A NOTE ON $f$-DERIVATIONS OF SUBTRACTION ALGEBRAS

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Abstract. In this paper, we introduced the concept of $f$-derivation which is a generalization of derivation in subtraction algebra, and some related properties are investigated.

1. Introduction

B. M. Schein [2] considered systems of the form $(\Phi; \circ, \setminus)$, where $\Phi$ is a set of functions closed under the composition “$\circ$” of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction “$\setminus$” (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [4] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In this paper, we introduced the concept of $f$-derivation which is a generalization of derivation in subtraction algebra, and some related properties are investigated.

2. Preliminaries.

We first recall some basic concepts which are used to present the paper.

By a subtraction algebra we mean an algebra $(X; -)$ with a single binary operation “$-$” that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on $X$: $a \leq b \iff a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \land b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$b \lor c = (b' \land c')' = a - ((a - b) \land (a - c)) = a - ((a - b) - ((a - b) - (a - c))).$$

In a subtraction algebra, the following are true:

(p1) $(x - y) - y = x - y$.
(p2) $x - 0 = x$ and $0 - x = 0$.
(p3) $(x - y) - x = 0$.
(p4) $x - (x - y) \leq y$.
(p5) $(x - y) - (y - x) = x - y$.

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(p6) \( x - (x - (x - y)) = x - y. \)
(p7) \( (x - y) - (z - y) \leq x - z. \)
(p8) \( x \leq y \) if and only if \( x = y - w \) for some \( w \in X. \)
(p9) \( x \leq y \) implies \( x - z \leq y - z \) and \( z - y \leq z - x \) for all \( z \in X. \)
(p10) \( x, y \leq z \) implies \( x - y = x \land (z - y). \)
(p11) \( (x \land y) - (x \land z) \leq x \land (y - z). \)
(p12) \( (x - y) - z = (x - z) - (y - z). \)

A mapping \( d \) from a subtraction algebra \( X \) to a subtraction algebra \( Y \) is called a morphism if \( d(x - y) = d(x) - d(y) \) for all \( x, y \in X. \) A self map \( d \) of a subtraction algebra \( X \) which is a morphism is called an endomorphism.

**Lemma 2.1** Let \( X \) be a subtraction algebra. Then the following properties hold:

1. \( x \land y = y \land x, \) for every \( x, y \in X. \)
2. \( x - y \leq x \) for all \( x, y \in X. \)

**Lemma 2.2** Every subtraction algebra \( X \) satisfies the following property.

\[ (x - y) - (x - z) \leq z - y \]

for all \( x, y, z \in X. \)

Proof. Using (S3) and (p7), we have

\[
((x - y) - (x - z)) - (z - y) = ((x - (x - z)) - y) - (z - y) \\
\leq (x - (x - z)) - z \\
(x - z) - (x - z) = 0
\]

for all \( x, y, z \in X. \)

**Definition 2.3** Let \( X \) be a subtraction algebra and \( Y \) a non-empty set of \( X. \) Then \( Y \) is called a subalgebra if \( x - y \in Y \) whenever \( x, y \in Y. \)

3. \( f \)-derivations of subtraction algebras.

**Definition 3.1.** ([3]) Let \( X \) be a subtraction algebra. By a derivation of \( X, \) a self-map \( d \) of \( X \) satisfying the identity \( d(x - y) = (d(x) - d(y)) \) for all \( x, y \in X \) is meant.

**Example 3.2.** Let \( X = \{0, a, b\} \) be a subtraction algebra with the following Cayley table

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a map \( d : X \to X \) by

\[ d(x) = \begin{cases} 
0 & \text{if } x = 0, b \\
b & \text{if } x = a 
\end{cases} \]

Then it is easily checked that \( d \) is a derivation of subtraction algebra \( X. \)

**Definition 3.3** Let \( X \) be a subtraction algebra. A function \( d : X \to X \) is called an \( f \)-derivation on \( X \) if there exists a function \( f : X \to X \) such that

\[ d(x - y) = (d(x) - f(y)) \land (f(x) - d(y)) \]

for all \( x, y \in X. \)

**Example 3.4.** Let \( X = \{0, 1, 2, 3\} \) in which “−” is defined by
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It is easy to check that $(X; -)$ is a subtraction algebra. Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0, 3 \\ 1 & \text{if } x = 2 \\ 2 & \text{if } x = 1 \end{cases}$$

and define a map $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 2 \\ 2 & \text{if } x = 1, 3 \end{cases}$$

Then it is easily checked that $d$ is an $f$-derivation of a subtraction algebra $X$.

Example 3.5. Let $X = \{0, a, b\}$ be a subtraction algebra with the following Cayley table

<table>
<thead>
<tr>
<th>−</th>
<th>0</th>
<th>a</th>
<th>b</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>b</td>
<td>0</td>
</tr>
</tbody>
</table>

Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b \end{cases}$$

and define a map $f : X \to X$ by $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b \end{cases}$$

Then it is easily checked that $d$ is an $f$-derivation of subtraction algebra $X$.

Example 3.6. In Example 3.4, $f$-derivation $d$ is not a derivation of $X$ since $2 = d(1) = d(1 - 2) \neq (d(1) - 2) \land (1 - d(2)) = (2 - 2) \land (1 - 1) = 0 \land 0 = 0$. 

\[
\begin{array}{ccc}
& 3 & \\
1 & & 2 \\
& 0 & \\
\end{array}
\]
**Proposition 3.7.** Let $X$ be a subtraction algebra and $d$ an $f$-derivation. Then the following identities hold:

1. $d(x) \leq f(x)$ for all $x, y \in X$,
2. $d(x - y) \leq f(x)$ for all $x, y \in X$.

Proof. (1) By definition of $f$-derivation and Proposition 3.7 (1), we have $d(x - y) \leq d(x) - d(y)$ for all $x, y \in X$.

(2) By definition of $f$-derivation, we have

\[
\begin{align*}
    d(x - y) &= (d(x) - f(y)) \wedge (f(x) - d(y)) \\
    &\leq f(x) - d(y) \\
    &\leq f(x)
\end{align*}
\]

for all $x, y \in X$.

**Proposition 3.8.** Let $X$ be a subtraction algebra and $d$ an $f$-derivation. Then $d(0) = 0$.

Proof. By definition of $f$-derivation, we have

\[
\begin{align*}
    d(0) &= d(0 - 0) = (d(0) - f(0)) \wedge (f(0) - d(0)) \\
    &= (d(0) - f(0)) - ((d(0) - f(0)) - (f(0) - d(0))) \\
    &= (d(0) - f(0)) - (d(0) - f(0)) = 0
\end{align*}
\]

from (p5).

**Proposition 3.9.** Let $X$ be a subtraction algebra and $d$ an $f$-derivation. Then the following identities hold:

1. $d(x - y) \leq d(x) - d(y)$ for all $x, y \in X$,
2. $d(x) - f(y) \leq f(x) - d(y)$ for all $x, y \in X$.

Proof. (1) By definition of $f$-derivation and Proposition 3.7 (1), we have $d(x - y) \leq d(x) - f(y) \leq d(x) - d(y)$ for all $x, y \in X$.

(2) Since $d(x) \leq f(x)$ for all $x \in X$, we have $d(x) - f(y) \leq f(x) - f(y) \leq f(x) - d(y)$.

**Theorem 3.10.** Let $X$ be a subtraction algebra. If $d$ is an $f$-derivation of $X$, $d(x - y) = d(x) - f(y)$ for all $x, y \in X$.

Proof. Suppose that $d$ is an $f$-derivation of $X$. Then for any $x, y \in X$, we have $d(x) - f(y) \leq f(x) - d(y)$ by Proposition 3.9 (2) and

\[
    d(x - y) = (d(x) - f(y)) \wedge (f(x) - d(y)) = d(x) - f(y).
\]

**Definition 3.11.** Let $X$ be a subtraction algebra and $d$ a derivation on $X$. If $x \leq y$ implies $d(x) \leq d(y)$, $d$ is called an isotone derivation.

**Theorem 3.12.** Let $d$ be an $f$-derivation of $X$. Then $d$ is an isotone derivation.

Proof. Let $x \leq y$ for all $x, y \in X$. Then by (p8), $x = y - w$ for some $w \in X$. Hence we have

\[
    d(x) = d(y - w) = (d(y) - f(w)) \wedge (f(y) - d(w)) \leq d(y) - f(w) \leq d(y)
\]

by Lemma 2.3 (2).

Let $d$ be a $f$-derivation of $X$. Define a set by

\[
    F := \{x \mid d(x) = f(x)\}
\]
for all \( x \in X \).

**Proposition 3.13.** Let \( d \) be an \( f \)-derivation and \( f \) an endomorphism. Then \( F \) is a subalgebra of \( X \).

Proof. Let \( x, y \in F \). Then we get \( d(x) = f(x) \) and \( d(y) = f(y) \), and so \( d(x - y) = d(x) - f(y) \wedge f(x) = f(x) - f(y) = f(x - y) \). Hence \( x - y \in F \). This completes the proof.

**Theorem 3.14.** Let \( d \) be an \( f \)-derivation and \( f \) an increasing endomorphism. If \( x \leq y \) and \( y \in F \), then we have \( x \in F \).

Proof. Let \( x \leq y \) and \( y \in F \). Then we obtain \( f(x) \leq f(y) \) and \( f(y) = d(y) \), and so we have

\[
\begin{align*}
d(x) &= d(x \wedge y) = d(x - (x - y)) = d(y - (y - x)) \\
&= d(y) - f(y - x) = d(y) - (f(y) - f(x)) \quad \text{(by Theorem 3.10)} \\
&= f(y) - (f(y) - f(x)) = f(x) - (f(x) - f(y)) \\
&= f(x) - 0 \leq f(x).
\end{align*}
\]

This completes the proof.

**Definition 3.15.** Let \( X \) be a subtraction algebra and \( d \) an \( f \)-derivation. Define a \( \text{Kerd} \) by

\[
\text{Kerd} = \{ x \in X \mid d(x) = 0 \}.
\]

**Proposition 3.16.** Let \( X \) be a subtraction algebra and \( d \) an \( f \)-derivation. Then \( \text{Kerd} \) is a subalgebra of \( X \).

Proof. Let \( x, y \in \text{Kerd} \). Then \( d(x) = d(y) = 0 \), and so \( d(x - y) \leq d(x) - d(y) = 0 - 0 = 0 \) by Proposition 3.9 (1). Thus \( d(x - y) = 0 \) that is, \( x - y \in \text{Kerd} \). Hence \( \text{Kerd} \) is a subalgebra of \( X \).

**Proposition 3.17.** Let \( X \) be a subtraction algebra and \( d \) an \( f \)-derivation. If \( x \in \text{Kerd} \) and \( y \in X \), then \( x \wedge y \in \text{Kerd} \).

Proof. Let \( x \in \text{Kerd} \). Then we get \( d(x) = 0 \), and so

\[
\begin{align*}
d(x \wedge y) &= d(x - (x - y)) = d(x) - f(x - y) \\
&= 0 - f(x - y) \\
&= 0.
\end{align*}
\]

This completes the proof.

**References**


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