A NOTE ON BE-ALGEBRAS

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Abstract. In this paper, we introduce the notion of essence in BE-algebras and related properties are investigated. Also, we discuss relations among subalgebras, filters and essences. Finally, we consider the homomorphic image and inverse image of an essence.

1. Introduction. Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras [3, 4]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [5], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a dualization of a generation of a BCK-algebra. In this paper, we introduce the notion of essence in BE-algebras and related properties are investigated. Also, we discuss relations among subalgebras, filters and essences. Finally, we consider the homomorphic image and inverse image of an essence.

2. Preliminaries. In what follows, let $X$ denote an BE-algebra unless otherwise specified.

By a BE-algebra we mean an algebra $(X; *, 1)$ of type $(2, 0)$ with a single binary operation "*" that satisfies the following identities: for any $x, y, z \in X$,

- $(BE1)$ $x * x = 1$ for all $x \in X$,
- $(BE2)$ $x * 1 = 1$ for all $x \in X$,
- $(BE3)$ $1 * x = x$ for all $x \in X$,
- $(BE4)$ $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

We introduce a relation "$\leq$" on $X$ by $x \leq y$ imply $x * y = 1$. An BE-algebra $(X, *, 1)$ is said to be self-distributive if $x * (y * z) = (x * y) + (x * z)$ for all $x, y, z \in X$. A non-empty subset $S$ of an BE-algebra $X$ is said to be a subalgebra of $X$ if $x * y \in S$ whenever $x, y \in S$.

In an BE-algebra, the following identities are true:

- $(p1)$ $x * (y * x) = 1$.
- $(p2)$ $x * ((x * y) * y)) = 1$.

Definition 2.1. Let $(X, *, 1)$ be an BE-algebra and $F$ a non-empty subset of $X$. Then $F$ is said to be a filter of $X$ if

- $(F1)$ $1 \in F$,
- $(F2)$ If $x \in F$ and $x * y \in F$, then $y \in F$.

Example 2.1. Let $X = \{1, a, b, c, d\}$ in which "*" is defined by

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It is easy to know that $X$ is a BE-algebra, and $F_1 = \{1, a\}, F_2 = \{1, b\}, F_3 = \{1, c\}, F_4 = \{1, a, b\}$ are filters of $X$.

**Lemma 2.2.** If $F$ is a filter of an BE-algebra $X$, then $F$ is a subalgebra of $X$.

**Proof.** If $x, y \in F$, then we have $y*(x*y) = 1 \in F$, and so $x*y \in F$ by (F2). This completes the proof. \qed

**Definition 2.3.** A non-empty subset $I$ of $X$ is called an ideal of $X$ if

1. $(I_1)$ If $x \in X$ and $a \in I$, then $x*a \in I$, i.e., $X*I \subseteq I$,
2. $(I_2)$ If $x \in X$ and $a, b \in I$, then $(a*(b*x)) * x \in I$.

The following Lemma 2.5 is well-known in BE-algebras (see [6]).

**Lemma 2.4.** Let $X$ be an BE-algebra. Then

1. Every ideal of $X$ contains 1,
2. If $I$ is an ideal of $X$, then $(a*x) * x \in I$ for all $a \in I$ and $x \in X$.

Let $I$ be an ideal of $X$. Define $I_w$ by

$$I_w = \{x \in X | w*x \in I\}$$

for any $w \in X$.

**Proposition 2.5.** Let $X$ be a self-distributive BE-algebra and $I$ an ideal of $X$. Then $I_w$ is a subalgebra of an BE-algebra $X$.

**Proof.** Let $a, b \in I_w$. Then $w*a \in I$ and $w*b \in I$, and so $w*(a*b) = (w*a)*(w*b) \subseteq I \subseteq X*I \subseteq I$. This implies $a*b \in I_w$. \qed

**Proposition 2.6.** Let $X$ be a self-distributive BE-algebra and $I$ an ideal of $X$. Then $I_w$ is an ideal of an BE-algebra $X$.

**Proof.** Let $x \in X$ and $a \in I_w$. Then we have $w*a \in I$, and so $w*(x*a) = (w*x)*(w*a) \in X*I \subseteq I$ from (I1). This implies $x*a \in I_w$. Now let $a, b \in I_w$ and $x \in X$. Then we obtain $w*a \in I$ and $w*b \in I$. Thus we get $w*((a*(b*x)) * x) = (w*((a*(b*x)))) * (w*x) = ((w*a)*(w*(b*x)))*(w*x) = ((w*a)*(w*(b*(w*x)))+(w*x)) \in I$ by (I2). This implies $(a*(b*x)) * x \in I_w$. This completes the proof. \qed

**Proposition 2.7.** Let $X$ be a self-distributive BE-algebra and $I$ an ideal of $X$. If $a \in I_w$ and $a \leq b$, then $b \in I_w$.

**Proof.** Let $a \in I_w$ and $a \leq b$. Then we have $w*a \in I$ and $a*b = 1$. Hence we get

$$w*b = w*(1*b) = w*((a*b)) = w*(w*(a*b)) = ((w*a)*(w*(b*x)))*(w*x) \in I,$$

from Lemma 2.5(2). This implies $b \in I_w$. \qed
Let $X$ be an BE-algebra and $x, y \in X$. Define $A(x, y)$ by

$$A(x, y) = \{ z \in X \mid x * (y * z) = 1 \}.$$ 

We call $A(x, y)$ an upper set of $x$ and $y$. It is easy to see that $1, x, y \in A(x, y)$ for all $x, y \in X$.

**Proposition 2.8.** Let $X$ be an BE-algebra. If $(X; *, 1)$ is a self distributive BE-algebra, then $A(x, y)$ is a subalgebra of $X$.

**Proof.** Let $m, n \in A(x, y)$. Then we have $x * (y * m) = 1$ and $x * (y * n) = 1$, and so $x * (y * (m * n)) = x * ((y * m) * (y * n)) = (x * (y * m)) * (x * (y * n)) = 1 * 1 = 1$. Thus $m * n \in A(x * y)$. This completes the proof. \(\square\)

**Definition 2.9.** Let $X$ be an BE-algebra and $a \in X$. Define $A(a)$ by

$$A(a) = \{ x \in X \mid a \leq x \}.$$ 

Then we call $A(a)$ the initial section of the element $a$.

**Example 2.2.** Let $X = \{1, a, b, c\}$ in which “*” is defined by

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Then $X$ is an BE-algebra. Also, we have $A(1) = \{1\}$, $A(a) = \{1, a\}$, $A(b) = \{1, a, b\}$ and $A(c) = \{1, a, c\}$.

**Lemma 2.10.** Let $X$ be an BE-algebra and $x \leq y$. If $x \in A(a)$, then $y \in A(a)$.

**Proof.** Since $x \in A(a)$, we have $a \leq x$. Hence $a \leq x \leq y$, that is, $a \leq y$. This implies $y \in A(a)$. \(\square\)

**Proposition 2.11.** Let $X$ be a self-distributive BE-algebra and $a \in X$. Then $A(a)$ is a filter of $X$.

**Proof.** Clearly $1 \in A(a)$ because $a * a = 1$. Let $x \in A(a)$ and $x * y \in A(a)$. Then we have $a \leq x$ and $a \leq x * y$. Hence we have $a * x = 1$ and $a * (x * y) = 1$, and so $a * (x * y) = (a * x) * (a * y) = 1 * (a * y) = a * y = 1$, that is, $y \in A(a)$. This completes the proof. \(\square\)

**Proposition 2.12.** Let $X$ be a self-distributive BE-algebra and $x, y, z \in X$. If $z \leq x * y$ and $z \leq x$, then $z \leq y$.

**Proof.** Suppose that $z \leq x * y$ and $z \leq x$ for all $x, y, z \in X$. Then we have $x * y \in A(z)$ and $x \in A(z)$. Since $A(z)$ is a filter, it follows that $y \in A(z)$ or $z \leq y$. This completes the proof. \(\square\)

**Theorem 2.13.** Let $X$ be an BE-algebra, $F$ a filter and $x \in F$. Then $A(x) \subseteq F$.

**Proof.** If $y \in A(x)$, then we have $x \leq y$. Hence $x * y = 1$. Since $F$ is a filter of $X$ and $x \in X$, we obtain $y \in F$. Therefore $A(x) \subseteq F$. \(\square\)
3. **Essences.** Let $X$ be an BE-algebra. For any subsets $A$ and $B$ of $X$, we define

$$A \ast B := \{a \ast b \mid a \in A, y \in B\}.$$

We use the notation $A \ast b$ (resp. $a \ast B$) instead of $A \{b\}$ (resp. $\{a\} \ast B$). Note that $A \ast B = \bigcup_{a \in A} (a \ast b) = \bigcup_{b \in B} (A \ast b)$.

**Lemma 3.1.** For any subsets $A, B$ and $E$ of an BE-algebra $X$, we have

1. $A \subseteq B \Rightarrow A \ast E \subseteq B \ast E, E \ast A \subseteq E \ast B$,
2. $(A \cap B) \ast E \subseteq (A \ast E) \cap (B \ast E)$,
3. $E \ast (A \cap B) \subseteq (E \ast A) \cap (E \ast B)$,
4. $(A \cup B) \ast E = (A \ast E) \cup (B \ast E)$,
5. $E \ast (A \cup B) = (E \ast A) \cup (E \ast B)$.

**Proof.** (1) Let $x \in A \ast E$. Then $x = a \ast e$ for some $a \in A$ and $e \in E$. Since $A \subseteq B$, it follows that $x = a \ast e$ for some $a \in B$ and $e \in E$ so that $x \in B \ast E$. Therefore $A \ast E \subseteq B \ast E$.

Similarly, we obtain $E \ast A \subseteq E \ast B$.

(2) Since $A \cap B \subseteq A, B$, it follows from (1) that $(A \cap B) \ast E \subseteq A \ast E$ and $(A \cap B) \ast E \subseteq B \ast E$ so that $(A \cap B) \ast E \subseteq (A \ast E) \cap (B \ast E)$. Similarly, (3) is valid.

(4) Since $A, B \subseteq A \cup B$, we get $A \ast E \subseteq (A \cup B) \ast E$ and $B \ast E \subseteq (A \cup B) \ast E$ by (1), and so $(A \ast E) \cup (B \ast E) \subseteq (A \cup B) \ast E$. If $x \in A \cup B \ast E$, then $x = y \ast e$ for some $y \in A \cup B$ and $e \in E$. It follows that $x = y \ast e$ for some $y \in A$ and $e \in E$; or $x = y \ast e$ for some $y \in B$ and $e \in E$ so that $x = y \ast e \in A \ast E$ or $x = y \ast e \in B \ast E$. Hence $x \in (A \ast E) \cup (B \ast E)$, which shows that $(A \cup B) \ast E \subseteq (A \ast E) \cup (B \ast E)$. Therefore (4) is valid. Similarly, we can prove that (5) is valid. \qed

**Definition 3.2.** If a non-empty subset $A$ of an BE-algebra $X$ satisfies the following equality

$$X \ast A = A,$$

then we say that $A$ is an **essence** of $X$. Note that $\{1\}$ and $X$ itself are essences of $X$.

**Example 3.1.** Let $X = \{1, a, b, c\}$ in which “$\ast$” is defined by

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Then $X$ is an BE-algebra. It is easy to check that $A_1 = \{1, a\}, A_2 = \{1, c\}, A_3 = \{1, a, c\}$ are an essence of $X$ but $A_4 = \{1, b\}, A_5 = \{1, a, b\}$ are not essences of $X$.

**Proposition 3.3.** Every essence contains the constant 1.

**Proof.** Let $A$ be an essence of $X$. Then $\phi \neq A = X \ast A$, and so there exists $a \in A$ such that and thus $1 = a \ast a \in X \ast A = A$. This completes the proof. \qed

**Theorem 3.4.** Every essence is a subalgebra of $X$.

**Proof.** Let $A$ be an essence of $X$ and $x, y \in A$. Then

$$x \ast y \in A \ast A \subseteq X \ast A = A$$

by Lemma 3.1(1), and so $A$ is a subalgebra of $X$. \qed

The converse of Theorem 3.5 is not true. For Example, the set $D := \{1, b\}$ in Example 3.3 is a subalgebra which is not an essence of $X$. 

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Theorem 3.5. Every filter is an essence of $X$.

Proof. Let $F$ be a filter of $X$. Then $1 \in F$, and so $F \neq \emptyset$. Since $a \leq a * d$ for all $a \in F$ and $d \in X$, we have $b * d \in F$. Thus $X * F \subseteq F$. Obviously, $F = \{1\} * F \subseteq X * F$ by Lemma 3.1(1). Therefore $X * F = F$, i.e., $F$ is an essence of $X$. \qed

The converse of Theorem 3.6 may not be true. For example, the set $C = \{1, a, c\}$ in Example 2.11 is an essence which is not a filter of $X$ since $a * c = a \in C$ and $c \notin C$.

Theorem 3.6. If $F$ is a filter of an BE-algebra $X$, then $A * F$ is an essence of $X$ for every non-empty subset $A$ of $X$.

Proof. Let $A$ be a non-empty subset of $X$ and $F$ a filter of $X$. Then $F$ is an essence of $X$ (see Theorem 3.6). Using (BE4), we have

$$X * (A * F) = A * (X * F) = A * F,$$

and hence $A * F$ is an essence of $X$. \qed

Corollary 3.7. If $A$ is a non-empty proper subset of an BE-algebra $X$, then $A * X$ is an essence of $X$.

Theorem 3.8. Let $A$ and $B$ be essences of $X$. Then $A \cap B$ and $A \cup B$ are essences of $X$.

Proof. Let $K = A \cap B$. Then

$$K = 1 * K \subseteq X * K = X * (A \cap B) \subseteq (X * A) \cap (X * B) = A \cap B = K,$$

and so $X * K = K$, that is, $K = A \cap B$ is an essence of $X$. Now let $L = A \cup B$. Then

$$L = 1 * L \subseteq X * L = X * (A \cup B) = (X * A) \cup (X * B) = A \cup B = K,$$

and thus $X * L = L$, that is, $L = A \cup B$ is an essence of $X$. \qed

Generally, we have the following results.

Theorem 3.9. If $\{A_i \mid i \in \Lambda \subseteq \mathbb{N}\}$ is a family of an BE-algebra $X$, then $\bigcup_{i \in \Lambda}$ and $\bigcap_{i \in \Lambda}$ are essences of $X$.

In general, the union of two filters of BE-algebras $X$ may not be a filter of $X$. For example, in Example 2.2, $F_1 = \{1, b\}$ and $F_2 = \{1, c\}$ are filters, but $F_1 \cup F_2$ is not a filter of $X$. But we know that the following result is derived from Theorem 3.6 and 3.9.

Corollary 3.10. The union of two filters of an BE-algebra $X$ is an essence of $X$.

Let $A$ be an essence and $B$ a subalgebra of an BE-algebra $X$. Then $A \cup B$ is not an essence of $X$ in general as seen in the following example.

Example 3.2. In Example 2.2, it is easy to check that $A = \{1, a, c\}$ is an essence of $X$ and $B = \{1, d\}$ is a subalgebra of $X$. But $A \cup B = \{1, a, c, d\}$ is not an essence of $X$.

Theorem 3.11. Let $X$ be an BE-algebra. If $A$ is an essence of $X$ and $B$ is a subalgebra of $X$, then $A \cup B$ is an essence of $B$.

Proof. Using Lemma 3.1 (3), we have

$$B * (A \cap B) \subseteq (B * A) \cap (B * B) \subseteq (X * A) \cap B = A \cap B \subseteq B * (A \cap B),$$

and so $B * (A \cap B) = A \cap B$. Therefore $A \cap B$ is an essence of $B$. \qed

Proposition 3.12. Let $A$ be an essence of an BE-algebra of $X$. If $1 \in B \subseteq X$, then $B * A = A$. 

Proof. Let $A$ be an essence of an BE-algebra of $X$. Then 
\[ A = 1 * A \subseteq B * A \subseteq X * A = A. \]

Let $X$ and $Y$ be BE-algebras. A mapping $f : X \to Y$ is called a homomorphism if 
\[ f(x * y) = f(x) * f(y) \]
for all $x, y \in X$. Note that $f(1) = 1$.

**Lemma 3.13.** Let $X$ be an BE-algebra. If $1 \in A \subseteq X$, then $B$ is contained in $A * B$ for every subset $B$ of $X$.

**Proof.** Let $b \in B$. Then $b = 1 * b \in A * B$, and so $B$ is contained in $A * B$. \(\square\)

**Theorem 3.14.** Let $f : X \to Y$ be a homomorphism of BE-algebras.

1. If $f$ is onto and $A$ is an essence of $X$, then $f(A)$ is an essence of $Y$.
2. If $B$ is an essence of $Y$, then $f^{-1}(B)$ is an essence of $X$.

**Proof.** Suppose that $f$ is onto and $A$ is an essence of $X$. Using Lemma 3.1(2) and 3.15, we have $f(A) \subseteq Y * f(A)$. Let $b \in f(A)$ and $y \in Y$. Then $b = f(a)$ and $y = f(x)$ for some $a \in A$ and $x \in X$. Thus
\[ y * b = f(x) * f(a) = f(x * a) \in f(X * A) = f(A), \]
and so $Y * f(A) \subseteq f(A)$. Therefore $f(A)$ is an essence of $Y$.

(ii) Using Lemma 3.1(1), we have $f^{-1}(B) \subseteq X * f^{-1}(B)$. Let $a \in f^{-1}(B)$ and $x \in X$. Then $f(a) \in B$ and $f(x) \in Y$. It follows that
\[ f(x * a) = f(x) * f(a) \in Y * B = B \]
so that $x * a \in f^{-1}(B)$, i.e., $X * f^{-1}(B) \subseteq f^{-1}(B)$. Hence $f^{-1}(B)$ is an essence of $X$. \(\square\)

**Corollary 3.15.** If $f : X \to Y$ is a homomorphism of BE-algebras, then $f^{-1}$ is an essence of $X$.

**References**


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