MODULAR GROUP ALGEBRAS WITH ALMOST MAXIMAL LIE NILPOTENCY INDICES, II

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Abstract. Let $K$ be a field of positive characteristic $p$ and $KG$ the group algebra of a group $G$. It is known that, if $KG$ is Lie nilpotent, then its upper (or lower) Lie nilpotency index is at most $|G'| + 1$, where $|G'|$ is the order of the commutator subgroup. Previously we determined the groups $G$ for which the upper/lower nilpotency index is maximal or the upper nilpotency index is ‘almost maximal’ (that is, of the next highest possible value, namely $|G'|-p+2$). Here we determine the groups for which the lower nilpotency index is ‘almost maximal’.

Let $R$ be an associative algebra with identity. The algebra $R$ can be regarded as a Lie algebra, called the associated Lie algebra of $R$, via the Lie commutator $[x,y] = xy - yx$, for every $x, y \in R$. Set $[x_1, \ldots, x_n] = [x_1, \ldots, x_{n-1}], x_n]$, where $x_1, \ldots, x_n \in R$. The $n$-th lower Lie power $R^{[n]}$ of $R$ is the associative ideal generated by all the Lie commutators $[x_1, \ldots, x_n]$, where $R^{[1]} = R$ and $x_1, \ldots, x_n \in R$. By induction, we define the $n$-th upper Lie power $R^{(n)}$ of $R$ as the associative ideal generated by all the Lie commutators $[x,y]$, where $R^{(1)} = R$ and $x \in R^{(n-1)}$, $y \in R$.

The algebra $R$ is called Lie nilpotent (respectively upper Lie nilpotent) if there exists $m$ such that $R^{[m]} = 0$ ($R^{(m)} = 0$). The algebra $R$ is called Lie hypercentral if for each sequence $\{a_i\}$ of elements of $R$ there exists some $n$ such that $[a_1, \ldots, a_n] = 0$. The minimal integers $m, n$ such that $R^{[m]} = 0$ and $R^{(n)} = 0$ are called the lower Lie nilpotency index and the upper Lie nilpotency index of $R$ and they are denoted by $t_L(R)$ and $t_U(R)$, respectively.

Let $U(KG)$ be the group of units of a group algebra $KG$. For the noncommutative modular group algebra $KG$ the following Theorem due to A.A. Bovdi, I.I. Khripta, I.B.S. Passi, D.S. Passman and etc. (see [3, 10]) is well known: The following statements are equivalent: (a) $KG$ is Lie nilpotent; (b) $KG$ is Lie hypercentral; (c) $KG$ is upper Lie nilpotent; (d) $U(KG)$ is nilpotent; (e) char($K$) = $p > 0$, $G$ is nilpotent and its commutator subgroup $G'$ is a finite $p$-group.

It is well known (see [12, 14]) that, if $KG$ is Lie nilpotent, then

\[ t_L(KG) \leq t_U(KG) \leq |G'| + 1. \]

Moreover, according to [1], if char($K$) > 3, then $t_L(KG) = t_U(KG)$. But the question of when does $t_L(KG) = t_U(KG)$ hold for char($K$) = 2, 3 is in general still open. Using the program packages GAP and LAGUNA (see [5, 9]), A.Konovalov in [11] verified that $t_L(KG) = t_U(KG)$ for all 2-groups of order at most 256 and char($K$) = 2. Several important results on this topic were obtained in [4].

We say that a Lie nilpotent group algebra $KG$ has

- upper maximal Lie nilpotency index, if $t_U(KG) = |G'| + 1$;
- lower maximal Lie nilpotency index, if $t_L(KG) = |G'| + 1$;

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• upper almost maximal Lie nilpotency index, if \( t_L(K) = |G'| - p + 2 \);
• lower almost maximal Lie nilpotency index, if \( t_L(K) = |G'| - p + 2 \).

A. Shalev in [13] began to study the question when do the Lie nilpotent group algebras \( KG \) have lower maximal Lie nilpotency index. In [6, 13] there was given the complete description of the Lie nilpotent group algebras \( KG \) with lower/upper maximal Lie nilpotency index. In [7] the characterization of such \( KG \) with upper almost maximal Lie nilpotency index was obtained. In the present paper we prove the following

0.1. Theorem. Let \( KG \) be a Lie nilpotent group algebra over a field \( K \) of positive characteristic \( p \). Then \( KG \) has lower almost maximal Lie nilpotency index if and only if one of the following conditions holds:

(i) \( p = 2 \), \( cl(G) = 2 \) and \( \gamma_2(G) \) is noncyclic of order 4;
(ii) \( p = 2 \), \( cl(G) = 4 \), \( \gamma_2(G) \cong C_4 \times C_2 \) and \( \gamma_3(G) \cong C_2 \times C_2 \);
(iii) \( p = 2 \), \( cl(G) = 4 \) and \( \gamma_2(G) \) is elementary abelian of order 8;
(iv) \( p = 3 \), \( cl(G) = 3 \) and \( \gamma_2(G) \) is elementary abelian of order 9.

Now using results of [6, 7, 13] we obtain

0.2. Corollary. Let \( KG \) be a Lie nilpotent group algebra over a field \( K \) of positive characteristic \( p \). The group algebra \( KG \) has lower almost maximal Lie nilpotency index if and only if it has upper almost maximal Lie nilpotency index.

According to Du’s and Khripta’s Theorems (see [8, 10]) we have

0.3. Corollary. Let \( KG \) be the group algebra of a finite \( p \)-group \( G \) over a field \( K \) of positive characteristic \( p \) and \( U(KG) \) its group of units. Then the nilpotency class of \( U(KG) \) is equal to \( |G'| - p + 1 \) if and only if \( G \) and \( K \) satisfy one of the conditions (i)–(iv) of Theorem 1.

As a consequence, we obtain that the Theorem 3.9 of [13] can not be extent for \( p = 2 \) and \( p = 3 \):

0.4. Corollary. Let \( K \) be a field of positive characteristic \( p \) and \( G \) a nilpotent group such that \( |G'| = p^n \).

(i) If \( p = 2 \) and \( t_L(KG) < 2^n + 1 \), then \( t_L(KG) \leq 2^n \).
(ii) If \( p = 3 \) and \( t_L(KG) < 3^n + 1 \), then \( t_L(KG) \leq 3^n - 1 \).

We shall use the following results:

0.5. Proposition. ([6, 13]) Let \( KG \) be a Lie nilpotent group algebra over a field \( K \) of positive characteristic \( p \). Then \( t^L(KG) = |G'| + 1 \) if and only if either \( G' \) is cyclic or \( p = 2 \) and \( G' \) is noncyclic of order 4 such that \( \gamma_3(G) \neq 1 \). Moreover, if \( t^L(KG) = |G'| + 1 \) then \( t_L(KG) = t^L(KG) \).

0.6. Proposition. ([7]) Let \( KG \) be a Lie nilpotent group algebra over a field \( K \) of positive characteristic \( p \). Then \( KG \) has upper almost maximal Lie nilpotency index if and only if one of the conditions of Theorem 1 holds. Moreover, if \( t^L(KG) < |G'| + 1 \) then \( t^L(KG) \leq |G'| - p + 2 \).

Let \( KG \) be a Lie nilpotent group algebra over a field \( K \) of char\( (K) = p \) and \( t_L(KG) = |G'| - p + 2 \). Obviously \( t_L(KG) \leq t^L(KG) \leq |G'| + 1 \). If \( t^L(KG) > |G'| - p + 2 \), then by Propositions 0.5 and 0.6 we get \( t^L(KG) = |G'| + 1 \) and also \( t_L(KG) = t^L(KG) = |G'| + 1 \), a contradiction. Thus by Proposition 0.6 we obtain that \( t^L(KG) = |G'| - p + 2 \) and \( G \) satisfies one of the conditions of our Theorem.
Conversely, let condition (i) of the Theorem holds. Since \( \text{cl}(G) = 2 \), by Theorem 3.2 of [4] we obtain that \( t_L(KG) \geq 4 \) and \( t_L(KG) = t_L(KG) \).

First, let \( G \) be a nilpotent group of class \( \text{cl}(G) = 4 \), such that either \( \gamma_2(G) \cong C_4 \times C_2 \) or \( \gamma_2(G) \cong C_2 \times C_2 \times C_2 \). For \( g_1, \ldots, g_n \in G \) we set \( (g_1, g_2) = g_1^{-1}g_2^{-1}g_1g_2 \) and \( (g_1, \ldots, g_n) = (g_1, \ldots, g_{n-1}, g_n) \). If \( G \) is finite, then by [2] there exist \( g, h \in G \) with the properties

\[
(1) \quad a = (g, h), \quad b = (g, h, h), \quad c = (g, h, h, h),
\]

such that \( \gamma_2(G) = \langle a, b, c \rangle, \quad \gamma_3(G) = \langle b, c \rangle, \quad \gamma_4(G) = \langle c \rangle \), where for the case \( \gamma_2(G) \cong C_4 \times C_2 \) we put \( c = a^2 \).

Some finitely generated subgroup will have the same lower central series, so there is no harm in assuming that \( G \) itself is finitely generated and therefore residually finite. Let \( N \) be the normal subgroups of finite index which avoid \( \gamma_2(G) \): then \( G/N \) is a finite 2-group, and so there exist \( g, h \in G \) such that the commutator \( (g, h) \) lies in the coset \( aN \), \( (g, h, h) \in bN \), and \( (g, h, h) \in cN \). Now \( a^{-1}(g, h) \in \gamma_2(G) \cap N = 1 \) shows that in fact \( (g, h) = a \) and similar arguments show that also \( (g, h, h) = b \) and \( (g, h, h, h) = c \).

Let \( G \) be a finitely generated nilpotent group of class \( \text{cl}(G) = 4 \), such that \( \gamma_2(G) = \langle a \rangle \times \langle b \rangle \cong C_4 \times C_2 \). Therefore we have

\[
(a, g) = f, \quad (a, h) = b, \quad (b, g) = t, \quad (f, g) = z_1, \quad (f, h) = z_2,
\]

where \( f, t \in \gamma_3(G) \), \( z_1, z_2 \in \gamma_4(G) \). Since \( a^gh = a^ {hga} \), we get \( t = z_2 \), so

\[
(2) \quad a^g = af, \quad f^g = f z_1, \quad f^h = f z_2, \quad b^h = a^2b, \quad b^g = b z_2.
\]

We consider the following two cases:

Case 1. Let \( f \in \{ b, a^2b \} \). By (1) and (2), using the well known equality \( (ab, c) = (a, c)(a, b)(b, c) \), we get \( (g^2, h) = a^2f \) so \( (g^2h^2)^{-1}gh^2g = a^2b \) and \( (h^2g^2)^{-1}gh^2h^3 = a^2f \).

It follows that

\[
[h, gh, g] = [g^2h^2(a^2b + 1), g] = gh^2(a^2b(a^2f b z_2 + 1) + a^3b + 1) \\
\in \{ g^2h^2(1 + a^2), \quad g^2h^2(1 + (a + a^2 + a^3)b) \};
\]

\[
[h, gh, g, h] = h^2g^2h^2(1 + a^2) - [h, gh, g] + [h, gh, g] \\
\in \{ h^2g^2h^2(1 + a^2), \quad h^2g^2h^2(1 + a^2)b \}.
\]

Now, since \( (ghg^2h^2)^{-1}gh^2g = a^3b \) and \( (hghg^2h^2)^{-1}gh^2h^3 = abf z_1 \), we obtain that

\[
[h, gh, g, h, g] = (hgh^2h^2 + hghg^2h^2)(1 + a^2) \\
= gh^2h^2(1 + ab)(1 + a^2),
\]

\[
[h, gh, g, h, g, h] = (ghg^2h^3(1 + a^2) + hghg^2h^2(1 + ab))(1 + a^2) \\
= hghg^2h^2a(1 + b)(1 + a^2).
\]

Finally, by \( (h^2g^2h^2)^{-1}ghg^2h^3 = abf z_1 \) we get

\[
[h, gh, g, h, h] = (hghg^2h^3 + h^2g^2h^2a(1 + a^2)(1 + b) \\
= qa(1 + a)(1 + a^2)(1 + b) = \eta \cdot \tilde{a} \cdot \tilde{b} \neq 0,
\]

where \( \eta = h^2g^2h^2 \) and \( \tilde{g} = \sum_{h \in \langle g \rangle} h \).

Case 2. Let \( f \in \{ 1, a^2 \} \). By (1) and (1) it yields that

\[
a^g = af, \quad a^h = ab, \quad b^g = b, \quad b^h = a^2b.
\]
Clearly, that $[gh, g, gh] = [g^2h(a^3 + 1), gh] = g^2hgh(abf + a^3)$. Since 
$(ghg^2hg)^{−1}g^2hghgh = a^3$ and $(g^2hg^2hgh)^{−1}ghg^2hghg = a^3b$, this yields

$[gh, g, gh, gh] = ghg^2hgh(a^3 + a^3bf) + abf + a^3$;

$[gh, g, gh, gh, g] = α(a^3b(a^2 + a^2bf + ab + a^3f) + a^2 + a^2bf + abf + a^3)$

where $α = g^2hg^2hgh$. Obviously, $(h^2g^2hgh)^{−1}g^2hgh^2hgh = ab$ and

$(h^2g^2hgh)^{−1}g^2hgh^2hgh = ab$, so it follows that

$[gh, g, gh, gh, g] ∈ \{ α(1 + a^2 + a^3), α(1 + a^2)(1 + ab) \}$,

where $β = h^2g^2hgh$, $γ = h^2g^2hgh$ and $\hat{g} = \sum_{h ∈ \langle g \rangle} h$.

Therefore in both cases, the lower Lie nilpotency index is at least 8. Since $t^L(KG) = 8$, we obtain that $t_L(KG) = t^L(KG) = 8$.

Let $G$ be a finitely generated nilpotent group of class $cl(G) = 4$, such that $γ_2(G) = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \cong C_2 \times C_2 \times C_2$. The proof is similar to the previous case, using the same commutators.

Let condition (iv) of the Theorem holds. Obviously, similarly to the previous cases, we can assume that $G$ is finitely generated and, according to [2], there exist $g, h ∈ G$ such that

$\hat{g} \wedge = a, \quad (g, h) = (a, h) = b, \quad (a, g) = t ∈ \langle b \rangle$.

Therefore, $a^h = ab, a^g = at$ and we consider the following cases:

Case 1. Let $t = 1$. By (3), using a simple computation we obtain that

$[gh, g, g] = g^3h \cdot \hat{a}$;

$[gh, g, g, h] = h^3g(a^2b^2 + ab - a^2 - a)$;

$[gh, g, g, h, g] = g^2h^2(a + b + a^2b^2 - b - a^2 - ab)$;

$[gh, g, g, h, g, h] = g^2h^2h^2h^2h(1 - a^2)(1 + b + b^2)$;

$[gh, g, g, h, g, h] = h^2g^2h^2h \cdot \hat{a} \cdot \hat{b} \neq 0$.

Case 2. Let $t = b$. By (3) it is easy to check that

$[h, g, gh] = g^2gh(gh)(b - 1)$;

$[h, g, gh, g] = g^2hgh(1 - a^2)(b - 1)$;

$[h, g, gh, g, h] = ghg^2hgh(a^2 + ab - 1 - b^2)(b - 1)$;

$[h, g, gh, g, h, g] = ghg^2hgh(a^2b + ab - a^2 - b^2)(b - 1)$;

$[h, g, gh, g, h, g] = -ghg^2hgh \cdot \hat{a} \cdot \hat{b} \neq 0$.

Case 3. Let $t = b^2$. Similarly to the previous two cases we have

$[g, gh, g] = g^3(-1 - a - a^2b)$;

$[g, gh, g, h] = ghg^2hgh(a^2b + 1 - b^2 - a^2b^2)$;

$[g, gh, g, h, g] = ghg^2hgh(a + b + a^2b^2 - ab^2 - a^2b - 1)$;

$[g, gh, g, h, g, h] = ghg^2hgh(a - a^2)(1 + b + b^2)$;

$[g, gh, g, h, g, h] = -h^2ghg^2hgh \cdot \hat{a} \cdot \hat{b} \neq 0$.

Therefore the lower Lie nilpotency index is at least 8. Since $t^L(KG) = 8$ we obtain that $t_L(KG) = t^L(KG) = 8$ and the proof is complete.

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