On the number of the non-equivalent Cm-spanning subgraphs of the complete graph with order mk

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Abstract. Let m be greater than or equal to 3 and n be a multiple of m. An m-vertex cycle graph is denoted C_m. We will call a spanning subgraph whose components are C_m of the complete graph K_n a C_m-spanning subgraph of K_n. The Dihedral group D_n acts on the complete graph K_n naturally. This action of D_n induces the action on the set of the C_m-spanning subgraphs of the complete graph K_n . In [4], we calculated the number of the equivalence classes of the K_m-spanning subgraphs of the complete graph K_n by using Burnside's Lemma. In this paper we calculate the number of the non-equivalent C_m-spanning subgraphs of K_n for all m and n. In the special case we have the number of the non-equivalent Hamiltonian cycles of K_m for all m.

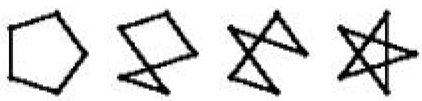
Let m be greater than or equal to 3 and let n be a multiple of m. Let $\{v_0; v_1; v_2; \cdots; v_{n-1}\}$ be the vertices of the complete graph K_n. The action to K_n of the Dihedral group $D_n = \{\rho_0; \rho_1; \cdots; \rho_{n-1}; \sigma_0; \sigma_1; \cdots; \sigma_{n-1}\}$ is defined by

$$\rho_i(v_k) = v_{(k+i) \pmod n} \quad \text{for } 0 \le i \le n-1;\ 0 \le k \le n-1$$

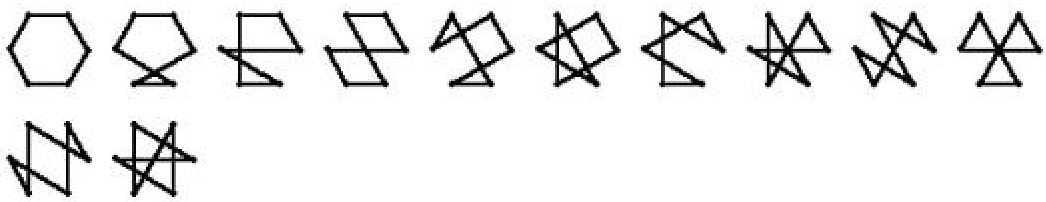
$$\sigma_i(v_k) = v_{(n+i-k) \pmod n} \quad \text{for } 0 \le i \le n-1;\ 0 \le k \le n-1$$

An m-vertex cycle graph is denoted C_m. We call a spanning subgraph whose components are C_m of the complete graph K_n a C_m-spanning subgraph of K_n. Let X_n^m be the set of the C_m-spanning subgraphs of K_n. Then the above action induces the action on X_n^m of the Dihedral group D_n. In the special case that n = m, X_m^m is the set of the Hamiltonian cycles of K_m.

For example, the equivalence classes of X_5^5 are given with the next figure.



The equivalence classes of X_6^6 are given with the next figure.



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We calculate the number of the equivalence classes of X_n^m by this group action. These computations can be done by using Burnside's lemma.

Theorem 1. (Burnside's lemma) Let G be a group of permutations acting on a set S. Then the number of orbits induced on S is given by

$$\frac{1}{|G|}\sum_{\pi\in G}|fix(\pi)|$$

where $fix(\pi)=\{x\in S \mid \pi(x)=x\}$.

Notation 1. The Euler function $\phi(m)$ is defined by

$$\phi(m)=|\{k \mid 0<k\le m, (k,m)=1\}|.$$

Notation 2. An integer function $\mu(p,q)$ is defined by

$$\mu(p,q)=\begin{cases}1 & \text{if } p\equiv 0 \pmod q\\ 0 & \text{otherwise}\end{cases}$$

Notation 3. For each integer i such that $0\le i\le n$, let $d=(n,i)$ and $R^m_{n,i}$ be

$$R^m_{m,i}=2^{\frac{m-4}{2}}\times\left(\left(\frac{m}{2}\right)!+\left(\frac{m-2}{2}\right)!\right)$$

if $n=m$ and $m=2i$

$$R^m_{m,i}=\frac{(d-1)!}{2}\times\left(\frac{m}{d}\right)^{d-1}\times\phi\left(\frac{m}{d}\right)$$

if $n=m$ and $m\neq 2i$

$$R^m_{n,i}=\sum_{\substack{d=\sum_{j=1}^{l}s_jp_j\\ s_j\ge 1,\ p_j|m \text{ for } 1\le j\le l}}\left(\frac{d!}{\prod_{j=1}^{l}(p_j!)^{s_j}s_j!}\prod_{j=1}^{l}\mu\left(\frac{n}{d},\frac{m}{p_j}\right)\left(\left(\frac{p_jn}{dm}\right)^{(p_j-1)}\times R^m_{m,p_j}\right)^{s_j}\right)$$

if $n>m$

Notation 4. $S^m_{n,i}, 0\le i\le n-1$ is given by the following recursive formula:

If n is odd then

$S^m_{n,k}=S^m_{n,0}$ for $1\le k\le n-1$.

If n is even then

$S^m_{n,2k}=S^m_{n,0}$ for $1\le k\le\frac{n}{2}-1$ and $S^m_{n,2k+1}=S^m_{n,1}$ for $1\le k\le\frac{n}{2}-1$.

If m is odd then

$$S^m_{m,0} = 2^{\frac{m-3}{2}} \times \left(\frac{m-1}{2}\right)!$$

$$S^m_{2m,1} = 2^{m-1} \times \frac{(m-1)!}{2}$$

$$S^m_{n,0} = \binom{\frac{n-1}{2}}{\frac{m-1}{2}} \times S^m_{n-m,1} \times S^m_{m,0} \qquad \text{if } n \text{ is odd and } n \geq 2m$$

$$S^m_{n,0} = \binom{\frac{n-2}{2}}{\frac{m-1}{2}} \times S^m_{n-m,0} \times S^m_{m,0} \qquad \text{if } n \text{ is even and } n \geq 2m$$

$$S^m_{n,1} = \left(\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!}\right) \times S^m_{n-2m,1} \times \frac{(m-1)!}{2} \qquad \text{if } n \text{ is even and } n \geq 3m$$

If m is even then

$$S^m_{m,0} = 2^{\frac{m-4}{2}} \times \left(\frac{m-2}{2}\right)! \text{ and}$$

$$S^m_{m,1} = 2^{\frac{m-4}{2}} \times \left\{\left(\frac{m}{2}\right)! + \left(\frac{m-2}{2}\right)!\right\} \text{ and}$$

$$S^m_{2m,1} = 2^{m-1} \times \frac{(m-1)!}{2} + \binom{\frac{2m-2}{2}}{\frac{m-2}{2}} \times S^m_{m,1} \times S^m_{m,1}$$

$$S^m_{n,0} = \binom{\frac{n-2}{2}}{\frac{m-2}{2}} \times S^m_{n-m,1} \times S^m_{m,0} \text{ if } n \geq 2m$$

$$S^m_{n,1} = \binom{\frac{n-2}{2}}{\frac{m-2}{2}} \times S^m_{n-m,1} \times S^m_{m,1} + \left(\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!}\right) \times S^m_{n-2m,1} \times \frac{(m-1)!}{2} \text{ if } n \geq 3m$$

Our main Theorem is the following:

Theorem 2. The number of the non-equivalent C_m-spanning subgraphs of the complete graph K_n is given by

$$\frac{1}{2n}\left\{\sum_{k=0}^{n-1}\left(R^m_{n,i} + S^m_{n,i}\right)\right\}.$$

We must determine the numbers of the fixed points of each permutation ρ_i and σ_i to prove the Theorem by using Burnside's Lemma.

First of all we consider the special case that $n = m$.

Lemma 1. The number of the Hamiltonian cycles of K_m is $(m-1)!/2$. This is the number of the fixed points of ρ_0.

Proof. Since the number of the circular permutations of $v_0, v_1, v_2, \cdots, v_{m-1}$ is $(m-1)!$ and the cycles $< v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_{m-1}}, v_0 >$ and $< v_0, v_{k_{m-1}}, \cdots, v_{k_2}, v_{k_1}, v_0 >$ represent the same Hamiltonian cycle, the number of the Hamiltonian cycles of K_m is $(m-1)!/2$. □

Lemma 2. If $(m, i) = 1$ then the number of the fixed points of ρ_i is $\phi(m)/2$.

Proof. Let V be the set of the vertices of K_m and k be an integer such that $(m,k)=1$. Since $(m,i)=1$, there is an integer α such that $\alpha i \equiv 1 \pmod m$. Since $\rho_i^{\alpha\beta k}(v_0v_k) = v_{\beta k \bmod m}v_{(\beta+1)k \bmod m}, 0 \le \beta \le m-1$ and $\{v_{\beta k \bmod m} | 0 \le \beta \le m-1\} = V$, the walk $< v_0, v_k, v_{2k \bmod m}, v_{3k \bmod m}, \cdots, v_{(m-1)k \bmod m}, v_{mk \bmod m} >$ is a Hamiltonian cycle. We call this Hamiltonian cycle H. Since $(m,k)=1$, there is an integer γ such that $\gamma k \equiv \beta k + i \pmod m$. Then we have that $(\beta+1)k + i \equiv (\gamma+1)k \pmod m$ and $\rho_i(v_{\beta k \bmod m}v_{(\beta+1)k \bmod m}) = v_{\beta k+i \bmod m}v_{(\beta+1)k+i \bmod m} = v_{\gamma k \bmod m}v_{(\gamma+1)k \bmod m}$ and therefore $\rho_i(H)=H$. Conversely, let H be a Hamiltonian cycle such that $\rho_i(H)=H$ and $v_0v_k \in H$. Since $\rho_i^{\alpha\beta k}(v_0v_k) = v_{\beta k \bmod m}v_{(\beta+1)k \bmod m}, 0 \le \beta \le m-1$, $v_{\beta k \bmod m}v_{(\beta+1)k \bmod m} \in H$. If $(m,k)=d>1$ then $(\frac{m}{d})k = (\frac{k}{d})m \equiv 0 \pmod m$ and $< v_0, v_k, v_{2k \bmod m}, \cdots, v_{(\frac{m}{d})k \bmod m} = v_0 >$ is a cycle. This is contradict to the fact that H is Hamiltonian cycle. Then we have that $(m,k)=1$. In this case H is determined uniquely by joining $v_{\beta k \bmod m}$ and $v_{(\beta+1)k \bmod m}, 0 \le \beta \le m-1$. Since the Hamiltonian cycle generated by v_0v_k is coinside with the Hamiltonian cycle generated by v_0v_{n-k}, the number of the fixed points of ρ_i is $\varphi(m)/2$. □

Lemma 3. If $(m,i)=d>1$ then the number of the fixed points of ρ_i is given as follow: If $m=2d$ then

$$2^{\frac{m-4}{2}} \times \left(\left(\frac{m}{2}\right)! + \left(\frac{m-2}{2}\right)!\right).$$

If $m>2d$ then

$$\left(\frac{m}{d}\right)^{d-1} \times \varphi\left(\frac{m}{d}\right) \times \frac{(d-1)!}{2}.$$

Proof. We assume that m be equal to $2d$. Since $(m,i)=d>1$ and $0 \le i \le m-1$, we have that $i=d$. Let H be a Hamiltonian cycle of K_m fixed by ρ_i. Then H is point symmetry. If $v_jv_{j+d} \in H$ for some $0 \le j \le d-1$ and H has a path $< v_j, v_{k_1}, v_{k_2}, \cdots, v_{k_{d-1}} >$, where $v_{k_1} \ne v_{j+d}$, then by the symmetry H has a path $< v_{(j+d) \bmod m}, v_{(k_1+d) \bmod m}, v_{(k_2+d) \bmod m}, \cdots, v_{(k_{d-1}+d) \bmod m} >$. If $v_{k_p} = v_{(k_{p-1}+d) \bmod m}$ for some $1 \le p \le d-1$ then $< v_j, v_{k_1}, v_{k_2}, \cdots, v_{k_{p-1}}, v_{(k_{p-1}+d) \bmod m}, \cdots, v_{(k_2+d) \bmod m}, v_{(k_1+d) \bmod m}, v_{j+d}, v_j >$ is a cycle whose length is $2p \le 2d-2$. This is contradiction. Then $v_{k_p} \ne v_{(k_{p-1}+d) \bmod m}$ for all $1 \le p \le d-1$. Then H is the Hamiltonian cycle $< v_j, v_{k_1}, v_{k_2}, \cdots, v_{k_{d-1}}, v_{(k_{d-1}+d) \bmod m}, \cdots, v_{(k_2+d) \bmod m}, v_{(k_1+d) \bmod m}, v_{j+d}, v_j >$. These Hamiltonian cycles can be composed to join the antipodal points of the endpoints of the path that are mode with the permutation that took one of each from $\{v_0\}, \{v_1, v_{d+1}\}, \{v_2, v_{d+2}\}, \cdots, \{v_{d-1}, v_{m-1}\}$ and the point symmetric path. The number of these Hamiltonian cycles is $d! \times 2^{d-1}/2$.

We assume that $v_jv_{j+d} \notin H$ for all $0 \le j \le d-1$. If H has a path $< v_{k_0}, v_{k_1}, v_{k_2}, \cdots, v_{k_p} >$ such that $2 \le p \le d-1$ and $v_{k_0} \in V_j$ and $v_{k_p} \in V_j$ then the cycle $< v_{k_0}, v_{k_1}, v_{k_2}, \cdots, v_{k_p} = v_{(k_0+d) \bmod m}, v_{(k_1+d) \bmod m}, v_{(k_2+d) \bmod m}, \cdots, v_{(k_p+d) \bmod m} = v_{k_0} >$ is contained in H. This contradicts the fact that H is a Hamilton cycle. Accordingly, H is the Hamiltonian cycle which is the concatenation of the path P from vertex v_0 to vertex v_d through the permutation that took one of each from $\{v_1, v_{d+1}\}, \{v_2, v_{d+2}\}, \cdots, \{v_{d-1}, v_{m-1}\}$ with the point symmetric path of P. The number of these Hamiltonian cycles is $(d-1)! \times 2^{d-1}/2$.

Next we assume that $m>2d$. Let $V_0 = \{v_0, v_d, v_{2d}, \cdots, v_{m-d}\}, V_1 = \{v_1, v_{d+1}, v_{2d+1}, \cdots, v_{m-d+1}\}, V_2 = \{v_2, v_{d+2}, v_{2d+2}, \cdots, v_{m-d+2}\}, \cdots, V_{d-1} = \{v_{d-1}, v_{2d-1}, v_{3d-1}, \cdots, v_{m-1}\}$. Let H be a Hamiltonian cycle of K_m fixed by ρ_i. We assume that $v_jv_p \in H$ and $v_j \in V_s$ and $v_p \in V_s$. If $p \not\equiv (j+m/2) \bmod m$ then $\{\rho_i(v_jv_p), \rho_i^2(v_jv_p), \rho_i^3(v_jv_p), \cdots, \rho_i^{m/d}(v_jv_p)\}$ contains a cycle whose length is less than m. This is contradict the fact that H is a Hamiltonian cycle. If $p \equiv (j+m/2) \bmod m$ then $v_{(s+td) \bmod m}v_{(s+td+m/2) \bmod m} \in H$ for all

$0 \le t \le m/d - 1$ and H is point symmetry. Let $< v_j; v_{k_1}; v_{k_2}; \cdots; v_{k_t} >$ be a shortest path from the vertex v_j to the vertex in V_s of H which does not pass through the edge $v_j v_p$. If $v_{k_t} = v_p$ then $< v_j; v_{k_1}; v_{k_2}; \cdots; v_{k_t}; v_j >$ is a cycle in H. Since $m/d > 2$ and $|V_s| > 2$, we have that $v_{k_t} \ne v_p$. Here we assume that $\{v_{k_1}; v_{k_2}; \cdots; v_{k_{t-1}}\}$ contains the vertices v_{k_f} and v_{k_g} which belong to same V_r. If $f + 1 = g$ then $v_{k_g} = v_{(k_f + m/2) \bmod m}$ and $< v_j; v_{k_1}; \cdots; v_{k_{f-1}}; v_{k_f}; v_{(k_{f-1}+m/2) \bmod m}; \cdots; v_{(k_1+m/2) \bmod m}; v_p; v_j >$ is cycle. This is contradiction. If $f+1 \ne g$ then let P be the path $< v_{k_f}; v_{k_{f+1}}; \cdots; v_{k_g} >$. Then the union of the pathes $P; \rho_i(P); \rho_i^2(P); \cdots; \rho_i^{m/d-1}(P)$ contains a cycle which does not conatain any vertex of V_s. Therefore $v_{k_1}; v_{k_2}; \cdots; v_{k_{t-1}}$ are contained to the different vertex set V_r, respectively. In this case $< v_j; v_{k_1}; v_{k_2}; \cdots; v_{k_t}; v_{(k_t+m/2) \bmod m}; v_{(k_{t-1}+m/2) \bmod m}; \cdots; v_{(k_2+m/2) \bmod m}; v_{(k_1+m/2) \bmod m}; v_p; v_j >$ is a cycle which is not H. Accordingly, for each $0 \le j \le d-1$, the edge that joins the vertices of V_j does not exist in H. Let Q be a shortest path $< v_0; v_{k_1}; v_{k_2}; \cdots; v_{k_p} >$ from v_0 to the vertex of V_0 such that $v_0 \ne v_{k_p}$ and $v_{k_p} \in V_0$. Here we assume that $\{v_{k_1}; v_{k_2}; \cdots; v_{k_{p-1}}\}$ contains the vertices v_{k_f} and v_{k_g} which belong to same V_r. Let P be the path $< v_{k_f}; v_{k_{f+1}}; \cdots; v_{k_g} >$. Then the union of the pathes $P; \rho_i(P); \rho_i^2(P); \cdots; \rho_i^{m/d-1}(P)$ contains a cycle which does not conatain any vertex of V_0. If there is V_r whose vertex does not belong to Q then the union of the pathes $Q; \rho_i(Q); \rho_i^2(Q); \cdots; \rho_i^{m/d-1}(Q)$ contains a cycle which does not conatain any vertex of V_r. Therefore, Q contains one and only one vertex of $V_1; V_2; \cdots; V_{d-1}$, respectively. The union of the pathes $Q; \rho_i(Q); \rho_i^2(Q); \cdots; \rho_i^{m/d-1}(Q)$ becomes generally the sum of cycles and it becomes the Hamiltonian cycle if and only if $(m; k_p/d)$ is equal to one. Therefore, these H is generated by the path that begins with v_0 and ends with the vertex v_{dk} such that $(m; k) = 1$ and passes through the vertices which are the permutation that took one of each from $V_1; V_2; \cdots; V_{d-1}$. Therefore, the number of such H is

$$\left(\frac{m}{d}\right)^{d-1} \times \phi\left(\frac{m}{d}\right) \times (d-1)!/2$$

Then we have the results. □

Lemma 4. If m is odd then the number of the fixed points of σ_0 is equal to the number of the fixed points of σ_k for all $1 \le k \le m-1$.

Proof. We assume that k is even. Let H be a Hamiltonian cycle of K_m fixed by σ_0. Then it is easily verified that $\rho_{\frac{k}{2}}(H)$ is a Hamiltonian cycle of K_m fixed by σ_k. Conversely, if H is a Hamiltonian cycle of K_m fixed by σ_k then $\rho_{\frac{k}{2}}^{-1}(H)$ is a Hamiltonian cycle of K_n fixed by σ_0. Next we assume that k is odd. Let H be a Hamiltonian cycle of K_m fixed by σ_0. Then it is easily verified that $\rho_{\frac{m+k}{2}}(H)$ is a Hamiltonian cycle of K_m fixed by σ_k. Conversely, if H is a Hamiltonian cycle of K_m fixed by σ_k then $\rho_{\frac{m+k}{2}}^{-1}(H)$ is a Hamiltonian cycle of K_n fixed by σ_0. Then we have the results. □

Similarly, we have the next Lemma.

Lemma 5. If m is even then the number of the fixed points of σ_0 is equal to the number of the fixed points of σ_{2d} for all $1 \le d \le m/2 - 1$ and the number of the fixed points of σ_1 is equal to the number of the fixed points of σ_{2d+1} for all $1 \le d \le m/2 - 1$.

Lemma 6. If m is odd then the number of the fixed points of σ_0 is

$$2^{\frac{m-3}{2}} \times \left(\frac{m-1}{2}\right)!.$$

Proof. Let H be a Hamiltonian cycle of K_m fixed by σ_0. Since m is odd, the axis of the line symmetry is passing only vertex v_0. Let $< v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_{(m-1)/2}} >$ be a path in H. Then, by the symmetry, there is another path $< v_0, v_{m-k_1}, v_{m-k_2}, \cdots, v_{m-k_{(m-1)/2}} >$ in H. And therefore H must be $< v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_{(m-1)/2}}, v_{m-k_{(m-1)/2}}, \cdots, v_{m-k_2}, v_{m-k_1}, v_0 >$. Therefore, the number of H is able to calculate in the following manner. The number of the ways to choose one vertex from each $V_1 = \{v_1, v_{m-1}\}, V_2 = \{v_2, v_{m-2}\}, \cdots, V_{(m-1)/2} = \{v_{(m-1)/2}, v_{(m+1)/2}\}$ is $2^{\frac{m-1}{2}}$ and the number of its permutations is $\left(\frac{m-1}{2}\right)!$. Additionally, the cycle $< v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_{(m-1)/2}}, v_{m-k_{(m-1)/2}}, \cdots, v_{m-k_2}, v_{m-k_1}, v_0 >$ and the cycle $< v_0, v_{m-k_1}, v_{m-k_2}, \cdots, v_{m-k_{(m-1)/2}}, v_{k_{(m-1)/2}}, \cdots, v_{k_2}, v_{k_1}, v_0 >$ are the same Hamiltonian cycle. Then the number of the fixed points of σ_0 is $\left(\frac{m-1}{2}\right)! \times 2^{\frac{m-1}{2}}/2$ Then we have the results. □

Lemma 7. *If m is even then the number of the fixed points of σ_0 is*

$$2^{\frac{m-4}{2}} \times \left(\frac{m-2}{2}\right)!.$$

Proof. Let H be a Hamiltonian cycle of K_m fixed by σ_0. Since m is even, the axis of the line symmetry is passing vertices v_0 and $v_{m/2}$. Let $< v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_{m/2}} >$ be a path in H. Then, by the symmetry, there is another path $< v_0, v_{m-k_1}, v_{m-k_2}, \cdots, v_{k_{m-m/2}} >$ in H. And therefore H must be $< v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_{m/2}}, \cdots, v_{m-k_2}, v_{m-k_1}, v_0 >$. Therefore, the number of H is able to calculate in the following manner. The number of the ways to choose one vertex from each $V_1 = \{v_1, v_{m-1}\}, V_2 = \{v_2, v_{m-2}\}, \cdots, V_{(m-2)/2} = \{v_{(m-2)/2}, v_{(m+2)/2}\}$ is $2^{\frac{m-2}{2}}$ and the number of its permutations is $\left(\frac{m-2}{2}\right)!$. Additionally, the cycle $< v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_{(m-2)/2}}, v_{m/2}, v_{m-k_{(m-2)/2}}, \cdots, v_{m-k_2}, v_{m-k_1}, v_0 >$ and the cycle $< v_0, v_{m-k_1}, v_{m-k_2}, \cdots, v_{m-k_{(m-2)/2}}, v_{m/2}, v_{k_{(m-2)/2}}, \cdots, v_{k_2}, v_{k_1}, v_0 >$ are the same Hamiltonian cycle. Then the number of the fixed points of σ_0 is $\left(\frac{m-2}{2}\right)! \times 2^{\frac{m-2}{2}}/2$. Then we have the results. □

Lemma 8. *If m is even then the number of the fixed points of σ_1 is*

$$2^{\frac{m-4}{2}} \times \left(\left(\frac{m}{2}\right)! + \left(\frac{m-2}{2}\right)!\right).$$

Proof. Let H be a Hamiltonian cycle of K_m fixed by σ_1. Since m is even, the axis of the line symmetry is not passing any vertices. We assume that $v_j v_{(m+1-j) \bmod m} \in H$ for some $1 \leq j \leq m/2$. Let $< v_j, v_{k_1}, v_{k_2}, \cdots, v_{k_{(m-2)/2}} >$ be a path in H. Then, by the symmetry, there is another path $< v_{(m+1-j) \bmod m}, v_{(m+1-k_1) \bmod m}, v_{(m+1-k_2) \bmod m}, \cdots, v_{(m+1-k_{(m-2)/2}) \bmod m} >$. And therefore H must be $< v_j, v_{k_1}, v_{k_2}, \cdots, v_{k_{(m-2)/2}}, v_{(m+1-k_{(m-2)/2}) \bmod m}, \cdots, v_{(m+1-k_2) \bmod m}, v_{(m+1-k_1) \bmod m}, v_{(m+1-j) \bmod m}, v_j >$. Therefore, the number of H is able to calculate in the following manner. The number of the ways to choose one vertex from each $V_1 = \{v_1, v_0\}, V_2 = \{v_2, v_{m-1}\}, \cdots, V_{(m-2)/2} = \{v_{(m-2)/2}, v_{(m+4)/2}\}, V_{m/2} = \{v_{m/2}, v_{(m+2)/2}\}$ is $2^{\frac{m}{2}}$ and the number of its permutations is $\left(\frac{m}{2}\right)!$. Additionally, the cycle $< v_{k_0}, v_{k_1}, v_{k_2}, \cdots, v_{k_{(m-2)/2}}, v_{m+1-k_{(m-2)/2}}, \cdots, v_{m+1-k_2}, v_{m+1-k_1}, v_{m+1-k_0}, v_{k_0} >$ and the cycle $< v_{k_0}, v_{m+1-k_0}, v_{m+1-k_1}, v_{m+1-k_2}, \cdots, v_{m+1-k_{(m-2)/2}}, v_{k_{(m-2)/2}}, \cdots, v_{k_2}, v_{k_1}, v_{k_0} >$ and the cycle $< v_{m+1-k_{(m-2)/2}}, \cdots, v_{m+1-k_2}, v_{m+1-k_1}, v_{m+1-k_0}, v_{k_0}, v_{k_1}, v_{k_2}, \cdots, v_{k_{(m-2)/2}}, v_{m+1-k_{(m-2)/2}} >$ and the cycle $< v_{m+1-k_{(m-2)/2}}, v_{k_{(m-2)/2}}, \cdots, v_{k_2}, v_{k_1}, v_{k_0}, v_{m+1-k_0}, v_{m+1-k_1}, v_{m+1-k_2}, \cdots, v_{m+1-k_{(m-2)/2}} >$ are the same Hamiltonian cycle. Then the number of such H is $\left(\frac{m}{2}\right)! \times 2^{\frac{m}{2}}/4$. Next we assume that $v_j v_{(m+1-j) \bmod m} \notin H$ for all $1 \leq j \leq \frac{m}{2}$. Let

$<v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_s}, v_1>$ be a path in H. Then, by the symmetry, there is another path $<v_1, v_{(m+1-k_1) \bmod m}, v_{(m+1-k_2) \bmod m}, \cdots, v_{(m+1-k_s) \bmod m}, v_0>$ in H. Since the walk $<v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_s}, v_1, v_{(m+1-k_1) \bmod m}, v_{(m+1-k_2) \bmod m}, \cdots, v_{(m+1-k_s) \bmod m}, v_0>$ contains a cycle, we have $s = (m-2)/2$ and therefore H must be $<v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_{(m-2)/2}}, v_1, v_{(m+1-k_1) \bmod m}, v_{(m+1-k_2) \bmod m}, \cdots, v_{(m+1-k_{(m-2)/2}) \bmod m}, v_0>$. Therefore, the number of H is able to calculate in the following manner. The number of the ways to choose one vertex from each $V_2 = \{v_2, v_{m-1}\}, \cdots, V_{(m-2)/2} = \{v_{(m-2)/2}, v_{(m+4)/2}\}, V_{m/2} = \{v_{m/2}, v_{(m+2)/2}\}$ is $2^{\frac{m-2}{2}}$ and the number of its permutations is $\left(\frac{m-2}{2}\right)!$. Additionally, the cycle $<v_0, v_{k_1}, v_{k_2}, \cdots, v_{k_{(m-2)/2}}, v_1, v_{m+1-k_1}, v_{m+1-k_2}, \cdots, v_{m+1-k_{(m-2)/2}}, v_0>$ and the cycle $<v_0, v_{m+1-k_{(m-2)/2}}, \cdots, v_{m+1-k_2}, v_{m+1-k_1}, v_1, v_{k_{(m-2)/2}}, \cdots, v_{k_2}, v_{k_1}, v_0>$ are the same Hamiltonian cycle. Then the number of such H is $\left(\frac{m-2}{2}\right)! \times 2^{\frac{m-2}{2}}/2$. Then we have the results. □

The analysis of Hamiltonian cycles of K_m has been completed. Next we do the analysis of general C_m-spanning subgraphs of K_n.

Lemma 9. The number of the C_m-spanning subgraphs of K_n is

$$\left(\frac{(m-1)!}{2}\right)^{n/m} \times \prod_{k=1}^{n/m} \binom{mk-1}{m-1}.$$

This is the number of the fixed points of ρ_0.

Proof. The number of ways to select $\frac{n}{m}$ groups of size m from a collection of n items is $\prod_{i=1}^{n/m} \binom{mi-1}{m-1}$ by Lemma 1 in [4]. By Lemma 1, the number of applying C_m to m-set is $\frac{(m-1)!}{2}$. Then we have the results. □

Remark 1. It is easily checked that $R^m_{n,0}$ is equal to $\left(\frac{(m-1)!}{2}\right)^{n/m} \times \prod_{k=1}^{n/m} \binom{mk-1}{m-1}$.

Lemma 10. The fixed points of ρ_i for each $0 < i < n$ is $R^m_{n,0}$.

Proof. Let $d = (n, i)$ and $V_0 = \{v_0, v_d, v_{2d}, \cdots, v_{n-d}\}, V_1 = \{v_1, v_{d+1}, v_{2d+1}, \cdots, v_{n-d+1}\}$, $V_2 = \{v_2, v_{d+2}, v_{2d+2}, \cdots, v_{n-d+2}\}$, $\cdots$, $V_{d-1} = \{v_{d-1}, v_{2d-1}, v_{3d-1}, \cdots, v_{n-1}\}$.

Since $(n, i) = d$, the equation $xi \equiv m \pmod n$ has a solution if and only if d divides m. Then we have $\rho_i(V_k) = V_k$ for $0 \le k \le d-1$. Let H be a C_m-spanning subgraph of K_n which is fixed by ρ_i and let G be a K_m-spanning subgraph of K_n which change each component C_m of H into K_m. Then G is also fixed by ρ_i. We divide $\{V_0, V_1, V_2, \cdots, V_{d-1}\}$ into the subsets $W_1, W_2, W_3, \cdots, W_s$ in the following manner:

If $W_j = \{V^j_0, V^j_1, \cdots, V^j_{p_j-1}\}$ then each component of $G|V^j_0 \cup V^j_1 \cup \cdots \cup V^j_{p_j-1}$ is K_m and any component of the restriction to the proper subset of W_j of G is not K_m for each $1 \le j \le s$.

By the proof of Lemma 3 in [4] we have that $m \equiv 0 \pmod{p_k}$ and $\frac{n}{d} \equiv 0 \pmod{\frac{m}{p_k}}$ for $1 \le k \le s$ and the number of such G is $\prod_{k=1} \left(\frac{p_k n}{dm}\right)^{p_k-1}$. Each component C_m of $H|V^j_0 \cup V^j_1 \cup \cdots \cup V^j_{p_j-1}$ is fixed by ρ_{p_j} when we change the name of vertices properly.

The number of the way to taking of such C_m is R^m_{m,p_j}. Then the number of such H is $\prod_{k=1}\left(\left(\frac{p_k n}{dm}\right)^{p_k-1} \times R^m_{m,p_k}\right)$. In this case we have that $\sum_{k=1} p_k = d$ and p_k is a divisor of m and $\frac{n}{d} \equiv 0 \pmod{\frac{m}{p_k}}$ for $1 \le k \le s$.

Let $d = \sum_{j=1} s_j p_j$ be a representation of d as the sum of divisors p_j of m. The number of ways to divide $\{V_0, V_1, V_2, \cdots, V_{d-1}\}$ into s_1 pieces of p_1-element set, s_2 pieces of p_2-element set, s_3 pieces of p_3-element set, $\cdots$, s_l pieces of p_l-element set is

$$\frac{d!}{\prod_{j=1}(p_j!)^{s_j} s_j!}.$$

Accordingly, the number of all the possibilities of H is

$$\sum_{\substack{d=\sum_{j=1}^{l} s_j p_j \\ s_j \ge 1,\, p_j|m \text{ for } 1 \le j \le l}} \left(\frac{d!}{\prod_{j=1}(p_j!)^{s_j} s_j!} \prod_{j=1} \mu\left(\frac{n}{d}, \frac{m}{p_j}\right) \left(\left(\frac{p_j n}{dm}\right)^{p_j-1} R^m_{m,p_j} \right)^{s_j} \right).$$

We have the results. □

Notation 5. Let $S^m_{n,i}$ be the number of the fixed points of σ_i for X^m_n.

Remark 2. By the following lemmas we will see that $S^m_{n,i}$ agrees with the one which is given in Notation 4.

Lemma 11. If n is odd then the number of the fixed points of σ_0 is equal to the number of the fixed points of σ_k for all $1 \le k \le n-1$.

Proof. We assume that k is even. Let H be a C_m-spanning subgraph of K_n fixed by σ_0. Then it is easily verified that $\rho_{\frac{k}{2}}(H)$ is a C_m-spanning subgraph of K_n fixed by σ_k. Conversely, if H is a C_m-spanning subgraph of K_n fixed by σ_k then $\rho^{-1}_{\frac{k}{2}}(H)$ is a C_m-spanning subgraph of K_n fixed by σ_0. Next we assume that k is odd. Let H be a C_m-spanning subgraph of K_n fixed by σ_0. Then it is easily verified that $\rho_{\frac{n+k}{2}}(H)$ is a C_m-spanning subgraph of K_n fixed by σ_k. Conversely, if H is a C_m-spanning subgraph of K_n fixed by σ_k then $\rho^{-1}_{\frac{n+k}{2}}(H)$ is a C_m-spanning subgraph of K_n fixed by σ_0. Then we have the results. □

Similarly, we have the next Lemma.

Lemma 12. If n is even then the number of the fixed points of σ_0 is equal to the number of the fixed points of σ_{2d} for all $1 \le d \le n/2 - 1$ and the number of the fixed points of σ_1 is equal to the number of the fixed points of σ_{2d+1} for all $1 \le d \le n/2 - 1$.

Lemma 13. If n is odd and m is odd then

$$S^m_{m,0} = 2^{\frac{m-3}{2}} \times \left(\frac{m-1}{2}\right)! \qquad \text{and}$$

$$S^m_{n,0} = \binom{\frac{n-1}{2}}{\frac{m-1}{2}} \times S^m_{n-m,1} \times S^m_{m,0} \qquad \text{if } n \ge 2m.$$

Proof. By Lemma 6 we have that

$$S^m_{m;0} = 2^{\frac{m-3}{2}} \times \left(\frac{m-1}{2}\right)!.$$

We assume that $n \geq 2m$. Let H be a C_m-spanning subgraph of K_n fixed by σ_0. Let C be the component of H which contains vertex v_0. $H - C$ naturally becomes C_m-spanning subgraph of K_{n-m} fixed by σ_1 when we change the name of the vertices. Conversely, let H be a C_m-spanning subgraph of K_{n-m} fixed by σ_1. Since $n - m$ is even, the axis of the line symmetry is not passing any vertices. If we take one vertex of C_m in the position of v_0 of the graph which we will construct and divide the remaining vertices of C_m into halves and distribute them between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a C_m-spanning subgraph of K_n fixed by σ_0 when we join the vertices of C_m such that it is fixed by σ_0. The number of ways to distribute the vertices of C_m is $\binom{\frac{n-1}{2}}{\frac{m-1}{2}}$ and the number of the way of joinning of new vertices is $S^m_{m;0}$. Then we have the results. □

Lemma 14. If n is even and m is odd then

$$S^m_{n;0} = \binom{\frac{n-2}{2}}{\frac{m-1}{2}} \times S^m_{n-m;0} \times S^m_{m;0}.$$

Proof. Let H be a C_m-spanning subgraph of K_n fixed by σ_0. Since n is even, the axis of σ_0 passes v_0 and $v_{\frac{n}{2}}$. Let C be the component of H which contains vertex $v_{\frac{n}{2}}$. Since m is odd, C does not contain the vertex v_0. $H - C$ naturally becomes C_m-spanning subgraph of K_{n-m} fixed by σ_0 when we change the name of the vertices. Conversely, let H be a C_m-spanning subgraph of K_{n-m} fixed by σ_0. Since $n - m$ is odd, the axis of σ_0 passes the vertex v_0. If we take one vertex of C_m in the position of $v_{\frac{n}{2}}$ of the graph which we will construct and divide the remaining vertices of C_m into halves and distribute them between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a C_m-spanning subgraph of K_n fixed by σ_0 when we join the vertices of C_m such that it is fixed by σ_0. The number of ways to distribute the vertices of C_m is $\binom{\frac{n-2}{2}}{\frac{m-1}{2}}$ and the number of the way of joining of new vertices is $S^m_{m;0}$. Then we have the results. □

Lemma 15. If n is even and m is odd then

$$S^m_{2m;1} = 2^{m-1} \times \frac{(m-1)!}{2} \quad \text{and}$$

$$S^m_{n;1} = \left(\sum_{k=0}^{m-1} \frac{\left(\frac{n-2}{2}\right)!}{k!(m-k-1)!(\frac{n-2m}{2})!}\right) \times S^m_{n-2m;1} \times \frac{(m-1)!}{2} \qquad \text{if } n \geq 4m.$$

Proof. We assume that $n = 2m$. If we take one vertex of C_m in the position of $v_{\frac{n}{2}}$ and one vertex of another C_m in the position of $v_{\frac{n}{2}+1}$ of the graph which we will construct and distribute the remaining vertices of two C_m to both sides of the perpendicular bisector of $v_{\frac{n}{2}-1}$ and $v_{\frac{n}{2}-1}$ permitting redundancy and symmetrically regarding the perpendicular bisector then the resulting graph becomes a C_m-spanning subgraph of K_{2m} fixed by σ_1 when we join the vertices of two C_m as it becomes symmetric regarding the perpendicular bisector of $v_{\frac{n}{2}-1}$ and $v_{\frac{n}{2}-1}$. The number of ways to distribute the vertices of two C_m is

$\sum_{k=0}^{m-1} \frac{(m-1)!}{k!(m-k-1)!} = 2^{m-1}$ and the number of the way to joining the vertices of two C_m is $\frac{(m-1)!}{2}$. Then we have that

$$S^m_{2m,1} = 2^{m-1} \times \frac{(m-1)!}{2}.$$

We assume that $n \geq 4m$. Let H be a C_m-spanning subgraph of K_n fixed by σ_1. Since n is even, the axis of σ_1 does not pass any vertices. Since m is odd, there is no component which contains both $v_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$. Let L_0 be the component which contains vertex $v_{\frac{n}{2}}$ and L_1 be the component which contains vertex $v_{\frac{n}{2}+1}$. $H - L_0 - L_1$ naturally becomes C_m-spanning subgraph of K_{n-2m} fixed by σ_1 when we change the name of the vertices. Conversely, let H be a C_m-spanning subgraph of K_{n-2m} fixed by σ_1. Since $n-2m$ is even, the axis of σ_1 does not pass any vertices. If we take one vertex of C_m in the position of $v_{\frac{n}{2}}$ and one vertex of another C_m in the position of $v_{\frac{n}{2}+1}$ of the graph which we will construct and distribute the remaining vertices of two C_m between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a C_m-spanning subgraph of K_n fixed by σ_1 when we join the vertices of two C_m as it becomes symmetric regarding the axis. The number of ways to distribute the vertices of two C_m is $\sum_{k=0}^{m-1} \frac{\left(\frac{n-2}{2}\right)!}{k!(m-k-1)!\left(\frac{n-2m}{2}\right)!}$ and the number of the ways to joining the vertices of two C_m is $\frac{(m-1)!}{2}$. Then we have the results. □

Lemma 16. If n is even and m is even then

$$S^m_{m,0} = 2^{\frac{m-4}{2}} \times \left(\frac{m-2}{2}\right)! \qquad \text{and}$$

$$S^m_{n,0} = \binom{\frac{n-2}{2}}{\frac{m-2}{2}} \times S^m_{n-m,1} \times S^m_{m,0} \qquad \text{if } n \geq 2m.$$

Proof. By Lemma 7 we have that

$$S^m_{m,0} = 2^{\frac{m-4}{2}} \times \left(\frac{m-2}{2}\right)!.$$

We assume that $n \geq 2m$. Let H be a C_m-spanning subgraph of K_n fixed by σ_0. Since n is even, the axis of σ_0 passes v_0 and $v_{\frac{n}{2}}$. Let C be the component of H which contains vertex v_0 and $v_{\frac{n}{2}}$. $H - C$ naturally becomes C_m-spanning subgraph of K_{n-m} fixed by σ_1 when we change the name of the vertices. Conversely, let H be a C_m-spanning subgraph of K_{n-m} fixed by σ_1. Since $n-m$ is even, the axis of σ_1 does not pass any vertices. If we take two vertices of C_m in the positions of v_0 and $v_{\frac{n}{2}}$ of the graph which we will construct and divide the remaining vertices of C_m into halves and distribute them between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a C_m-spanning subgraph of K_n fixed by σ_0 when we join the vertices of C_m as it becomes symmetric regarding the axis. The number of ways to distribute the vertices of C_m is $\binom{\frac{n-2}{2}}{\frac{m-2}{2}}$ and the number of the ways to joining the vertices of C_m is $S^m_{m,0}$. Then we have the results. □

Lemma 17. If n is even and m is even then

$$S^m_{m;1} = 2^{\frac{m-4}{2}} \times \left(\left(\frac{m}{2}\right)! + \left(\frac{m-2}{2}\right)!\right) \quad \text{and}$$

$$S^m_{2m;1} = 2^{m-1} \times \frac{(m-1)!}{2} + \binom{\frac{2m-2}{2}}{\frac{m-2}{2}} \times S^m_{m;1} \times S^m_{m;1} \quad \text{and}$$

$$S^m_{n;1} = \binom{\frac{n-2}{2}}{\frac{m-2}{2}} \times S^m_{n-m;1} \times S^m_{m;1} + \left(\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!}\right) \times S^m_{n-2m;1} \times \frac{(m-1)!}{2} \quad \text{if } n \geq 3m.$$

Proof. By Lemma 8 we have that

$$S^m_{m;1} = 2^{\frac{m-4}{2}} \times \left(\left(\frac{m}{2}\right)! + \left(\frac{m-2}{2}\right)!\right).$$

We assume that $n \geq 3m$. We study two kinds of constitutions that compose C_m-spanning subgraphs of K_n fixed by σ_1 inductively.

The first method is the following:
Let H be a C_m-spanning subgraph of K_{n-m} fixed by σ_1. Since $n-m$ is even, the axis of σ_1 does not pass any vertices. If we take two vertices of C_m in the positions of $v_{\frac{n}{2}}$ and $v_{\frac{n}{2}+1}$ of the graph which we will construct and divide the remaining vertices of C_m into halves and distribute them between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a C_m-spanning subgraph of K_n fixed by σ_1 when we join the vertices of C_m as it becomes symmetric regarding the axis. The number of ways to distribute the vertices of C_m is $\binom{\frac{n-2}{2}}{\frac{m-2}{2}}$ and the number of the ways to joining the vertices of C_m is $S^m_{m;1}$. Similarly, if we take two vertices of C_m in the positions of v_0 and v_1 of the graph which we will construct and divide the remaining vertices of C_m into halves and distribute them between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a C_m-spanning subgraph of K_n fixed by σ_1 when we join the vertices of C_m as it becomes symmetric regarding the axis. The number of ways to distribute the vertices of C_m is $\binom{\frac{n-2}{2}}{\frac{m-2}{2}}$ and the number of the ways to joining the vertices of C_m is $S^m_{m;1}$. Accordingly, it is possible $2 \times \binom{\frac{n-2}{2}}{\frac{m-2}{2}} \times S^m_{n-m;1} \times S^m_{m;1}$ C_m-spanning subgraph of K_n fixed by σ_1 as a whole with these constitutions.

The second method is the following:
Let H be a C_m-spanning subgraph of K_{n-2m} fixed by σ_1. Since $n-m$ is even, the axis of σ_1 does not pass any vertices. If we take one vertex of C_m in the position of $v_{\frac{n}{2}}$ and one vertex of another C_m in the position of $v_{\frac{n}{2}+1}$ of the graph which we will construct and distribute the remaining vertices of two C_m between the vertices of H permitting redundancy and symmetrically regarding the axis then the resulting graph becomes a C_m-spanning subgraph of K_n fixed by σ_1 when we join the vertices of two C_m as it becomes symmetric regarding the axis. The number of ways to distribute the vertices of two C_m is $\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!}$ and the number of the ways to joining the vertices of two C_m is $\frac{(m-1)!}{2}$. Similarly, if we take one vertex of C_m in the position of v_0 and one vertex of another C_m in the position of v_1 of the graph which we will construct and distribute the remaining vertices of two C_m between the vertices of H permitting redundancy and symmetrically regarding the axis

then the resulting graph becomes a C_m-spanning subgraph of K_n fixed by σ_1 when we join the vertices of two C_m as it becomes symmetric regarding the axis. The number of ways to distribute the vertices of two C_m is $\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!}$ and the number of the ways to joining the vertices of two C_m is $\frac{(m-1)!}{2}$. Therefore, by this construction, we can construct $2 \times \sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!} \times \frac{(m-1)!}{2}$ C_m-spanning subgraph of K_n fixed by σ_1. By these two constructions, we can construct

$$2 \times \binom{\frac{n-2}{2}}{\frac{m-2}{2}} S^m_{n-m,1} S^m_{m,1} + 2 \times \left(\sum_{k=0}^{m-1} \frac{(\frac{n-2}{2})!}{k!(m-k-1)!(\frac{n-2m}{2})!} \right) \times S^m_{n-2m,1} \times \frac{(m-1)!}{2}$$

C_m-spanning subgraphs of K_n fixed by σ_1. Clearly there are doubling two pieces of each. Also, it is clear to be able to compose all the C_m-spanning subgraphs of K_n fixed by σ_1 by these methods. Next we assume that n is equal to $2m$. Then we can similarly construct all C_m-spanning subgraphs of K_{2m} fixed by σ_1 by these two constructions if we set H be a empty graph in the case of the second constitution. We have the results. □

Then we completely proved Theorem 2.

Remark 3. We calculated the non-equivarent Hamiltonian cycles of K_m, $m \le 11$ by computer. The numbers agreed with the numbers that are given by Theorem 2. The results is as follows:

n=3	1
n=4	2
n=5	4
n=6	12
n=7	39
n=8	202
n=9	1219
n=10	9468
n=11	83435

Remark 4. We calculated the non-equivalent C_4-spanning subgraphs of K_n, $n \le 12$ by computer. The numbers agreed with the numbers that are given by Theorem 2. The results is as follows:

n=4	2
n=8	39
n=12	7003

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