  $\overline{k}(T,\overline{\beta},\overline{\gamma},\beta_0,\gamma_0) > 0$  if  $(\overline{\beta},\overline{\gamma}) = (\beta_{\phi}(T),\gamma_{\phi}(T)) \notin G$ .

We are now in a position to consider the controllability of system (2.14) in the space  $C([0, T], \mathbb{R})$ . System (2.14) becomes

(2.16) 
$$\begin{cases} \beta'_{\phi}(t) = \phi(t) + g_1(\beta_{\phi}(t), \gamma_{\phi}(t)), \\ \gamma'_{\phi}(t) = g_2(\beta_{\phi}(t), \gamma_{\phi}(t)). \end{cases}$$

We then have the following claims.

**Claim 1.** Let  $\gamma_0 > \gamma_1$ . Since  $\lim_{\beta \to \infty} \rho_1^{-1}(\beta + \gamma_0) = \lim_{\beta \to \infty} \rho_2^{-1}(\beta) = \infty$ , it is easy to see that  $\lim_{\beta \to \infty} g_2(\beta, \gamma) = -\infty$  uniformly in  $\gamma \in [\gamma_1, \gamma_0]$ . Then, there exists  $\beta_0$  such that  $g_2(\beta_0, \gamma) \leq -1$  for any  $\gamma \in [\gamma_1, \gamma_0]$ . Choose  $\phi(t) = -g_1(\beta_0, \gamma(t))$ , where  $\gamma(t)$  is a solution of the following equation

$$\begin{cases} \gamma'(t) = g_2(\beta_0, \gamma(t)), \\ \gamma(0) = \gamma_0. \end{cases}$$

Then, system (2.16) has a solution  $(\beta_{\phi}(t), \gamma_{\phi}(t)) = (\beta_0, \gamma(t))$  and  $\gamma_{\phi}(0) = \gamma_0$ . Since  $\gamma'_{\phi} = g_2(\beta_0, \gamma_{\phi}) \leq -1$  whenever  $\gamma_{\phi} \in [\gamma_1, \gamma_0]$ , we can find T > 0 such that  $\gamma_{\phi}(T) = \gamma_1$ .

**Claim 2.** Let  $\gamma_0 < \gamma_1$ . Since  $\lim_{\beta \to -\infty} \rho_1^{-1}(\beta + \gamma_0) = \lim_{\beta \to -\infty} \rho_2^{-1}(\beta) = -\infty$ , it is easy to see that  $\lim_{\beta \to -\infty} g_2(\beta, \gamma) = \infty$  uniformly in  $\gamma \in [\gamma_0, \gamma_1]$ . Similarly to Claim 1, there exist  $\beta_0$ , a function  $\phi(t)$  and T > 0 such that (2.16) has a solution  $(\beta_{\phi}(t), \gamma_{\phi}(t)) = (\beta_0, \gamma(t)), \gamma_{\phi}(0) = \gamma_0$  and  $\gamma_{\phi}(T) = \gamma_1$ .

Claim 3. Fix  $y_0 \in \mathbb{R}, L > 0, A_1 > A_0$  and  $\epsilon > 0$  such that  $\epsilon < \min\{\frac{L}{4}, \frac{A_1 - A_0}{4}\}$ . Let

$$m^* = \max\{|g_1(\beta,\gamma)| + |g_2(\beta,\gamma)| : (\beta,\gamma) \in [\beta_0,\beta_0+L] \times [A_0,A_1]\},\$$

 $t_0 = \frac{\epsilon}{m^*}$  and  $\phi = \frac{3m^*L}{4\epsilon}$ . For every  $\gamma_0 \in [A_0 + \epsilon, A_1 - \epsilon]$  the solution of system (2.16) with  $(\beta_{\phi}(0), \gamma_{\phi}(0)) = (\beta_0, \gamma_0)$  satisfies

(2.17) 
$$\begin{cases} \beta_{\phi}(t_0) \in \left(\beta_0 + \frac{L}{2}, \beta_0 + L\right), \\ \gamma_{\phi}(t) \in [\gamma_0 - \epsilon, \gamma_0 + \epsilon], t \in [0, t_0]. \end{cases}$$

Indeed, it follows from system (2.16) that

$$\begin{cases} \beta'_{\phi}(t) \geqslant \phi - m^* > \frac{m^*L}{2\epsilon} > 0, \\ \beta'_{\phi}(t) \leqslant \phi + m^* = m^* \left(\frac{3L}{4\epsilon} + 1\right) \end{cases}$$

Therefore, for every  $0 \leq s \leq t \leq t_0$ , we have

$$\begin{cases} \beta_{\phi}(t_0) > \beta_0 + \frac{m^*L}{2\epsilon} t_0 = \beta_0 + \frac{L}{2}, \\ \beta_{\phi}(t) > \beta_{\phi}(s), \\ \beta_{\phi}(t_0) \leqslant m^* t_0(\frac{3L}{4\epsilon} + 1) + \beta_0 < \beta_0 + L, \end{cases}$$

and

$$|\gamma_{\phi}(t) - \gamma_{\phi}(0)| = \left| \int_{0}^{t} g_{2}(\beta_{\phi}(s), \gamma_{\phi}(s)) ds \right| \leqslant \int_{0}^{t_{0}} m^{*} ds = m^{*} t_{0} = \epsilon.$$

It now follows from (2.17) that for  $(\beta_1, \gamma_1) \in (\beta_0, \beta_0 + \frac{L}{2}] \times [A_0 + 2\epsilon, A_1 - 2\epsilon]$  there exists  $\gamma_0 \in [\gamma_1 - \epsilon, \gamma_1 + \epsilon]$  and  $T \in (0, t_0)$  such that  $\beta_{\phi}(T) = \beta_1$  and  $\gamma_{\phi}(T) = \gamma_1$ . The same proof works for  $\beta_1 \in (\beta_0 - \frac{L}{2}, \beta_0]$  and gives that: for all  $(\beta_1, \gamma_1) \in (\beta_0 - \frac{L}{2}, \beta_0 + \frac{L}{2}) \times [A_0 + 2\epsilon, A_1 - 2\epsilon]$ , there exist  $\gamma_0 \in [\gamma_1 - \epsilon, \gamma_1 + \epsilon]$  and  $T \in (0, t_0)$  such that  $\beta_{\phi}(T) = \beta_1, \gamma_{\phi}(T) = \gamma_1$ .

Summing up these claims, we get

**Lemma 2.9.** System (2.16) is controllable in  $\mathbb{R}^2$  by piecewise continuous controls, i.e., for any  $(\beta_0, \gamma_0), (\beta_1, \gamma_1) \in \mathbb{R}^2$  there exist a piecewise continuous control function  $\phi$  and T > 0such that  $(\beta_{\phi}(0), \gamma_{\phi}(0)) = (\beta_0, \gamma_0)$  and  $(\beta_{\phi}(T), \gamma_{\phi}(T)) = (\beta_1, \gamma_1)$ .

Combining two Lemmas 2.8-2.9 yields

**Lemma 2.10.** Let  $(\xi^0, \eta^0), (\beta^0, \gamma^0) \in \mathbb{R}^2$  be given.

- For almost every  $(\beta, \gamma) \in \mathbb{R}^2$ , there exists  $\overline{T} > 0$  such that  $\overline{k}(\overline{T}, \beta, \gamma, \beta^0, \gamma^0) > 0$ .
- For almost every  $(\xi, \eta) \in \mathbb{R}^2$ , there exists T > 0 such that  $k(T, \xi, \eta, \xi^0, \eta^0) > 0$ .

We have the following theorem.

**Theorem 2.11.** Under Hypothesis 2.6, let  $(x_t, y_t)$  be the solution of (1.2) with initial value  $(x_0, y_0) \in \mathbb{S}$ . Then for every t > 0,  $(x_t, y_t)$  has a density  $\varphi(t, x, y)$  satisfying the Fokker-Planck equation  $L^*\varphi(t, x, y) = 0$ , where  $L^*$  is the adjoint operator of L in (1.3) which is given by

$$\begin{split} L^*\varphi(t,x,y) &= -\frac{\partial\varphi}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma_1^2(x)x^2(1-x)^2\varphi] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma_2^2(y)y^2(1-y)^2\varphi] \\ &+ \frac{\partial^2}{\partial x\partial y}[\sigma_1(x)\sigma_2(y)xy(1-x)(1-y)\varphi] - \frac{\partial}{\partial x}[\{-ae^{-by^2(3-2y)}x - cx + h(1-x)\}\varphi] \\ &- \frac{\partial}{\partial y}[qy(x+y-1)(1-y)\varphi]. \end{split}$$

Furthermore, for any compact set  $K \subset \mathbb{S}$ ,  $\lim_{t\to\infty} \mathbb{P}\{(x_t, y_t) \in K\} = 0$ .

*Proof.* By virtue of Lemma 2.7,  $\{P(t)\}_{t\geq 0}$  is an integral Markov semigroup with a continuous kernel  $k(t, \xi, \eta, \xi_0, \eta_0)$  for t > 0. Then, the distribution of  $(\xi_t, \eta_t)$  has a density  $u(t, \xi, \eta)$  satisfying the Fokker-Planck equation (2.13). This implies the first statement of theorem. Next, using Lemma 2.10, we have for  $v \in \mathcal{D}$ 

$$\int_0^\infty P(t)vdt > 0, \qquad \text{a.e. on } \mathbb{R}^2.$$

Therefore, from Theorem 2.5,  $\{P(t)\}_{t\geq 0}$  has the Foguel alternative, i.e., one of two following assertions holds true:

- (A)  $\{P(t)\}_{t\geq 0}$  is asymptotically stable on  $\mathbb{R}^2$ .
- (B)  $\{P(t)\}_{t\geq 0}$  is sweeping with respect to compact sets.

We shall show that the assertion (A) is not the case. Indeed, if (A) takes place then there exists a unique stationary density  $u_*(\xi, \eta)$  of the Fokker-Planck equation (2.13) such that

$$\lim_{t \to \infty} \iint_{\mathbb{R}^2} |u(t,\xi,\eta) - u_*(\xi,\eta)| d\xi d\eta = 0.$$

As a consequence,  $\lim_{t\to\infty} \mathbb{P}\{(\xi_t,\eta_t)\in K_1\} = \iint_{K_1} u_*(\xi,\eta)d\xi d\eta$  for any  $K_1\in \mathcal{B}(\mathbb{R}^2)$ . We choose a compact set  $K_1$  satisfying  $\iint_{K_1} u_*(\xi,\eta)d\xi d\eta > 1-\varrho_1$ . On the other hand, from Theorem 2.3 we have  $\mathbb{P}[\lim_{t\to\infty} \eta_t = \infty] = \varrho_1$ . Therefore,  $1 \ge \mathbb{P}[\lim_{t\to\infty} \eta_t = \infty] + \lim_{t\to\infty} \mathbb{P}\{(\xi_t,\eta_t)\in K_1\} > \varrho_1 + (1-\varrho_1) = 1$ . This contradiction implies that (B) must hold. It means that  $\lim_{t\to\infty} \mathbb{P}\{(\xi_t,\eta_t)\in K_1\} = 0$  for any compact set  $K_1 \subset \mathbb{R}$ . Therefore,  $\lim_{t\to\infty} \mathbb{P}\{(x_t,y_t)\in K\} = 0$  for any compact set  $K \subset \mathbb{S}$ . Thus, the proof is complete.  $\Box$  **3** Numerical examples and conclusion In this section, we present numerical examples. Set  $a = 4, b = 5, c = 0.2, h = 1, q = 1, (x_0, y_0) = (0.8, 0.45)$ , and  $(\sigma_1, \sigma_2) = (0.5, 0.2)$ . In the following figures,  $d_1$  and  $d_2$  denote, respectively, the lines  $x = \frac{h}{a+c+h}$  and  $x = \frac{h}{ae^{-b}+c+h}$ . In Figure 1(a),  $y_t \to 1$  as  $t \to \infty$ . It follows that  $\frac{1}{t} \int_0^t x_s ds \to \frac{h}{ae^{-b}+c+h}$  as  $t \to \infty$ , see Figure 1(b). In this case, the trajectory of  $x_t$  oscillates around  $d_2$ . Similarly, in Figure 2(a), the trajectory of  $y_t$  approaches to 0 and in Figure 2(b) one of  $x_t$  oscillates around  $d_1$  as  $t \to \infty$ . The model (1.2) is sweeping with respect to compact sets.

In conclusion, we recall that this work provides some results about the asymptotic behavior of a stochastic adsorbate-induced phase transition model (1.2). The mathematical analysis shows that the square  $(0, 1) \times (0, 1)$  is an invariant set of this model and the model is sweeping with respect to compact sets of  $(0, 1) \times (0, 1)$ . Furthermore, we show that  $y_t \to 0, y_t \to 1, \frac{1}{t} \int_0^t x_s ds \to \frac{h}{ae^{-b}+c+h}$ , and  $\frac{1}{t} \int_0^t x_s ds \to \frac{h}{a+c+h}$  as  $t \to \infty$  with positive probability. These results agree to those of Hildebrand et al. [6, 7] where they confirm that  $(\frac{h}{ae^{-b}+c+h}, 0)$  and  $(\frac{h}{ae^{-b}+c+h}, 1)$  are stable homogeneous stationary solutions.



2 1 х 0.8 d1 1.5 0.6 1 × 0.4 0.5 0.2 0 0 5 10 15 10 20 0 0 30 t t (b) (a)

Figure 1: Case when  $y_t \to 1$  and  $\frac{1}{t} \int_0^t x_s ds \to \frac{h}{ae^{-b} + c + h}$ 

Figure 2: Case when  $y_t \to 0$  and  $\frac{1}{t} \int_0^t x_s ds \to \frac{h}{a+c+h}$ 

## References

- L. Arnold, Stochastic Differential Equations: Theory and Applications, Wiley, New York, 1972.
- [2] S. Aida, S. Kusuoka, D. Stroock, On the support of Wiener functionals, in Asymptotic Problems in Probability Theory: Wiener functionals and Asymptotics (K. D. Elworthy and N. Ikeda, eds.), Longman, New York, 1993, 3-34.
- [3] G. Ben Arous, R. Leandre, Exponential decay of the heat kernel on the diagonal, Probab. Theory Related Fields 90 (1991), 175-202.
- [4] G. Ertl, Oscillatory kinetics and spatio-temporal self-organization in reactions at solid surfaces, Science 254 (1991), 1750-1755.
- [5] A. Friedman, Stochastic Differential Equations and their Applications, Academic press, New York, 1976.
- [6] M. Hildebrand, M. Kuperman, H. Wio, A. S. Mikhailov, G. Ertl, Self-organized chemical nanoscale microreactors, Phys. Rev. Lett. 83 (1999), 1475-1478.
- [7] M. Hildebrand, M. Ipsen, A. S. Mikhailov, G. Ertl, Localized nonequilibrium nanostructures in surface chemical reactions, New J. Phys. 5 (2003), 61.1-61.28.
- [8] N. Ikeda, S. Wantanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland, Amsterdam, 1981.
- [9] S. Jakubith, H. H. Rotermund, W. Engel, A. V. Oertzen, G. Ertl, Spatiotemporal concentration patterns in a surface reaction: Propagating and standing waves, rotating spirals, and turbulence, Phys. Rev. Lett. 65 (1990), 3013-3016.
- [10] I. Karatzas, S. E. Shreve, Brownian Motion and Stochastic Calculus, Springer-Verlag, Berlin, 1991.
- [11] P. Malliavin, C<sup>k</sup> hypoellipticity with degeneracy, Stochastic Analysis (A. Friedman and M. Pinsky, eds.), Academic Press, New York, 1978, 199-214.
- [12] J. Norris, Simplified Malliavin calculus, Lecture Notes in Math. 1204, Springer-Verlag, Berlin, 1986, 101-130.
- [13] D. Nualart, Malliavin calculus and its applications, CBMS Regional Conference Series in Math. 110, Amer. Math. Soc., Providence, 2009, 1-85.
- [14] R. Rudnicki, Long-time behaviour of a stochastic prey-predator model, Stochastic processes and their applications 108 (2003), 93-107.
- [15] D. W. Stroock, S. R. S. Varadhan, On the support of diffusion processes with applications to the strong maximum principle, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability Vol. 3, Univ. of Cal. Press, Berkeley, 1972, 333-359.
- [16] Y. Takei, Adsorbate-induced phase transition model and pattern formation, Doctoral thesis, Osaka University, 2004 (in Japanese).
- [17] Y. Takei, T. Tsujikawa, A. Yagi, Numerical computation and pattern formation for adsorbateinduced phase transition model, Sci. Math. Jpn. 61 (2005), 525-534.
- [18] A. Yagi, Abstract Parabolic Evolution Equations and their Applications, Springer-Verlag, Berlin, 2009.

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