ON BE-ALGEBRAS

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Abstract. In this paper, as a generalization of a BCK-algebra, we introduce the notion of a BE-algebra, and using the notion of upper sets we give an equivalent condition of the filter in BE-algebras.

1. Introduction.

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([9]) introduced the notion of d-algebras which is another generalization of BCK-algebras, and also they introduced the notion of B-algebras ([10, 11]), i.e., (I) $x * x = 0$; (II) $x * 0 = x$; (III) $(x * y) * z = x * (z * (0 * y))$, for any $x, y, z \in X$, which is equivalent in some sense to the groups. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim ([7]) introduced a new notion, called an BH-*algebra*, which is a generalization of $BCH/BCI/BCK$-algebras, i.e., (I); (II) and (IV) $x * y = 0$ and $y * x = 0$ imply $x = y$ for any $x, y \in X$. A. Walendziak obtained the another equivalent axioms for B-algebra ([12]). H. S. Kim, Y. H. Kim and J. Neggers ([6]) introduced the notion a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. C. B. Kim and H. S. Kim ([5]) introduced the notion of a BM-algebra which is a specialization of B-algebras. They proved that the class of BM-algebras is a proper subclass of B-algebras and also showed that a BM-algebra is equivalent to a 0-commutative B-algebra. In this paper, as a generalization of a BCK-algebra, we introduce the notion of a BE-algebra, and using the notion of upper sets we give an equivalent condition of the filter in BE-algebras.

Definition 1. An algebra $(X; *, 1)$ of type (2, 0) is called a BE-*algebra* if

(BE1) $x * x = 1$ for all $x \in X$;

(BE2) $x * 1 = 1$ for all $x \in X$;

(BE3) $1 * x = x$ for all $x \in X$;

(BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*)

We introduce a relation "$\leq$" on X by $x \leq y$ if and only if $x * y = 1$.

Proposition 2. *If $(X; *, 1)$ is a BE-algebra, then $x * (y * x) = 1$ for any $x, y \in X$.*

Proof. Given $x, y \in X$, we have $1 = y * 1 = y * (x * x) = x * (y * x)$, proving the proposition. □

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Example 3. Let $X := \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra.

Definition 4. Let $(X; *, 1)$ be a BE-algebra and let F be a non-empty subset of X. Then F is said to be a *filter* of X if

(F1) $1 \in F$;

(F2) $x * y \in F$ and $x \in F$ imply $y \in F$.

In Example 3, $F_1 := \{1, a, b\}$ is a filter of X, but $F_2 := \{1, a\}$ is not a filter of X, since $a * b \in F_2$ and $a \in F_2$, but $b \notin F_2$.

Definition 5. Let X be a BE-algebra and let $x, y \in X$. Define

$$A(x, y) := \{z \in X \mid x * (y * z) = 1\}$$

We call $A(x, y)$ an *upper set* of x and y. It is easy to see that $1, x, y \in A(x, y)$ for any $x, y \in X$. The set $A(x, y)$, where $x, y \in X$, need not be a filter of X in general. In Example 3, it is easy to check that $A(1, a) = \{1, a\} = F_2$, which means that $A(1, a)$ is not a filter of X.

Proposition 6. *Let X be a BE-algebra. If $y \in X$ satisfies $y * z = 1$ for all $z \in X$, then $A(x, y) = X = A(y, x)$ for all $x \in X$.*

Proof. Straightforward. □

Definition 7. A BE-algebra $(X, *, 1)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Example 8. Let $X := \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

It is easy to see that X is a BE-algebra satisfying self distributivity.

Note that the BE-algebra in Example 3 is not self distributive, since $d*(a*0) = d*d = 1$, while $(d * a) * (d * 0) = 1 * a = a$.

Theorem 9. *Let $(X;*,1)$ be a self distributive BE-algebra. Then the upper set $A(x,y)$ is a filter of X, where $x,y\in X$.*

Proof. Let $a*b\in A(x,y)$ and $a\in A(x,y)$. Then $1=x*(y*(a*b))$ and $1=x*(y*a)$. It follows from the self distributivity law that

$$\begin{aligned}1 &= x*(y*(a*b)) && \\ &= x*((y*a)*(y*b)) && \text{[self distributive]}\\ &= (x*(y*a))*(x*(y*b)) && \text{[self distributive]}\\ &= 1*(x*(y*b)) && [a\in A(x,y)]\\ &= x*(y*b), && \text{[(BE3)]}\end{aligned}$$

whence $b\in A(x,y)$. This proves that $A(x,y)$ is a filter of X. □

Using the notion of upper set $A(x,y)$ we give an equivalent condition of the filter in BE-algebras.

Theorem 10. *Let F be a non-empty subset of a BE-algebra X. Then F is a filter of X if and only if $A(x,y)\subseteq F$ for all $x,y\in F$.*

Proof. Assume that F is a filter of X and let $x,y\in F$. If $z\in A(x,y)$, then $x*(y*z)=1\in F$. Since $x,y\in F$, by applying $(F2)$ we have $z\in F$. Hence $A(x,y)\subseteq F$. Conversely, suppose that $A(x,y)\subseteq F$ for all $x,y\in F$. Since $x*(y*1)=x*1=1$, $1\in A(x,y)\subseteq F$. Assume $a*b, a\in F$. Since $(a*b)*(a*b)=1$, we have $b\in A(a*b,a)\subseteq F$. Hence F is a filter of X. □

Theorem 11. *If F is a filter of a BE-algebra X, then $F=\cup_{x,y\in F}A(x,y)$.*

Proof. Let F be a filter of X and let $z\in F$. Since $z*(1*z)=z*z=1$, we have $z\in A(z,1)$. Hence

$$F\subseteq \cup_{z\in F}A(z,1)\subseteq \cup_{x,y\in F}A(x,y).$$

If $z\in\cup_{x,y\in F}A(x,y)$, then there exist $a,b\in F$ such that $z\in A(a,b)$. It follows from Theorem 10 that $z\in F$. This means that $\cup_{x,y\in F}A(x,y)\subseteq F$. This completes the proof. □

Corollary 12. *If F is a filter of a BE-algebra X, then $F=\cup_{x\in F}A(x,1)$.*

Proof. If $z\in\cup_{x\in F}A(x,1)$, then there exists $a\in F$ such that $z\in A(a,1)$, which means that $a*z=a*(1*z)=1\in F$. Since F is a filter of X and $a\in F$, we have $z\in F$. This proves $\cup_{x\in F}A(x,1)\subseteq F$. The converse was proved in the proof of Theorem 11. □

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