ATOMS IN CI-ALGEBRAS AND SINGULAR CI-ALGEBRAS

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ABSTRACT. In the present paper we continue to study CI-algebras. At first we introduce the notion of atoms in CI-algebras and investigate its elementary properties. Next we introduce the notion of singular CI-algebras and give a number of its properties. Especially we discuss relations between singular CI-algebras and Abelian groups.

1. Introduction.

The study of BCK/BCI-algebras was initiated by K.Iséki in 1966 as a generalization of propositional logic (see[4, 5, 6]). There exist several generalizations of BCK/BCI-algebras, as such BCH-algebras[3], dual BCK-algebras, dual BCI-algebras, $d$-algebras[12], etc. Especially, H.S.Kim and Y.H.Kim[7] introduced the notion of BE-algebras as another generalization of dual BCK-algebras. They provided an equivalent condition of the filters in BE-algebras. S.S.Ahn and Y.H.Kim in [1], S.S.Ahn and K.S.So in [2] introduced the notion of ideals in BE-algebras and gave several descriptions of ideals. H.S.Kim and K.J.Lee in [8] generalized the notions of upper sets and generalized upper sets and introduced extended upper sets, by using this notion they gave several descriptions of filters in BE-algebras. A. Walendziak in [13] introduced the notion of commutative BE-algebras and discussed some of its properties. Recently we in [9] and [10] introduced the notion of CI-algebras as a generalization of BE-algebras and dual BCI/BCH-algebras, and studied some of its important properties and relations with BE-algebras, especially proved the notion of ideals is equivalent to one of filters in transitive BE-algebras. In [11] we introduced the notion of closed filters in CI-algebras and built elementary theory of closed filter. We give a procedure to generate a closed filter by a nonempty subset of a CI-algebra. In the present paper we continue to study CI-algebras. At first we introduce the notion of atoms in CI-algebras and investigate its important properties. Next we introduce the notion of singular CI-algebras and give a number of its properties. Especially we discuss relations between CI-algebras and Abelian groups. The definitions and terminologies used in this paper are standard.

2. Preliminaries

Definition 2.1[9]. A CI-algebra is an algebra $(X; *, 1)$ of type $(2,0)$ satisfying the following axioms: for any $x, y, z \in X$

(CI1) $x * x = 1$;

(CI2) $1 * x = x$;

(CI3) $x * (y * z) = y * (x * z)$.

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In any CI-algebra $X$ one can define a binary relation $\leq$ by $x \leq y$ if and only if $x \ast y = 1 \forall x, y \in X$.

**Lemma 2.3**[9]. In a CI-algebra $X$, the following hold: for any $x, y \in X$

1. $x \ast ((x \ast y) \ast y) = 1$;
2. $(x \ast y) \ast 1 = (x \ast 1) \ast (y \ast 1)$;
3. $1 \leq x$ implies $x = 1$.

**Definition 2.4**[9]. Let $X$ be a CI-algebra. A nonempty subset $S$ of $X$ is said to be a **subalgebra** of $X$ if it satisfies:

$x, y \in S$ implies $x \ast y \in S$ for any $x, y \in X$.

**A nonempty subset** $F$ of $X$ is said to be a **filter** of $X$ if it satisfies

1. $1 \in F$;
2. for any $x, y \in X$, $x \ast y \in F$ and $x \in F$ imply $y \in F$.

A filter $F$ of $X$ is said to be **closed** if $x \in F$ implies $x \ast 1 \in F$.

**Lemma 2.5**[11]. Let $X$ be a CI-algebra. A filter $F$ of $X$ is closed if and only if $F$ is a subalgebra of $X$.

3. **Atoms in CI-algebras**

In this section we first introduce the notion of atoms in CI-algebras and next study some of its elementary properties.

**Definition 3.1.** Let $a$ be an element of a CI-algebra $X$. $a$ is said to be an **atom** in $X$ if for any $x \in X$, $a \ast x = 1$ implies $a = x$.

Denote the set of all atoms in $X$ by $A(X)$, which is called the **singular part** of $X$.

Obviously $1 \in A(X)$, so $A(X) \neq \emptyset$.

**Example 3.2.** Let $X = \{1, a, b, c, d\}$ with the following Cayley table:

<table>
<thead>
<tr>
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<th>1</th>
<th>a</th>
<th>b</th>
<th>c</th>
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</table>

1 and $d$ are atoms in $X$, but $a, b, c$ are not atoms of $X$.

**Proposition 3.3.** Let $X$ be a CI-algebra. Then $a \in X$ is an atom in $X$ if and only if it satisfies for any $x \in X$, $a = (a \ast x) \ast x$.

**Proof.** Let $a$ be an atom in $X$ and $x \in X$. It follows from $a \ast ((a \ast x) \ast x) = 1$ that $a = (a \ast x) \ast x$.

Conversely suppose that $a \in X$ satisfies for any $x \in X$, $a = (a \ast x) \ast x$. If $a \ast x = 1$, then

$$a = (a \ast x) \ast x = 1 \ast x = x,$$

hence $a$ is an atom in $X$. The proof is complete.

**Proposition 3.4.** Let $X$ be a CI-algebra. If $a, b \in X$ are atoms in $X$, then the following are true:
(1) \(a = (a * 1) * 1\);

(2) \((a * b) * 1 = b * a\);

(3) \(((a * b) * 1) * 1 = a * b\).

**Proof.** (1) is an immediate consequence of Proposition 3.3.

By Lemma 2.3(2), (CI3) and (1) we have

\((a * b) * 1 = (a * 1) * (b * 1) = b * ((a * 1) * 1) = b * a\).

(2) holds.

(3) follows from (2). The proof is complete. \(\square\)

**Proposition 3.5.** Let \(X\) be a CI-algebra. If \(a\) and \(b\) are atoms in \(X\), then the following are true:

(1) for any \(x \in X\), \((a * x) * (b * x) = b * a\);

(2) for any \(x \in X\), \((a * x) * b = (b * x) * a\);

(3) for any \(x \in X\), \((a * x) * (y * b) = (b * x) * (y * a)\).

**Proof.** Since \(a, b \in A(X)\), it follows from Proposition 3.3 that

\((a * x) * (b * x) = b * ((a * x) * x) = b * a\).

(1) holds.

By using (1) and Proposition 3.2 we have

\((a * x) * b = (a * x) * ((b * x) * x) = (b * x) * ((a * x) * x) = (b * x) * a\).

(2) holds.

By (2) we have

\((a * x) * (y * b) = y * [(a * x) * b] = y * [(b * x) * a] = (b * x) * (y * a)\).

The proof is complete. \(\square\)

**Open problem:** Let \(X\) be a CI-algebra. Is the set \(A(X)\) of all atoms of \(X\) a subalgebra of \(X\)? When?

4. **Singular CI-algebras**

In this section we introduce the notion of singular CI-algebras and give a number of its properties. Especially we discuss relations between CI-algebras and Abelian groups.

**Definition 4.1.** A CI-algebra \(X\) is said to be **singular** if every element of \(X\) is an atom of \(X\).

**Example 4.2.** Let \(X = \{1, a, b, c\}\) with the following Cayley table:

<table>
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</tbody>
</table>

Then \(X\) is a singular CI-algebra. \(X\) in the example 3.2 is not singular, because \(a\) and \(b\) are not atoms.

**Proposition 4.3.** Let \(X\) be a CI-algebra. Then \(X\) is singular if and only if \(X\) satisfies the condition

\((a * x) * (b * x) = b * a\) for any \(x \in X\).
(D) for any \(x, y, z \in X\), \((x * y) * z = (z * y) * x\).

Proof. Suppose \(X\) is singular. It follows from Proposition 3.4(2) that the condition (D) holds for \(X\).

Conversely suppose that \(X\) satisfies the condition (D). If \(x * y = 1\), then by (D) we have
\[
\begin{align*}
x = 1 * x & = (y * y) * x = (x * y) * y = 1 * y = y.
\end{align*}
\]

Hence \(x\) is an atom of \(X\), and \(X\) is singular. The proof is complete.

Proposition 4.4. Let \((X; *, 1)\) be a singular CI-algebra. Define \(x + y := (x * 1) * y\) for any \(x, y \in X\). Then \((X; +, 1)\) is an Abelian group with identity 1.

Proof. It follows from the condition (D) that \(x + y = y + x\) for any \(x, y \in X\), and so the operation \(+\) is commutative.

Because for any \(x \in X\), \(x + 1 = (x * 1) * 1 = x\), so 1 is identity.

Since for any \(x, y, z \in X\),
\[
(x + y) + z = \{(x + 1) * y\} * 1 + z = \{(x * 1) * y\} * z = \{(x + 1) * y\} * z
\]
by Definition
\[
= \{(x * 1) * (y + 1)\} * 1 = \{(x * 1) * y\} * (z + 1)
\]
by (D)
\[
= \{(x * 1) * (y + 1)\} * (z + 1) = \{(x * 1) * y\} * z
\]
by (CI3)
\[
= \{(x * 1) * y\} * z = x + (y + z)
\]
by (D)
and so the operation \(+\) satisfies associative law.

Because \(x + (x * 1) = (x * 1) * (x * 1) = 1\), so \(-x = x * 1\) is the inverse of \(x\). Therefore \((X; +, 1)\) is an Abelian group with identity 1.

The group \((X; +, 1)\) is called the adjoint group of CI-algebra \((X; *, 1)\).

Proposition 4.5. Let \((X; +, 1)\) be an Abelian group with identity 1. Define \(x + y := y - x\) for any \(x, y \in X\). Then \((X; *, 1)\) is a singular CI-algebra, whose adjoint group is exactly \((X; +, 1)\).

Proof. Because for any \(x \in X\), \(x + x = x - x = 1\) and \(1 * x = x - 1 = x\), so \((X; *, 1)\) satisfies (CI1) and (CI2).

For any \(x, y, z \in X\), we have \(x * (y * z) = (z - y) - x = (z - x) - y = y * (x * z)\), hence \((X; *, 1)\) satisfies (CI3). Therefore \((X; *, 1)\) is a CI-algebra.

For any \(x, y, z \in X\), we have
\[
(x * y) * z = (z - y - x) = (z - y - x + y + z) = (x - y - z) * x = (z * y) * x.
\]

Hence \((X; *, 1)\) is singular.

For the singular CI-algebra \((X; *, 1)\), define \(x + y := (x * 1) * y\) for \(x, y \in X\). By Proposition 3.8 we know that \((X; \oplus, 1)\) is the adjoint group of \((X; *, 1)\). Because
\[
x \oplus y = (x * 1) * y = y - (x * 1) = y - (1 - x) = y + x = x + y,
\]
\((X; \oplus, 1)\) is exactly \((X; +, 1)\). The proof is complete.

The CI-algebra \((X; *, 1)\) is called the adjoint algebra of group \((X; +, 1)\).

Proposition 4.6. Let \((X; *, 1)\) be a singular CI-algebra, \((X; +, 1)\) the adjoint group of \((X; *, 1)\). Then the adjoint algebra of \((X; +, 1)\) is exactly \((X; *, 1)\).
Proof. For the adjoint group \((X; +, 1)\), define \(x * y = y - x\) for \(x, y \in X\). By Proposition 3.9 we know that \((X; *, 1)\) is the adjoint algebra of \((X; +, 1)\). Because
\[
x * y = y - x = y + (x * 1) = (y * 1) * (x * 1) = x * ((y * 1) * 1) = x * y,
\]
hence \((X; *, 1)\) is exactly \((X; +, 1)\). The proof is complete.

**Proposition 4.7.** Let \((X; *, 1)\) and \((X'; *, 1')\) be singular CI-algebras. Let \((X; +, 1)\) and \((X'; +', 1')\) be adjoint groups of \((X; *, 1)\) and \((X'; *, 1')\), respectively. Then \((X; *, 1)\) is isomorphic to \((X'; *, 1')\) if and only if \((X; +, 1)\) is isomorphic to \((X; +, 1)\).

**Proof.** Suppose that \((X; *, 1)\) is isomorphic to \((X'; *, 1')\). Let \(\varphi : X \to X'\) be an isomorphism from \((X; *, 1)\) to \((X'; *, 1')\) such that \(\varphi(x) = x'\) for any \(x \in X\). Then for any \(x, y \in X\),
\[
\varphi(x + y) = \varphi((x * 1) * y) = (\varphi(x) *' \varphi(1)) *' \varphi(y) = \varphi(x) +' \varphi(y),
\]
and so \((X; +, 1)\) is isomorphic to \((X'; +', 1')\).

Conversely, suppose \((X; *, 1)\) is isomorphic to \((X'; *, 1')\). Let \(\varphi : X \to X'\) be an isomorphism from \((X; *, 1)\) to \((X'; *, 1')\) such that \(\varphi(x) = x'\) for any \(x \in X\). Then for any \(x, y \in X\),
\[
\varphi(x * y) = \varphi(y - x) = (\varphi(y) -' \varphi(x) = \varphi(x) *' \varphi(y),
\]
and so \((X; *, 1)\) is isomorphic to \((X'; *, 1')\). The proof is complete.

**Proposition 4.8.** Every subalgebra of a singular CI-algebra is a closed filter.

**Proof.** Let \(X\) be a CI-algebra and \(F\) a subalgebra of \(X\). By Lemma 2.6 it suffices to prove that \(F\) is a filter of \(X\). Obviously, \(1 \in F\). If \(x * y \in F\) and \(x \in F\), then \(1 * x \in F\) because \(F\) is a subalgebra \(X\). Also \(x * y \in F\) implies \(y * x = 1 * (x * y) \in F\) by Proposition 3.4(2). Thus by Proposition 3.4(3) we have
\[
y = 1 * y = (x * x) * (1 * y) = (y * x) * (1 * x) \in F,
\]
hence \(F\) is a filter of \(X\).

**References**


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