ON BRANCHES IN POSITIVE IMPLICATIVE BCI-ALGEBRAS WITH CONDITION (S)

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Abstract. In this paper we show that given a positive implicative BCI-algebra X with condition (S), every branch V(a) of X with respect to the BCI-ordering \(\leq\) on X forms an upper semilattice \((V(a); \leq)\); especially, if \(V(a)\) is a finite set, \((V(a); \leq)\) forms a lattice; moreover, if \((V(a); \leq)\) is a lattice, it must be distributive. We also obtain some interesting identities on \(V(a)\).


In this paper we will continue our discussion of [3], [4] and [10]. We will first consider the relations between lattices and the branches of a positive implicative BCI-algebra with condition (S), and next give several interesting identities on such a branch.

0 Preliminaries For the notations and elementary properties of BCK and BCI-algebras, we refer the reader to [7], [6] and [8]. And we will use some familiar notions and properties of lattices without explanation.

Recall that given a \(BCI\)-algebra \((X; *, 0)\), the following identities hold:

\[
\begin{align*}
x \ast x &= 0, \quad x \ast 0 = x \quad \text{and} \quad (x \ast y) \ast x = 0 \ast y, \\
(x \ast y) \ast z &= (x \ast z) \ast y, \\
0 \ast (x \ast y) &= (0 \ast x) \ast (0 \ast y).
\end{align*}
\]
(0.1)

And \(X\) with respect to its \(BCI\)-ordering \(\leq\) forms a partially ordered set \((X; \leq)\) satisfying the following quasi-identities:

\[
\begin{align*}
(x \ast y) \ast (x \ast z) \leq z \ast y, \\
(x \ast z) \ast (y \ast z) \leq x \ast y, \\
x \ast (x \ast y) \leq y,
\end{align*}
\]
(0.3)

(0.4)

(0.5)

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where the binary relation ≤ on X is defined as follows: \( x \leq y \) if and only if \( x \ast y = 0 \). Moreover, the following assertions are valid: for any \( x, y, z \in X \),

\[
\begin{align*}
  x \leq y & \quad \text{implies} \quad z \ast y \leq z \ast x, \\
  x \leq y & \quad \text{implies} \quad x \ast z \leq y \ast z, \\
  x \ast y \leq z & \quad \text{implies} \quad x \ast z \leq y.
\end{align*}
\]

A branch \( V(a) \) of a BCI-algebra \( X \) is such set \( \{ x \in X \mid x \geq a \} \) in which \( a \) is a minimal element of \( X \) in the sense that \( x \leq a \) implies \( x = a \) for all \( x \in X \). It has been known (see, e.g., \[8\], §1.3) that the collection \( \{ V(a) \mid a \in L(X) \} \) of branches of \( X \) forms a partition of \( X \), that is, \( X = \bigcup_{a \in L(X)} V(a) \) and \( V(a) \cap V(b) = \emptyset \) whenever \( a \neq b \), where \( L(X) \) is the set of the entire minimal elements of \( X \). And the following assertions are true:

\[
\begin{align*}
  x \in V(a) & \quad \text{implies} \quad 0 \ast x = 0 \ast a, \\
  x \in V(a) & \quad \text{and} \quad y \in V(b) \quad \text{imply} \quad x \ast y \in V(a \ast b), \\
  x \leq y & \quad \text{implies} \quad x \ast y \leq y \ast z \quad \text{if and only if} \quad x \leq y \ast z.
\end{align*}
\]

It has been known (see, e.g., \[8\], §2.8) that a BCI-algebra \( X \) is with condition \( (S) \) if and only if there is a binary operation \( \circ \) on \( X \) such that \((X; \circ, 0)\) is a commutative monoid satisfying the identity

\[
x \ast (y \circ z) = (x \ast y) \ast z.
\]

Moreover, if \( X \) is with condition \( (S) \), the following hold: for any \( x, y, z \in X \),

\[
\begin{align*}
  (x \circ y) \ast x & \leq y, \\
  x \ast y & \leq z \quad \text{if and only if} \quad x \leq y \circ z.
\end{align*}
\]

A BCI-algebra \( X \) is called positive implicative if it satisfies the identity

\[
(x \ast (x \ast y)) \ast (y \ast x) = x \ast (x \ast (y \ast (y \ast x)));
\]

it is called weakly positive implicative if it satisfies the identity

\[
(x \ast y) \ast z = ((x \ast z) \ast z) \ast (y \ast z).
\]

It is known (see, \[4\], Theorem 2) that a BCI-algebra is positive implicative if and only if it is weakly positive implicative. Thus, if \( X \) is positive implicative, \( (0.15) \) is valid. Replacing \( y \) by \( 0 \) and \( z \) by \( y \) in \( (0.15) \), the following holds: for any \( x, y \in X \),

\[
x \ast y = ((x \ast y) \ast y) \ast (0 \ast y).
\]

Moreover, if \( y \) is in the branch \( V(b) \) of \( X \), by \( (0.16) \) and \( (0.9) \), we obtain

\[
x \ast y = ((x \ast y) \ast y) \ast (0 \ast b).
\]

**Proposition 0.1.** Let \( V(a) \) be a branch of a positive implicative BCI-algebra \( X \). Then the following is true: for any \( x \in V(a) \),

\[
x = (x \ast a) \ast (0 \ast a),
\]

or equivalently,

\[
x = (x \ast (0 \ast a)) \ast a.
\]

**Proof.** For any \( x \in V(a) \), we have \((x \ast a) \ast (0 \ast a) \leq x \) by \( (0.4) \). Denote

\[
u = (x \ast a) \ast (0 \ast a).
\]

Then \( u \leq x \). So, by \( (0.11) \) and \( (0.9) \), we obtain \( u \in V(a) \) and \( 0 \ast u = 0 \ast a \). Also, by \( (0.4) \) and \( (0.5) \), the following holds:

\[
(x \ast (0 \ast a)) \ast ((x \ast a) \ast (0 \ast a)) \leq x \ast (x \ast a) \leq a.
\]

Since \( u = (x \ast a) \ast (0 \ast a) \) and \( 0 \ast u = 0 \ast a \), it follows \((x \ast (0 \ast u)) \ast u \leq a \). Then the face that \( a \) is a minimal element of \( X \) gives \((x \ast (0 \ast u)) \ast u = a \). So, by \( u \in V(a) \) (i.e., \( a \leq u \)), we derive

\[
((x \ast (0 \ast u)) \ast u) \ast u = a \ast u = 0.
\]
Hence \((x \ast u) \ast (0 \ast u) = 0\) by (0.1). Thus (0.16) implies \(x \ast u = 0\), i.e., \(x \leq u\). In addition, \(u \leq x\). Therefore \(x = u\). We have shown that \(x = (x \ast a) \ast (0 \ast a)\), in other words, \(x = (x \ast (0 \ast a)) \ast a\) by (0.1).

1 Relations between lattices and branches
Let’s begin our discussion with various relations between lattices and the branches of a positive implicative BCI-algebra with condition (S).

**Theorem 1.1.** Let \(X\) be a positive implicative BCI-algebra with condition (S). Then every branch \(V(a)\) of \(X\) with respect to the BCI-ordering \(\leq\) on \(X\) forms an upper semilattice \((V(a); \leq)\) with \(x \lor y = (x \circ y) \ast a\) for any \(x, y \in V(a)\).

**Proof.** For any \(x, y \in V(a)\), by (0.12) and (0.9), we have \(x \ast (x \circ y) = (x \ast x) \ast y = 0 \ast y = 0 \ast a\).

Then (0.14) and the commutativity of \(\circ\) give
\[
x \leq (x \circ y) \circ (0 \ast a) = (0 \ast a) \circ (x \circ y).
\]

So, (0.7) and (0.13) imply
\[
x \ast (0 \ast a) \leq ((0 \ast a) \circ (x \circ y)) \ast (0 \ast a) \leq x \circ y.
\]

Using (0.7) once more, it follows \((x \ast (0 \ast a)) \ast a \leq (x \circ y) \ast a\). Hence \(x \leq (x \circ y) \ast a\) by (0.19). Similarly, \(y \leq (x \circ y) \ast a\). It is easy to see from (0.11) that \((x \circ y) \ast a \in V(a)\). Therefore \((x \circ y) \ast a\) is an upper bound of \(x\) and \(y\). Next, let \(u \in V(a)\) be any upper bound of \(x\) and \(y\). Then \(x \leq u\) and \(y \leq u\). By \(x \leq u\) and (0.6), we obtain \((x \circ y) \ast u \leq (x \circ y) \ast x\). By (0.13) and \(y \leq u\), the following holds: \((x \circ y) \ast x \leq y \leq u\). Comparison gives \((x \circ y) \ast u \leq u\), i.e., \((x \circ y) \ast u \ast u = 0\). So,
\[
((x \circ y) \ast a) \ast u = (((((x \circ y) \ast a) \ast u) \ast u) \ast (0 \ast a) \ast (0 \ast a) \ast (0 \ast a) = 0.
\]

Hence \((x \circ y) \ast a \leq u\). We have shown that \((x \circ y) \ast a\) is the least upper bound of \(x\) and \(y\). Therefore \((V(a); \leq)\) is an upper semilattice with \(x \lor y = (x \circ y) \ast a\).

It is known that the zero element is the only minimal element of a BCK-algebra.

**Corollary 1.2** ([5], Theorem 1). If \(X\) is a positive implicative BCK-algebra with condition (S), then \((X; \leq)\) forms an upper semilattice with \(x \lor y = x \circ y\) for any \(x, y \in X\).

It is interesting that if the branch \(V(a)\) in Theorem 1.1 is a finite set, we have a nice result as follows.

**Proposition 1.3.** Let \(V(a)\) be a branch of a positive implicative BCI-algebra \(X\) with condition (S). If \(V(a)\) is a finite set, then \((V(a); \leq)\) forms a lattice.

**Proof.** From Theorem 1.1, \((V(a); \leq)\) is an upper semilattice, and we only need to prove that \((V(a); \leq)\) is a lower semilattice. For any \(x, y \in V(a)\), let \(\Omega\) denote the set consisting of the whole lower bounds of \(x\) and \(y\). Then \(\Omega\) is nonempty by \(a \in \Omega\). It is easily seen from (0.11) that \(\Omega \subseteq V(a)\). Now, since \(V(a)\) is a finite set, so is \(\Omega\). There is no harm in assuming \(\Omega = \{b_1, b_2, \cdots, b_n\}\). Put \(b = b_1 \lor b_2 \lor \cdots \lor b_n\). It is not difficult to verify that \(b\) is just the greatest lower bound of \(x\) and \(y\). Therefore \((V(a); \leq)\) is a lower semilattice.

However, if \(V(a)\) is an infinite set, Proposition 1.3 is false. In fact, a counter example has been given in Example 3 of [3]. That is because every BCK-algebra \(X\) is a BCI-algebra with the condition \(V(0) = X\).
In the following let’s turn to consider the distributivity of \((V(a); \leq)\) if \((V(a); \leq)\) is a lattice.

**Theorem 1.4.** Let \(V(a)\) be a branch of a positive implicative BCI-algebra \(X\) with condition \((S)\). If \((V(a); \leq)\) is a lattice, it must be distributive.

**Proof.** From lattice theory, a lattice is distributive if and only if it contains neither a rhombus sublattice nor a pentagon sublattice (see, e.g., [2]). Now, if our assertion is not true, the lattice \((V(a); \leq)\) contains either a rhombus sublattice or a pentagon sublattice whose Hasse diagrams are respectively assumed as follows.

![Hasse diagrams](image)

As to the first diagram, it is easy to see from Theorem 1.1 that
\[
u = x \lor y = (x \circ y) \ast a.
\]

Then (0.4) and (0.13) together give
\[
u \ast (x \ast a) = ((x \circ y) \ast a) \ast (x \ast a) \leq (x \circ y) \ast x \leq y.
\]

In a similar fashion we can prove \(u \ast (x \ast a) \leq z\). So, \(u \ast (x \ast a) \leq y \land z\). Observing our diagram, we have \(y \land z = e\). Hence \(u \ast (x \ast a) \leq e\). Thus \(u \ast e \leq x \ast a\) by (0.8). Thereby (0.7) implies that \((u \ast e) \ast x \leq (x \ast a) \ast x\), namely, \((u \ast x) \ast e \leq 0 \ast a\). It follows from (0.12) that \(u \ast (x \circ e) \leq 0 \ast a\). Therefore \(u \ast (0 \ast a) \leq x \circ e\) by (0.8). Now, using (0.7) once more, we obtain
\[
(u \ast (0 \ast a)) \ast a \leq (x \circ e) \ast a,
\]

which means from (0.19) and Theorem 1.1 that \(u \leq x \lor e\). Note that \(e \leq x\), we have \(x \lor e = x\). Hence \(u \leq x\), a contradiction with \(u > x\).

As to the second diagram, we have \((y \circ z) \ast a = y \lor z = u\) by Theorem 1.1. Then
\[
((x \ast a) \ast a) \ast ((y \circ z) \ast a) = ((x \ast a) \ast a) \ast u = ((x \ast u) \ast a) \ast a.
\]

(1.1)

By (0.15), the left side of (1.1) is equal to \((x \ast (y \circ z)) \ast a\); by \(x \leq u\), the right side to \((0 \ast a) \ast a\). So, \((x \ast (y \circ z)) \ast a = (0 \ast a) \ast a\). Hence
\[
((x \ast (y \circ z)) \ast a) \ast (0 \ast a) = ((0 \ast a) \ast a) \ast (0 \ast a) = 0 \ast a.
\]

Also, by (0.1) and (0.18), the following holds:
\[
((x \ast (y \circ z)) \ast a) \ast (0 \ast a) = ((x \ast a) \ast (0 \ast a)) \ast (y \circ z) = x \ast (y \circ z).
\]

Comparison gives \(x \ast (y \circ z) = 0 \ast a\). Thus \((x \ast y) \ast z = 0 \ast a\) by (0.12). Thereby (0.8) implies \((x \ast y) \ast (0 \ast a) \leq z\). On the other hand, by (0.9) and (0.4), we have
\[
(x \ast y) \ast (0 \ast a) = (x \ast y) \ast (0 \ast y) \leq x.
\]

Then \((x \ast y) \ast (0 \ast a) \leq z \land x\). Because of \(z \land x = e\), it follows \((x \ast y) \ast (0 \ast a) \leq e\), that is, \((x \ast (0 \ast a)) \ast y \leq e\). Thus \(x \ast (0 \ast a) \leq y \circ e\) by (0.14). Hence (0.7) implies
\[
(x \ast (0 \ast a)) \ast a \leq (y \circ e) \ast a,
\]

which means from (0.19) and Theorem 1.1 that \(x \leq y \lor e\). Note that \(e \leq y\), we have \(y \lor e = y\). Therefore \(x \leq y\), a contradiction with \(x > y\).

Summarizing the above arguments, the lattice \((V(a); \leq)\) is distributive.

**Corollary 1.5.** Let \(V(a)\) be a branch of a positive implicative BCI-algebra \(X\) with condition \((S)\). If \(V(a)\) is a finite set, then \((V(a); \leq)\) is a distributive lattice.
Corollary 1.6 ([3], Theorem 3). Let $X$ be a positive implicative BCK-algebra with condition $(S)$. If $(X; \leq)$ is a lattice, it must be distributive.

2 Several identities on a branch We now consider several identities on a branch of a positive implicative BCI-algebra with condition $(S)$, which are similar to those on a positive implicative BCK-algebra with condition $(S)$.

Proposition 2.1. Let $V(a)$ be a branch of a positive implicative BCI-algebra $X$ with condition $(S)$. Then the following are valid:

1. $x = (x \circ x) \ast a$ for any $x \in V(a)$;
2. $x \leq y$ implies $y = (x \circ y) \ast a$ for any $x, y \in V(a)$;
3. $(x \circ y) \ast (x \ast z) = (y \ast (x \circ z)) \ast (0 \ast a)$ for any $x \in V(a)$ and $y, z \in X$.

Proof. (1) and (2) are two immediate results of Theorem 1.1, and we only need to show (3). Assume that $x$ is any element in $V(a)$, and $y, z$ in $X$. By (0.13), we have $(x \circ y) \ast x \leq y$.

Using (0.7) two times, we obtain

$$(((x \circ y) \ast x) \ast x) \ast (0 \ast a) \leq (y \ast x) \ast (0 \ast a).$$

Then (0.17) implies

$$(x \circ y) \ast x \leq (y \ast x) \ast (0 \ast a).$$

(2.1)

Using (0.7) once more and applying (0.1), it follows

$$(x \circ y) \ast (x \ast z) \leq ((y \ast x) \ast z) \ast (0 \ast a),$$

which means from (0.12) that

$$y \ast (x \circ y) \ast (x \ast z) \leq (y \ast (x \circ z)) \ast (0 \ast a).$$

Next, by (0.12) and (0.9), one has

$$y \ast (x \circ y) = (y \ast x) \ast y = 0 \ast x = 0 \ast a.$$

Then (0.4) gives

$$(y \ast (x \circ z)) \ast ((x \circ y) \ast (x \circ z)) \leq y \ast (x \circ y) = 0 \ast a.$$

So, (0.8) implies

$$(y \ast (x \circ z)) \ast (0 \ast a) \leq (y \ast (x \circ z)).$$

(2.3)

Combining (2.2) with (2.3), it yields

$$(x \circ y) \ast (x \circ z) = (y \ast (x \circ z)) \ast (0 \ast a).$$

(2.4)

Theorem 2.2. Let $V(a)$ be a branch of a positive implicative BCI-algebra $X$ with condition $(S)$. Then for any $x, y \in V(a)$ and any $z \in X$, the least upper bound $(x \ast z) \lor (y \ast z)$ of $x \ast z$ and $y \ast z$ exists, and $(x \ast z) \lor (y \ast z) = (x \lor y) \ast z$.

Proof. For any $x, y \in V(a)$ and any $z \in X$, there is no harm in assuming $z \in V(b)$, then $x \ast z \in V(a \ast b)$ and $y \ast z \in V(a \ast b)$ by (0.10). So, by Theorem 1.1, the least upper bound $(x \ast z) \lor (y \ast z)$ of $x \ast z$ and $y \ast z$ exists. It is easy to see from (0.7) that $(x \lor y) \ast z$ is an upper bound of $x \ast z$ and $y \ast z$. Then

$$(x \ast z) \lor (y \ast z) \leq (x \lor y) \ast z.$$

(2.5)

It remains to show that the opposite inequality of (2.4) holds. Denote

$$t = (x \lor y) \ast z \quad \text{and} \quad u = (x \ast z) \lor (y \ast z).$$

Then we have $u \leq t$ by (2.4), and we only need to show $t \leq u$. We first assert that the following are valid:

$$t = (t \ast z) \ast (0 \ast b),$$

(5.5)

$$t = ((t \ast (0 \ast (a \ast b))) \ast (a \ast b)) \ast z,$$

(2.6)

$$t = (x \circ y) \ast (a \ast b),$$

(2.7)

$$u = (x \ast z) \circ (y \ast z) \ast (a \ast b).$$

(2.8)
In fact, by (0.17), we have
\[
 t = (x \lor y) \ast z = ((x \lor y) \ast z) \ast (0 \ast b) = (t \ast z) \ast (0 \ast b),
\]
(2.5) holding. Because \( t \in V(a \ast b) \), (2.6) is a direct result of (0.19). Finally, (2.7) and (2.8) can be seen from Theorem 1.1, as asserted. Now, combining (2.6) with (2.8) and noticing (0.4), we obtain
\[
 t \ast u \leq (t \ast (0 \ast (a \ast b))) \ast ((x \ast z) \circ (y \ast z)).
\]
By (0.1) and (0.12), (2.9) is equivalent to
\[
 t \ast u \leq ((t \ast (x \ast z)) \ast (y \ast z)) \ast (0 \ast (a \ast b)).
\]
Also, by (0.13), one has \((x \circ y) \ast x \leq y\), then \((x \circ y) \ast x \ast z \leq y \ast z\) by (0.7). So,
\[
 ((x \circ y) \ast x) \ast z \ast (y \ast z) = 0.
\]
Right \(*\) multiplying both sides of (2.11) by \(a\) and applying (0.1), one obtains
\[
 (((x \circ y) \ast a) \ast z) \ast (y \ast z) = 0 \ast a.
\]
Hence (2.7) gives
\[
 (t \ast x) \ast (y \ast z) = 0 \ast a.
\]
Moreover, by (0.4), we have \((t \ast z) \ast (x \ast z) \leq t \ast x\). Then (0.7) implies
\[
 ((t \ast z) \ast (x \ast z)) \ast (0 \ast b) \leq (t \ast x) \ast (0 \ast b).
\]
That is,
\[
 ((t \ast z) \ast (0 \ast b)) \ast (x \ast z) \leq (t \ast x) \ast (0 \ast b).
\]
So, by (2.5), we obtain \((t \ast z) \ast (x \ast z) \leq (t \ast x) \ast (0 \ast b)\). Hence
\[
 (t \ast (x \ast z)) \ast (y \ast z) \leq ((t \ast x) \ast (0 \ast b)) \ast (y \ast z) \quad \text{[by (0.7)]}
\]
\[
 = ((t \ast x) \ast (y \ast z)) \ast (0 \ast b) \quad \text{[by (0.1)]}
\]
\[
 = (0 \ast a) \ast (0 \ast b) \quad \text{[by (2.12)]}
\]
\[
 = 0 \ast (a \ast b). \quad \text{[by (0.2)]}
\]
From this, we derive
\[
 ((t \ast (x \ast z)) \ast (y \ast z)) \ast (0 \ast (a \ast b)) = 0.
\]
Comparing (2.10) with (2.13), it yields \( t \ast u \leq 0 \), in other words, \( t \ast u = 0 \) by 0 being a minimal element of \(X\). Consequently, \( t \leq u \). The proof is complete. \( \square \)

**Theorem 2.3.** Let \( V(a) \) be a branch of a positive implicative BCI-algebra \(X\) with condition \((S)\). Then the following hold: for any \(x, y, z \in V(a)\),
\begin{enumerate}
  \item \(x = (x \ast (x \ast y)) \lor ((x \ast y) \ast (0 \ast a))\);
  \item \(x \lor y = x \lor ((y \ast x) \ast (0 \ast a))\);
  \item \((x \lor y) \ast x = y \ast x\) and \((x \lor y) \ast y = x \ast y\);
  \item \(z \ast (x \lor y) = (z \ast x) \ast (z \ast y)\).
\end{enumerate}

**Proof.** (1) For any \(x, y \in V(a)\), we have \(x \ast y \in V(a \ast a) = V(0)\) by (0.10). Then \(0 \leq x \ast y\). So, by (0.6) and (0.11), we obtain
\[
 x \ast (x \ast y) \leq x \quad \text{and} \quad x \ast (x \ast y) \in V(a).
\]
Also, by (0.4), one has \((x \ast y) \ast (0 \ast y) \leq x\). So, by (0.9) and (0.11), one obtains
\[
 (x \ast y) \ast (0 \ast a) \leq x \quad \text{and} \quad (x \ast y) \ast (0 \ast a) \in V(a).
\]
Since \((V(a); \leq)\) is an upper semilattice, it follows
\[
 (x \ast (x \ast y)) \lor ((x \ast y) \ast (0 \ast a)) \leq x. \quad \text{(2.14)}
\]
Next, by (0.3), we have
\[
 (x \ast (0 \ast a)) \ast (x \ast (x \ast y)) \leq (x \ast y) \ast (0 \ast a).
\]
Then (0.14) gives
\[ x \ast (0 \ast a) \leq (x \ast (x \ast y)) \circ ((x \ast y) \ast (0 \ast a)). \]
So, by (0.7), we obtain
\[ (x \ast (0 \ast a)) \ast a \leq ((x \ast (x \ast y)) \circ ((x \ast y) \ast (0 \ast a))) \ast a. \]
Hence (0.19) and Theorem 1.1 imply
\[ x \leq (x \ast (x \ast y)) \lor ((x \ast y) \ast (0 \ast a)). \quad (2.16) \]
Comparing (2.15) with (2.16), it yields \( x = (x \ast (x \ast y)) \lor ((x \ast y) \ast (0 \ast a)). \)

(2) Following the proof of (2.1), we have
\[ (y \ast x) \ast (0 \ast a) \leq y \quad \text{and} \quad (y \ast x) \ast (0 \ast a) \in V(a). \]
Since \((V(a); \leq)\) is an upper semilattice and \(x, y \in V(a)\), it follows
\[ x \lor ((y \ast x) \ast (0 \ast a)) \leq x \lor y. \quad (2.17) \]
Next, following the proof of (2.1), we have \((x \circ y) \ast x \leq (y \ast x) \ast (0 \ast a)\). Then (0.14) implies
\[ x \circ y \leq x \circ ((y \ast x) \ast (0 \ast a)). \]
By (0.7), we derive
\[ (x \circ y) \ast a \leq (x \circ ((y \ast x) \ast (0 \ast a))) \ast a. \]
Therefore \( x \lor y \leq x \lor ((y \ast x) \ast (0 \ast a)) \) by Theorem 1.1. Comparison with (2.17) gives
\[ x \lor y = x \lor ((y \ast x) \ast (0 \ast a)). \]

(3) It is a direct result of Theorem 2.2.

(4) By (0.19), we have \( z = (z \ast (0 \ast a)) \ast a; \) by (2) and Theorem 1.1, we obtain
\[ x \circ y = x \circ ((y \ast x) \ast (0 \ast a)) = (x \circ ((y \ast x) \ast (0 \ast a))) \ast a. \]
Then
\[
\begin{align*}
z \ast (x \lor y) &= ((z \ast (0 \ast a)) \ast a) \ast ((x \circ ((y \ast x) \ast (0 \ast a))) \ast a) \\
&\leq (z \ast (0 \ast a)) \ast (x \circ ((y \ast x) \ast (0 \ast a))) \quad \text{[by (0.4)]} \\
&= ((z \ast (0 \ast a)) \ast x) \ast ((y \ast x) \ast (0 \ast a)) \quad \text{[by (0.12)]} \\
&= ((z \ast x) \ast (0 \ast a)) \ast ((y \ast x) \ast (0 \ast a)) \quad \text{[by (0.1)]} \\
&\leq (z \ast x) \ast (y \ast x). \quad \text{[by (0.4)]}
\end{align*}
\]
That is,
\[ z \ast (x \lor y) \leq (z \ast x) \ast (y \ast x). \quad (2.18) \]
Next, by (0.3) and (3), one has
\[ (z \ast x) \ast (z \ast (x \lor y)) \leq (x \lor y) \ast x = y \ast x. \]
So, (0.8) implies
\[ (z \ast x) \ast (y \ast x) \leq z \ast (x \lor y). \quad (2.19) \]
Combining (2.18) with (2.19), it follows
\[ z \ast (x \lor y) = (z \ast x) \ast (y \ast x). \]

References


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