MOND-PEČARIĆ METHOD FOR A MEAN-LIKE TRANSFORMATION
OF OPERATOR FUNCTIONS

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Abstract. As a generalization of the quasi-arithmetic mean, we consider a mean-like transformation of operator functions. Let \( \Phi \) be a unital positive linear map of \( B(H) \), the algebra of all bounded linear operators on a Hilbert space \( H \), and \( f(t) \) (resp. \( g(t) \)) a continuous function on an interval \([m, M]\) (resp. \([f([m, M])]\)). Then it is defined by \((g \circ \Phi \circ f)(A)\) for a selfadjoint operator \( A \) with \( m \leq A \leq M \). We give a lower bound of the difference between \((g \circ \Phi \circ f)(A)\) and \( \Phi(A) \). Precisely we prove that if \( f(t) \) is concave on \([m, M] \) and \( g(t) \) is increasing and convex on \([f([m, M])]\), then for each \( \lambda \in \mathbb{R} \), \((g \circ \Phi \circ f)(A) - \lambda \Phi(A) \geq \min_{t \in [m, M]} \{g(\alpha_f t + \beta_f) - \lambda t\} \) where \( \alpha_f := \frac{f(M) - f(m)}{M - m} \) and \( \beta_f := Mf(m) - mf(M) \). It is an extension of our previous estimation for \( \Phi = \omega_x \), the vector state for a unit vector \( x \in H \).

1 Introduction

Let \( f(t) \) be a strictly monotone, continuous function on an interval \([m, M]\) and \( w = (w_1, \ldots, w_n) \) a weight, i.e., \( \sum_{i=1}^n w_i = 1 \) and \( w_i \geq 0 \). Then the quasi-arithmetic mean is defined by \( f^{-1}(\sum_{i=1}^n w_i f(t_i)) \) for \( t_1, \ldots, t_n \in [m, M] \), cf.[2], [1]. Moreover if \( f(t) \) is concave on \([m, M]\), then the quasi-arithmetic mean and arithmetic mean inequality

\[
    f^{-1}(\sum_{i=1}^n w_i f(t_i)) \leq \sum_{i=1}^n w_i t_i
\]

follows from the classical Jensen inequality (see (7)).

It can be expressed as

\[
    f^{-1}(\langle f(A)x, x \rangle) \leq \langle Ax, x \rangle
\]

for a selfadjoint operator \( A \) on \( H \) with \( m \leq A \leq M \) and a unit vector \( x \in H \). By the way, replacing \( f^{-1} \) in (1) to an increasing function \( g \) on \([f([m, M])]\), we considered a low bound of \( g(\langle f(A)x, x \rangle) \) by the arithmetic mean \( \langle Ax, x \rangle \) in our previous notes [6], [7] and [8]. As a matter of fact, we showed that for every real number \( \lambda > 0 \)

\[
    g(\langle f(A)x, x \rangle) - \lambda \langle Ax, x \rangle \geq \min_{t \in [m, M]} \{g(\alpha_f t + \beta_f) - \lambda t\}
\]

holds for all unit vectors \( x \in H \) where

\[
    \alpha_f := \frac{f(M) - f(m)}{M - m} \quad \text{and} \quad \beta_f := \frac{Mf(m) - mf(M)}{M - m}.
\]

It suggests us that a unital positive linear map \( \Phi \) on \( B(H) \), the algebra of all bounded linear operators on \( H \), is regarded as a mean-like transformation of operator functions.

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Namely we can define a new operator function $g \circ \Phi \circ f$ for given $f$ and $g$. For example, if we take $\Phi(A) = \langle Ax, x \rangle$ for some unit vector $x \in H$, then (1) is rephrased by $f^{-1} \circ \Phi \circ f)(A) \leq \Phi(A)$.

As a continuation of our previous notes, we estimate a lower bound of the difference between $(g \circ \Phi \circ f)(A)$ and $\Phi(A)$ in this note. As an application of Mond-Pečarić method [5], we prove: Let $A$ be a selfadjoint operator on a Hilbert space $H$ with $m \leq A \leq M$. Let $f(t)$ be a concave function on $[m, M]$ and $g(t)$ an increasing convex function on $f([m, M])$. Then for every real number $\lambda$

$$g(\Phi(f(A))) - \lambda \Phi(A) \geq \min_{t \in [m, M]} \{g(\alpha f t + \beta f) - \lambda t\}.$$  

2 Estimations of $g(\Phi(f(A))) - \lambda \Phi(A)$ We give a lower bound of $g(\Phi(f(A)))$ by $\Phi(A)$ without the convexity of $g(t)$.

**Theorem 1.** Let $A$ be a selfadjoint operator on a Hilbert space $H$ with $m \leq A \leq M$ for some $m < M$. Let $f(t)$ be a concave function on $[m, M]$ with $f(m) \neq f(M)$ and $g(t)$ be a continuous function on $f([m, M])$. Let $\Phi$ be a unital positive linear map on $B(H)$. Then for every real number $\lambda$ with $\lambda \alpha_f > 0$

$$g(\Phi(f(A))) - \lambda \Phi(A) \geq \min_{t \in [m, M]} \{g(\alpha f t + \beta f) - \lambda t\}.$$  

Precisely, if $\lambda > 0$ and $\alpha_f > 0$ (resp. $\lambda < 0$ and $\alpha_f < 0$), then

$$g(\Phi(f(A))) - \lambda \Phi(A) \geq \min_{t \in [\alpha_f, \beta_f]} \{g(\alpha f t + \beta f) - \lambda t\}.$$  

(4) 

where $f_{\max} := \max_{t \in [m, M]} f(t)$.

**Proof.** Since $f(t)$ is concave, we have $f(A) \geq \alpha_f A + \beta f$, and hence $\Phi(f(A)) \geq \alpha_f \Phi(A) + \beta f$. So it follows from $\lambda \alpha_f > 0$ that

$$\lambda \Phi(A) \leq \frac{\lambda}{\alpha_f} (\Phi(f(A)) - \beta f),$$

so that

$$g(\Phi(f(A))) - \lambda \Phi(A) \geq g(\Phi(f(A))) - \frac{\lambda}{\alpha_f} (\Phi(f(A)) - \beta f)$$

$$\geq \min_{t \in [m, M]} \{g(u) - \frac{\lambda}{\alpha_f} (u - \beta f)\}$$

by $\sigma(\Phi(f(A))) \subset f([m, M])$.

Moreover if $\lambda > 0$ (and $\alpha_f > 0$), then for every $u \in f([m, M]) = [f(m), f_{\max}]$ the equation $u = \alpha_f t + \beta f$ has a unique solution $t = t_u = \frac{u - \beta f}{\alpha_f} \in \left[\frac{\alpha_f, f_{\max} - \beta f}{\alpha_f}\right]$. Since

$$g(u) - \frac{\lambda}{\alpha_f} (u - \beta f) = g(\alpha f t_u + \beta f) - \lambda t_u,$$
Corollary 2. Let the hypothesis of Theorem 1 be satisfied and \( f(t) \) be monotone. Then for every real number \( \lambda \) with \( \lambda \alpha_f > 0 \)

\[
g(\Phi(f(A))) - \lambda \Phi(A) - \min_{t \in [m,M]} \{ g(\alpha_f t + \beta_f) - \lambda \} = \frac{7}{9}
\]

so that

\[
g(\Phi(f(A))) - \lambda \Phi(A) - \min_{t \in [m,M]} \{ g(\alpha_f t + \beta_f) - \lambda \} = \frac{1}{9} \left[ \begin{array}{ccc} 41 & -37 & 26 \\ -37 & 26 & -26 \\ 26 & -26 & 26 \end{array} \right] \ngeq 0.
\]

In Theorem 1, if \( f(t) \) is increasing (resp. decreasing), then \( f_{\text{max}} = f(M)(= \alpha_f M + \beta_f) \) (resp. \( f_{\text{max}} = f(m)(= \alpha_f m + \beta_f) \)). So the following corollary is easily obtained and corresponds to (2):

\[
g(\Phi(f(A))) - \lambda \Phi(A) \geq \min_{t \in [m,M]} \{ g(\alpha_f t + \beta_f) - \lambda \}.
\]
3 The main result Let \( f(t) \) be a concave function on \([m, M]\). In this section, we give a lower bound of \( g(\Phi(f(A))) - \lambda \Phi(A) \) assuming the convexity of \( g(t) \). First of all, suppose that \( g(t) \) is increasing and \( g(t) > 0 \). Since \( f(t) \) is concave, it follows that \( \Phi(f(A)) \geq \alpha_f \Phi(A) + \beta_f \).

By [3, Theorem 2.2], we have \( g(\Phi(f(A))) \geq \min_{t \in I_f} \frac{g(r)}{\alpha_f t + \beta_f} \) \( g(\alpha_f \Phi(A) + \beta_f) \) where

\[
\alpha_{fg} := \frac{g(f_{\max}) - g(f_{\min})}{f_{\max} - f_{\min}} \quad \text{and} \quad \beta_{fg} := \frac{g(f_{\max}) - g(f_{\min})}{f_{\max} - f_{\min}} \quad \text{for} \quad f_{\max} := \max_{t \in [m, M]} f(t) \quad \text{and} \quad f_{\min} := \min_{t \in [m, M]} f(t) \quad \text{and} \quad I_f \text{ is the closed interval by } f(m) \quad \text{and} \quad f(M). \]

Hence it follows that for every \( \lambda > 0 \)

\[
g(\Phi(f(A))) - \lambda \Phi(A) \geq \min_{t \in [m, M]} \left\{ \min_{r \in I_f} \frac{g(r)}{\alpha_f t + \beta_f} \right\} g(\alpha_f t + \beta_f) - \lambda t. \tag{6}
\]

As compared with (2), we think about the necessity of the constant \( \min_{t \in I_f} \frac{g(r)}{\alpha_f t + \beta_f} \) in (6). We pay our attention to the Jensen inequality in [4]: Let \( C \) be a selfadjoint operator on a Hilbert space \( H \). Let \( f(t) \) be a concave function on an interval containing \( \sigma(C) \). Then for every unit vector \( x \in H \)

\[
(f(C)x, x) \leq f((Cx, x)). \tag{7}
\]

**Theorem 3.** Let \( A \) be a selfadjoint operator on a Hilbert space \( H \) with \( m \leq A \leq M \) for some \( m < M \). Let \( f(t) \) be a concave function on \([m, M]\) and \( g(t) \) be an increasing convex function on \( f([m, M]) \). Let \( \Phi \) be a unital positive linear map on \( B(H) \). Then for every real number \( \lambda \)

\[
g(\Phi(f(A))) - \lambda \Phi(A) \geq \min_{t \in [m, M]} \{ g(\alpha_f t + \beta_f) - \lambda t \}. \tag{8}
\]

**Proof.** Since \( f(t) \) is concave, we have \( f(A) \geq \alpha_f A + \beta_f \). So the inequality \( \Phi(f(A)) \geq \alpha_f \Phi(A) + \beta_f \) holds, i.e.,

\[
\langle \Phi(f(A))x, x \rangle \geq \alpha_f \langle \Phi(A)x, x \rangle + \beta_f
\]

for all unit vectors \( x \in H \). Since \( g(t) \) is an increasing convex function on \( f([m, M]) \), we have

\[
\langle g(\Phi(f(A)))x, x \rangle \geq g(\langle \Phi(f(A))x, x \rangle) \geq g(\alpha_f \langle \Phi(A)x, x \rangle + \beta_f)
\]

by (7). Hence it follows from \( m \leq \Phi(A) \leq M \) that

\[
\langle g(\Phi(f(A)))x, x \rangle - \lambda \langle \Phi(A)x, x \rangle \geq g(\alpha_f \langle \Phi(A)x, x \rangle + \beta_f) - \lambda \langle \Phi(A)x, x \rangle \geq \min_{t \in [m, M]} \{ g(\alpha_f t + \beta_f) - \lambda t \},
\]

and the proof is complete. \( \square \)

Comparing with Remark of Theorem 1, we mention the case \( \lambda \alpha_f < 0 \) in Theorem 3.

**Corollary 4.** Let \( A \) be a selfadjoint operator on \( H \) with \( m \leq A \leq M \) for some \( m < M \). Let \( f(t) \) be a concave function on \([m, M]\) and \( g(t) \) be an increasing function on \( f([m, M]) \). Let \( \Phi \) be a unital positive linear map on \( B(H) \).

(i) If \( \alpha_f > 0 \), then for every \( \lambda < 0 \)

\[
g(\Phi(f(A))) - \lambda \Phi(A) \geq g(\alpha_f m + \beta_f) - \lambda m.
\]
(ii) If $\alpha_f < 0$, then for every $\lambda > 0$

$$g(\Phi(f(A))) - \lambda \Phi(A) \geq g(\alpha_f M + \beta_f) - \lambda M.$$ 

**Proof.** If $\alpha_f > 0$, then $f(t) \geq f(m)$ for all $t \in [m, M]$ by the concavity of $f(t)$. Hence we have $f(A) \geq f(m)$, and so $\Phi(f(A)) \geq f(m)$. Since $g(t)$ is increasing and for $\lambda < 0$,

$$g(\Phi(f(A))) - \lambda \Phi(A) \geq g(f(m)) - \lambda m = g(\alpha_f m + \beta_f) - \lambda m.$$ 

The latter is shown similarly. \qed 

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