DERIVATIONS OF WEAK BCC-ALGEBRAS

JANUS THOMYS

Received November 9, 2010

ABSTRACT. We describe derivations of weak BCC-algebras (called also BZ-algebras) in which the condition $(xy)z = (xz)y$ is satisfied only in the case when elements x, y belong to the same branch.

1. INTRODUCTION

BCC-algebras were introduced by Y.Komori [9] as a generalization of BCK-algebras. In view of strongly connections with a BIK$^+$-logic, BCC-algebras are also called BIK$^+$-algebras (cf. [12] or [13]). Nowadays, many mathematicians, especially from China, Japan and Korea, have been studying various generalizations of BCC-algebras. All these algebras have one distinguished element and satisfy some common identities playing a crucial role in these algebras.

One of very important identities is the identity $(xy)z = (xz)y$. It holds in BCK-algebras and in some generalizations of BCK-algebras, but not in BCC-algebras. BCC-algebras satisfying this identity are BCK-algebras (cf. [2] or [3]). Therefore, it makes sense to consider such BCC-algebras and some of their generalizations for which this identity is satisfied only by elements belonging to some subsets. Such study has been initiated by W.A. Dudek in [4].

In this paper we will study derivations of weak BCC-algebras in which the condition $(xy)z = (xz)y$ is satisfied only in the case when elements x, y belong to the same branch.

2. PRELIMINARIES

The BCC-operation will be denoted by juxtaposition. Dots will be used only to avoid repetitions of brackets. For example, the formula $((xy)(zy))(xz) = 0$ will be written in the abbreviated form as $(xy \cdot zy) \cdot xz = 0$.

Definition 2.1. A *weak BCC-algebra* is a system $(G; \cdot, 0)$ of type $(2, 0)$ satisfying the following axioms:

(i) $(xy \cdot zy) \cdot xz = 0$,
(ii) $xx = 0$,
(iii) $x0 = x$,
(iv) $xy = yx = 0 \Longrightarrow x = y$.

Weak BCC-algebras are called *BZ-algebras* by many mathematicians, especially from China and Korea (cf. [6], [11], [14] or [15]), but we save the first name because it coincides with the general concept of names presented in the book [7] for algebras of logic.

A weak BCC-algebra satisfying the identity

(v) $0x = 0$

is called a *BCC-algebra.* A BCC-algebra with the condition

2000 *Mathematics Subject Classification.* 03G25, 06F35.
Key Words and Phrases. weak BCC-algebra; solid weak BCC-algebra; BZ-algebra; derivation.

(vi) $(x \cdot xy)y = 0$

is called a *BCK-algebra*.

One can prove (see [2] or [3]) that *a BCC-algebra is a BCK-algebra if and only if it satisfies the identity*

(vii) $xy \cdot z = xz \cdot y$.

An algebra $(G; \cdot, 0)$ of type $(2, 0)$ satisfying the axioms (i), (ii), (iii), (iv) and (vi) is called a *BCI-algebra*. A weak BCC-algebra is a BCI-algebra if and only if it satisfies (vii).

In any weak BCC-algebra we can define a natural partial order $\leqslant$ by putting

$$x \leqslant y \Longleftrightarrow xy = 0.$$

Directly from the axioms of weak BCC-algebras we can see that the following two implications

$(viii)$ $x \leqslant y \Longrightarrow xz \leqslant yz$,

(ix) $x \leqslant y \Longrightarrow zy \leqslant zx$

are valid for all $x, y, z \in G$.

The set of all minimal (with respect to $\leqslant$) elements of G is denoted by $I(G)$. Elements belonging to $I(G)$ are called *initial*.

In the investigation of algebras connected with various types of logics an important role plays the so-called *Dudek's map* φ defined as $\varphi(x) = 0x$. The main properties of this map in the case of weak BCC-algebras are collected in the following theorem proved in [6].

Theorem 2.2. *Let G be a weak BCC-algebra. Then*

(1) $\varphi^2(x) \leqslant x$,

(2) $x \leqslant y \Longrightarrow \varphi(x) = \varphi(y)$,

(3) $\varphi^3(x) = \varphi(x)$,

(4) $\varphi^2(xy) = \varphi^2(x)\varphi^2(y)$,

for all $x, y \in G$. □

Theorem 2.3. $I(G) = \{a \in G : \varphi^2(a) = a\}$. □

The proof of this theorem is given in [5]. Comparing this result with Theorem 2.2 (4) we obtain

Corollary 2.4. *$I(G)$ is a subalgebra of G.* □

Corollary 2.5. *$I(G) = \varphi(G)$ for any weak BCC-algebra G.* □

The set

$$B(a) = \{x \in G : a \leqslant x\},$$

where $a \in I(G)$ is called a *branch* of G initiated by a. The branch initiated by 0 is the greatest BCC-algebra contained in G.

Definition 2.6. A weak BCC-algebra G is called *solid* if (vii) is valid for all x, y belonging to the same branch and arbitrary $z \in G$.

Such weak BCC-algebras were introduced in [4].

Definition 2.7. A non-empty subset A of a weak BCC-algebra G is called a *BCC-ideal* if

(1) $0 \in A$,

(2) $y \in A$ and $xy \cdot z \in A$ imply $xz \in A$.

Using $(viii)$ and (ix) it is not difficult to see that $B(0)$ is a BCC-ideal of each weak BCC-algebra. The relation $\sim$ defined by

$$x \sim y \iff xy,\, yx \in B(0)$$

is a congruence on G. Its equivalence classes coincide with branches of G, i.e., $B(a) = C_a$ for any $a \in I(G)$ (cf. [5]). So, $B(a)B(b) = B(ab)$ and $xy \in B(ab)$ for $x \in B(a)$, $y \in B(b)$.

In the following part of this paper, we will need those two propositions proved in [5].

Proposition 2.8. *Elements $x, y \in G$ are in the same branch if and only if $xy \in B(0)$.* □

Proposition 2.9. *If $x, y \in B(a)$, then also $x \cdot xy$ and $y \cdot yx$ are in $B(a)$.* □

One of important classes of weak BCC-algebras is the class of *group-like weak BCC-algebras* called also *anti-grouped BZ-algebras* [14], i.e., weak BCC-algebras containing only one-element branches. A special case of such algebras are group-like BCI-algebras described in [1].

The conditions under which a weak BCC-algebra is group-like are found in [5] and [14]. Below we present some of these conditions.

Theorem 2.10. *A weak BCC-algebra G is group-like if and only if at least one of the following conditions is satisfied:*

(1) $\varphi^2(x) = x$ *for all* $x \in G$,
(2) $\varphi(xy) = yx$ *for all* $x, y \in G$,
(3) $xy \cdot zy = xz$ *for all* $x, y, z \in G$,
(4) $\operatorname{Ker}\varphi = \{0\}$. □

3. Derivations of weak BCC-algebras

In the theory of rings, the properties of derivations play an important role. In [8] Jun and Xin applied the notion of derivations in ring and near-ring theory to BCI-algebras, and they also introduced a new concept called a regular derivation in BCI-algebras. In [10] Ch. Prabpayak and U. Leerawat applied the notion of regular derivation in BCI-algebras to BCC-algebras. Here we give some results for derivations in solid weak BCC-algebras.

Definition 3.1. Let G be a weak BCC-algebra. A map $d : G \longrightarrow G$ is called a *left-right derivation* (briefly, *(l, r)-derivation*) of G, if it satisfies the identity

$$d(xy) = d(x)y \wedge xd(y), \tag{1}$$

where $x \wedge y$ means $y \cdot yx$.

If d satisfies the identity

$$d(xy) = xd(y) \wedge d(x)y, \tag{2}$$

then it is called a *right-left derivation* (briefly, *(r, l)-derivation*) of G. A map d which is both a (l, r)- and a (r, l)-derivation is called a *derivation.* Any d with the property $d(0) = 0$ is called *regular.*

Example 3.2. Let $G = \{0, 1, 2, 3\}$ be a weak BCC-algebra with the operation $\cdot$ defined as follows:

$\cdot$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

Table 3.1.

Consider two maps d_1, $d_2 : G \longrightarrow G$ defined by

$$d_1(x) = \begin{cases} 0 & if \quad x \in \{0,1,3\}, \\ 2 & if \quad x = 2, \end{cases}$$

$$d_2(x) = \begin{cases} 0 & if \quad x \in \{0,1\}, \\ 2 & if \quad x \in \{2,3\}. \end{cases}$$

Then it can be easily checked that d_1 is both a (l,r)- and (r,l)-derivation of G and d_2 is a (r,l)-derivation but not a (l,r)-derivation. □

Example 3.3. Let $G = \{0,1,2,3,4,5\}$ be a weak BCC-algebra with the operation $\cdot$ defined as follows:

$\cdot$	0	1	2	3	4	5
0	0	0	2	2	2	2
1	1	0	2	2	2	2
2	2	2	0	0	0	0
3	3	2	1	0	0	0
4	4	2	1	1	0	1
5	5	2	1	1	1	0

Table 3.2.

Define a map $d : G \longrightarrow G$ by

$$d_1(x) = \begin{cases} 2 & if \quad x \in \{0,1\}, \\ 0 & if \quad x \in \{2,3,4,5\}. \end{cases}$$

Then it is easily checked that d is a non-regular derivation of G. □

Theorem 3.4. *The endomorphism φ^2, where φ is the Dudek's map, is a regular derivation of each solid weak BCC-algebra.*

Proof. From Theorem 2.2 it follows that the map $d(x) = \varphi^2(x)$ is a regular endomorphism and $d(x) = a$ for all $x \in B(a)$. Thus for $x \in B(a)$, $y \in B(b)$, according to Proposition 2.9, we have $d(xy) = ab$.

On the other hand,

$$d(x)y \wedge xd(y) = ay \wedge xb = xb \cdot (xb \cdot ay).$$

Since, by the assumption, a weak BCC-algebra G is solid and elements xb, ay are in the same branch, we have

$$(xb \cdot (xb \cdot ay)) \cdot ay = (xb \cdot ay)(xb \cdot ay) = 0,$$

which means that

$$d(x)y \wedge xd(y) = xb \cdot (xb \cdot ay) \leqslant ay \leqslant ab,$$

by (ix). Thus $d(x)y \wedge xd(y) \leqslant ab$. This, in view of Corollary 2.4, gives $d(x)y \wedge xd(y) = ab$. Hence $d(x)y \wedge xd(y) = ab = d(xy)$.

Now,

$$xd(y) \wedge d(x)y = xb \wedge ay = ay \cdot (ay \cdot xb)$$

Since $ay, ab \in B(ab)$ we have $(ay \cdot (ay \cdot xb)) \cdot ab = (ay \cdot ab)(ay \cdot xb) \leqslant by \cdot (ay \cdot xb)$ from (i) and $(viii)$. But $by = 0$ and we have $0 \cdot (ay \cdot xb) = 0$, because $ay \cdot xb \in B(0)$. So, $xd(y) \wedge d(x)y = ab = d(xy)$. Therefore d is a regular derivation of G. □

Corollary 3.5. *A map $d : G \longrightarrow G$ such that $d(x) = a$ for all $x \in B(a)$, is a regular derivation of each solid weak BCC-algebra.* □

Theorem 3.6. *For any derivation d of a solid weak BCC-algebra G elements x and $d(d(x))$ are in the same branch.*

Proof. Indeed, let $x \in B(a)$ and $y = d(x)$. Then from (1) we obtain

$$d(xy) = d(x)y \wedge xd(y) = yy \wedge xd(y) = 0 \wedge xd(y) = xd(y) \cdot (xd(y) \cdot 0) = 0.$$

Thus $d(xy) = 0$ for $y = d(x)$. This together with (2) and Theorem 2.2 gives

$$0 = d(xy) = xd(y) \wedge d(x)y = xd(y) \wedge 0 = 0(0 \cdot xd(y)) \leqslant xd(y).$$

Hence $xd(y) \in B(0)$, which shows (by Proposition 2.8) that elements x and $d(y) = d(d(x))$ are in the same branch. □

Example 3.3 shows that x and $d(x)$ may not be in the same branch if d is not regular.

Theorem 3.7. *A (l, r)-derivation d of a solid weak BCC-algebra G is regular if and only if for every $x \in G$ elements x and $d(x)$ belongs to the same branch.*

Proof. Let d be a regular (l, r)-derivation of a solid weak BCC-algebra G. Then for any $x \in G$ we have

$$0 = d(xx) = d(x)x \wedge xd(x) = xd(x) \cdot (xd(x) \cdot d(x)x),$$

which implies $xd(x) \leqslant xd(x) \cdot d(x)x$. From this, by $(viii)$, we obtain

$$0 = xd(x) \cdot xd(x) \leqslant (xd(x) \cdot d(x)x) \cdot xd(x) = (xd(x) \cdot xd(x)) \cdot d(x)x = 0 \cdot d(x)x.$$

So, $d(x)x \in B(0)$. This, according to Proposition 2.8, shows that x and $d(x)$ are in the same branch.

Conversely, if for every $x \in G$ elements x and $d(x)$ are in the same branch $B(a)$, then also $d(x), d(d(x)) \in B(a)$. So, $xd(x)$, $d(x)x$ and $d(d(x))x$ are in $B(0)$. Thus $0 = 0 \cdot d(x)x = 0 \cdot xd(x) = 0 \cdot d(d(x))x$. Hence

$$\begin{aligned}
d(0) &= d(0 \cdot d(x)x) = (d(0) \cdot d(x)x) \wedge (0 \cdot d(d(x)x)) \\
&= (d(0) \cdot d(x)x) \wedge (0 \cdot (d(d(x))x \wedge d(x)d(x))) \\
&= (d(0) \cdot d(x)x) \wedge (0 \cdot (d(d(x))x \wedge 0)) \\
&= (d(0) \cdot d(x)x) \wedge (0 \cdot 0(0 \cdot d(d(x))x)) \\
&= (d(0) \cdot d(x)x) \wedge 0 = 0(0 \cdot (d(0) \cdot d(x)x)) \\
&= \varphi^2(d(0) \cdot d(x)x) = \varphi^2(d(0)) \cdot \varphi^2(d(x)x) \\
&= \varphi^2(d(0)) \cdot 0 = \varphi^2(d(0)),
\end{aligned}$$

by Theorem 2.2. In this way we obtain $\varphi^2(d(0)) = d(0)$, which in view of Theorem 2.3 means that $d(0) \in I(G)$. Since 0 and $d(0)$ are in the same branch and $d(0) \in I(G)$, it must be $d(0) = 0$, i.e., d is regular. □

Theorem 3.8. *A (r, l)-derivation d of a solid weak BCC-algebra G is regular if and only if for every $x \in G$ elements x and $d(x)$ belong to the same branch.*

Proof. The proof is analogous to the proof of the previous theorem. □

Corollary 3.9. *A derivation d of a solid weak BCC-algebra G is regular if and only if for every $x \in G$ elements x and $d(x)$ are in the same branch.* □

Theorem 3.10. *Let d be a self-map of a weak BCC-algebra G. Then the following holds:*

(1) *if d is a regular (l, r)-derivation of G, then $d(x) = d(x) \wedge x$,*

(2) *if d is a (r, l)-derivation of G, then $d(x) = x \wedge d(x)$ for all $x \in G$ if and only if d is regular.*

Proof. (1) Let d be a regular (l, r)-derivation of G. Then for any $x \in G$ we have $d(x) = d(x0) = d(x)0 \wedge xd(0) = d(x) \wedge x$.

(2) If d is a (r, l)-derivation of G and $d(x) = x \wedge d(x)$ for all $x \in G$, then, in particular, $d(0) = 0 \wedge d(0) = 0$. Hence d is regular.

Conversely, if $d(0) = 0$, then $d(x) = d(x0) = xd(0) \wedge d(x)0 = x \wedge d(x)$. □

Corollary 3.11. *For a derivation d of solid weak BCC-algebra G the following conditions are equivalent:*

(1) *d is regular,*
(2) *$d(x) \leqslant x$ for every $x \in G$,*
(3) *$d(a) = a$ for every $a \in I(G)$.*

Proof. (1) $\Longrightarrow$ (2) For any regular derivation d from Theorem 3.10 (2) we have

$$d(x)x = (x \wedge d(x)) \cdot x = (d(x) \cdot d(x)x) \cdot x = d(x)x \cdot d(x)x = 0,$$

because elements x and $d(x)$ belong to the same branch (Corollary 3.9). This proves (2).

Implications (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are obvious. □

Corollary 3.12. *$d(B(a)) \subset B(a)$ for any regular derivation d of a solid weak BCC-algebra G.*

Proof. Let $x \in B(a)$. Since, by Theorem 3.7, x and $d(x)$ are in the same branch, we have $d(x) \in B(a)$. Thus $d(B(a)) \subset B(a)$. □

In general $d(B(a)) \neq B(a)$. A simple example of a derivation with this property is a derivation $d = \varphi^2$ used in Theorem 3.4.

Corollary 3.13. *Let d be a regular derivation of solid weak BCC-algebra G. Then:*

(1) *$d(x)y \leqslant xd(y)$,*
(2) *$d(xy) = d(x)y$*

for all $x, y \in G$.

Proof. (1) Applying $(viii)$ and (ix) to Corollary 3.11 (2) we obtain

$$d(x)y \leqslant xy \leqslant xd(y)$$

which proves (1).

(2) From the above

$$d(xy) = xd(y) \wedge d(x)y = d(x)y \cdot (d(x)y \cdot xd(y)) = d(x)y \cdot 0 = d(x)y.$$

This completes the proof. □

Theorem 3.14. *A solid weak BCC-algebra G is group-like if and only if* $\operatorname{Ker} d = \{0\}$ *for each regular derivation d of G.*

Proof. In a group-like weak BCC-algebra $B(0) = \{0\}$. Since for any $x \in \operatorname{Ker} d$ we have $d(x) = 0$ and $0 \in B(0)$, Corollary 3.9 implies $x \in B(0)$, So, $x = 0$, i.e., $\operatorname{Ker} d = \{0\}$.

Conversely, let $\operatorname{Ker} d = \{0\}$ for any regular derivation d of G. Then, in particular, for $d = \varphi^2$ (Theorem 3.4) we have $\operatorname{Ker}\varphi^2 = \{0\}$. But, as it is not difficult to see, $B(0) \subset \operatorname{Ker} \varphi^2$. Hence $B(0) = \{0\}$. Theorem 2.10 (4) completes the proof. □

Theorem 3.15. *A derivation d of a solid weak BCC-algebra G is regular if and only if $d(A) \subset A$ for all BCC-ideals A of G.*

Proof. For a regular derivation of a solid weak BCC-algebra by Theorem 3.10 (2) and Corollary 3.9 we have $d(x) = x \wedge d(x) \leqslant x$ for all $x \in G$. Let $x \in A$. Then $d(x)x = 0 \in A$, and consequently $d(x) \in A$ because A is a BCC-ideal. Hence $d(A) \subset A$ for any BCC-ideal A of G.

Conversely, if $d(A) \subset A$ for each BCC-ideal A of G, then also for $A = \{0\}$. Thus $d(\{0\}) \subset \{0\}$. Hence $d(0) = 0$, i.e., d is regular. □

References

[1] W.A. Dudek, *On group-like BCI-algebras*, Demonstratio Math., 21 (1988), 369 − 376.
[2] W.A. Dudek, *On BCC-algebras*, Logique et Analyse, 129-130 (1990), 103 − 111.
[3] W.A. Dudek, *On proper BCC-algebras*, Bull. Inst. Math. Acad. Sinica, 20 (1992), 137 − 150.
[4] W.A. Dudek, *Solid weak BCC-algebras*, Intern. J. Computer Math. (in print).
[5] W.A. Dudek, B. Karamdin, S.A. Bhatti, *Branches and ideals of weak BCC-algebras*, Algebra Coll. (2011), in print.
[6] W.A.Dudek, X.H.Zhang, Y.Q.Wang, *Ideals and atoms of BZ-algebras*, Math. Slovaca, 59 (2009), 387 − 404.
[7] A. Iorgulescu, *Algebras of logic as BCK algebras*, Acad. Econom. Studies, Bucharest, 2008.
[8] Y.B. Jun, X.L. Xin, *On derivators of BCI-algebras*, Information Sciences, 159 (2004), 167 − 176.
[9] Y. Komori, *The class of BCC-algebras is not variety*, Math. Japonica, 29 (1984), 391 − 394.
[10] Ch. Prabpayak, U. Lerrawat, *On derivations of BCC-algebras*, Kasetsart J. (Nat. Sci.), 43 (2009), 398 − 401.
[11] R.F. Ye, *On BZ-algebras*, Selected Papers on BCI, BCK-algebras and Computer Logics, Shanghai Jiaotong Univ. Press, (1991), 21 − 24.
[12] X.H. Zhang, *BIK$^+$-logic and non-commutative fuzzy logics*, Fuzzy Systems Math. 21 (2007), 31 − 36.
[13] X.H. Zhang, W.A. Dudek, *Fuzzy BIK$^+$-logic and non-commutative fuzzy logics*, Fuzzy Systems Math. 23 (2009), 9 − 20.
[14] X.H. Zhang, R. Ye, *BZ-algebras and groups*, J. Math. Phys. Sci. 29 (1995), 223 − 233.
[15] X.H. Zhang, Y.Q. Wang, W.A. Dudek, *T-ideals in BZ-algebras and T-type BZ-algebras*, Indian J. Pure Appl. Math. 34 (2003), 1559 − 1570.

Hinueberstrasse 13a, 30175 Hannover, Germany
E-mail address: janus.thomys@htp-tel.de