ON THE HERGLOTZ-PETROVSKII-LERAY FORMULA FOR THE BOUNDARY VALUE PROBLEM

KOICHI YURA

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Dedicated to Professor Norio Shimakura on the occasion of his 60th birthday

ABSTRACT. In this paper we show the Herglotz-Petrovskii-Leray formula for the boundary value problem, which will be useful for investigating the existence of the lacuna for the mixed problem.

1. INTRODUCTION.

Huygens' principle is one of the properties of the wave eqution. This phenomenon is that the fundamental solution is identically zero in the propagation cone when the space-time dimension is even (≥ 4). One of the generalization of Huygens' principle is the theory of lacuna. Let L be a maximal connected open set where the fundamental solution is holomorphic. We say that L is a lacuna when the fundamental solution has C^∞-extension to $\overline{L}$. In particular, if the fundamental solution is identically 0 in L, we say L is a strong lacuna. Hence, Huygens' principle means the fundamental solution of the wave equation on $\mathbf{R}^n$ has a strong lacuna in the propagation cone when n is even (≥ 4). The theory of lacuna begins with Petrovskii's article, and Leray, Atiyah, Bott and Gårding have developed it. In case of the initial value problem for homogeneous hyperbolic partial differential equations with constant coefficient, the propagation of the singularities and the theory of lacuna has been almost completed by their works. Wakabayashi investigated the propagation of the singularities for the mixed problem ([2],[3]). However, in case of the boundary (or mixed) value problem, the theory of lacuna is untouched untill now.

In this paper we can not establish the theory of lacuna for the boundary value problem, but we show the Herglotz-Petrovskii-Leray formula for the boundary value problem, which will be useful for investigating the existence of the lacuna for the mixed problem.

2. PROBLEM AND ASSUMPTIONS.

Let us state our problem and assumptions. We owe these to Wakabayashi [2],[3]. We denote $x' = (x_1, \ldots, x_{n-1})$ for $x = (x_1, \ldots, x_n)$ in $\mathbf{R}^n$ and consider the boundary value problem

$$\text{(BP)} \qquad \begin{cases} P(D)F_{k^0}(x) = 0, & x \in \{x \in \mathbf{R}^n \,;\, x_n > 0\}, \\ D_n^{j-1}F_{k^0}(x)\Big|_{x_n=0} = \delta_{jk^0}\delta(x'), & x' \in \mathbf{R}^{n-1},\ 1 \leq j \leq \mu. \end{cases}$$

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Here k_0 $(1 \leq k_0 \leq \mu)$ is fixed. $P(D)$ is a homogeneous differential operator of n variables whose order is m. The number μ of boundary conditions will be determined later. D means $\frac{1}{i}\frac{\partial}{\partial x}$. We shall assume

(i). $P(\xi)$ is fanctorized in the form

$$P(\xi) = p_1(\xi)^{\nu_1} \dots p_q(\xi)^{\nu_q},$$

where $p_j(\xi)$ $(1 \leq j \leq q)$ are different strictly hyperbolic irreducible polynomials with respect to $\vartheta = (1, 0, \dots, 0)$.

(ii). $\{x \in \mathbf{R}^n\, ;\, x_n = 0\}$ is non-characteristic with respect to $P(\xi)$, that is, $P(0, 1) \neq 0$D

$F_{k^0}(x)$ expresses the propagation of the wave in case the delta shock is given at $\{x \in \mathbf{R}^n\, ;\, x_n = 0\}$. Next we prepare to describe $F_{k^0}(x)$.

Let $\Gamma'(P, \vartheta)$ be the section of $\Gamma(P, \vartheta)$ by $\{\xi_n = 0\}$, that is,

$$\Gamma'(P, \vartheta) = \{\xi' \in \mathbf{R}^{n-1}\, ;\, (\xi', 0) \in \Gamma(P, \vartheta)\}. \tag{2.1}$$

Here $\Gamma(P, \vartheta)$ denotes the connected component of $\mathbf{R}^n \setminus \{\xi \in \mathbf{R}^n\, ;\, P(\xi) = 0\}$ which contains ϑ. $\Gamma(P, \vartheta)$ is called the hyperbolic cone of P.

If we put

$$P(\xi) = \sum_{j=0}^{m} P_{m-j}(\xi'){\xi_n}^j, \tag{2.2}$$

then $P_0(\xi') = P(0, 1)$ is non-zero constant by assumption (ii). Thus $P(\xi)$ is a polynomial of degree m with respect to ξ_n. When ζ' belongs to $\mathbf{R}^{n-1} - i\Gamma'(P, \vartheta)$, $P(\zeta', \lambda) = 0$ has no real roots with respect to λ. Therefore we can denote the roots by

$$\begin{gathered} \lambda_1^+(\zeta'), \dots, \lambda_\mu^+(\zeta'), \lambda_1^-(\zeta'), \dots, \lambda_{m-\mu}^-(\zeta'), \\ \operatorname{Im} \lambda_k^{\pm}(\zeta') \gtrless 0. \end{gathered} \tag{2.3}$$

Of course, μ is invariable when ζ' belongs to $\mathbf{R}^{n-1} - i\Gamma'(P, \vartheta)$. This number μ determines the number of boundary conditions of (BP).

We now define the Lopatinskii determinant $R(\zeta')$ for (BP). We put

$$R(\zeta') = \det L(\zeta'), \tag{2.4}$$

$$L(\zeta') = \left(\frac{1}{2\pi i}\oint \lambda^{j+k-2} P_+(\zeta', \lambda)^{-1}\, d\lambda\right)_{j,k=1,\dots,\mu}, \tag{2.5}$$

$$P_+(\zeta', \lambda) = \prod_{j=1}^{\mu}(\lambda - \lambda_j^+(\zeta')) \tag{2.6}$$

for $\zeta' \in \mathbf{R}^{n-1} - i\Gamma'(P, \vartheta)$. Here the path of integration in (2.5) is a Jordan curve which encloses all roots of $P_+(\zeta', \lambda) = 0$ in complex plain with respct to λ. (In our problem (BP), $R(\zeta') \equiv 1$.) Then the forward fundamental solution $F_{k^0}(x)$ is written in the form

$$F_{k^0}(x) = (2\pi)^{-n} i^{-1} \sum_{j=1}^{\mu} \int_{\mathbf{R}^n - i\sigma} e^{ix\zeta} R_{jk^0}(\zeta') \zeta_n^{j-1} P_+(\zeta)^{-1}\, d\zeta, \quad \sigma \in \Gamma'(P, \vartheta) \times \mathbf{R}. \tag{2.7}$$

Here $R_{jk^0}(\zeta')$ is the (k^0, j)-cofactor of $L(\zeta')$ and homogeneous of degree $\mu - k^0 - j + 1$. $\mathbf{R}^n - i\sigma$ is oriented by $d\zeta > 0$. $F_{k^0}(x)$ is interpreted in the distribution sense with respect to x. That is

$$\langle F_{k^0}(\cdot),\varphi\rangle = i^{-1}\sum_{j=1}^{\mu}\int_{\mathbf{R}^n} R_{jk^0}(\xi'-i\sigma')(\xi_n-i\sigma_n)^{j-1} \times P_+(\xi-i\sigma)^{-1}\mathcal{F}^{-1}\varphi(\xi-i\sigma)\,d\xi \tag{2.8}$$

for $\varphi\in C_0^\infty(\{x_n>0\})$. $\mathcal{F}^{-1}\varphi(\xi-i\sigma)$ denotes $(2\pi)^{-n}\int e^{ix(\xi-i\sigma)}\varphi(x)\,dx$. The word "forward" means

$$\operatorname{supp} F_{k^0}(x)\subset\{x\in\mathbf{R}^n\,;\,x\vartheta\geq 0\}. \tag{2.9}$$

The formula (2.8) of $F_{k^0}(x)$ is not suitable for investigating the existence of lacuna. Therefore, we shall change (2.8) into suitable form, Herglotz-Petrovskii-Leray formula. Herglotz-Petrovskii-Leray formula immediately gives sufficient condition for F_{k^0} to have lacuna. Hence, our aim is to transform (2.8) into the Herglotz-Petrovskii-Leray formula for the boundary value problem.

3. The vector field v.

In case of the initial value problem, we have to deform the integral path, keeping away from the characteristics of $P(\xi)$ when we derive the Herglotz-Petrovskii-Leray formula from the integral formula of the fundamental solution. However, in case of the boundary value problem, $P_+(\xi)$ in integrand may have the branch point when it is analytically continued to a wider domain than $\mathbf{R}^n - i\Gamma'(P,\vartheta)$. So we have to keep away from not only the characteristics but also the branch set of $P_+(\xi)$. Paying attention to this point, let us define the local hyperbolic cone $\Gamma_\xi(P_+,\vartheta)$ of P_+ (due to Wakabayashi) and define a vector field v such that $v(\xi)\in\Gamma_\xi(P_+,\vartheta)$ for all ξ.

First we give the definition of the localization $P_\xi(\zeta)$ of homogeneous hyperbolic polynomial $P(\xi)$ and the local hyperbolic cone.

Definition 3.1. *The localization $P_\xi(\zeta)$ of the homogeneous hyperbolic polynomial $P(\xi)$ is defined by the first non-vanishing homogeneous term in Taylor expansion*

$$P(\xi+\nu\zeta)=\nu^p P_\xi(\zeta)+O(\nu^{p+1})\quad \textit{as } \nu\to+0. \tag{3.1}$$

The local hyperbolic cone $\Gamma_\xi(P,\vartheta)$ is defined by the connected component of $\mathbf{R}^n\setminus\{\zeta\in\mathbf{R}^n\,;\,P_\xi(\zeta)=0\}$ which contains ϑ.

Then, we can get the following lemma by the lower semi-continuity of the local hyperbolic cone.

Lemma 3.2. *Let $\xi^0\in\mathbf{R}^n\setminus\{0\}$ and let M be a compact set in $\Gamma_{\xi^0}(P,\vartheta)$. Then there exist a conic neighborhood Δ of ξ^0 and a positive number t_0 such that*

$$P(\xi-it|\xi|\eta)\neq 0\quad \textit{for } \xi\in\Delta,\ \eta\in M,\ 0<t\leq t_0. \tag{3.2}$$

Let $\xi'\in\mathbf{R}^{n-1}$ be arbitrarily fixed and let $\{j_k\}_{1\leq k\leq r_1}$ be the set of suffixes such that $p_{j_k}(\xi',\lambda)=0$ has a real multiple root λ_k. Then we put

$$\dot{\Gamma}_{\xi'}\times\mathbf{R}=\bigcap_{k=1}^{r_1}\Gamma_{(\xi',\lambda_k)}(p_{j_k},\vartheta). \tag{3.3}$$

When $r_1=0$, we put $\dot{\Gamma}_{\xi'}=\mathbf{R}^{n-1}$. $\dot{\Gamma}_{\xi'}$ corresponds to the local hyperbolic cone of the branch set.

P_+ is holomorphic in $\mathbf{R}^n-i\Gamma'(P,\vartheta)$. However, we can analytically continue P_+ to a wider domain.

Lemma 3.3. *Let $\xi^{0'} \in \mathbf{R}^{n-1} \setminus \{0\}$ and let M be a compact set in $\dot{\Gamma}_{\xi^{0'}}$. Then there exist a conic neighborhood Δ of $\xi^{0'}$ and positive numbers C, t_0 such that $P_+(\zeta)$ is holomorphic in $\zeta \in \Lambda \times \mathbf{C}$, where*

$$\Lambda = \{\zeta' = \xi' - it|\xi'|\eta' \,;\, \xi' \in \Delta,\ \eta' \in M,\ 0 < t \leq t_0\}. \tag{3.4}$$

RemarkF$\Lambda \supset \mathbf{R}^{n-1} - i\Gamma'(P, \vartheta)$. Therefore, P_+ is analytically continued to a wider domain $\Lambda \times \mathbf{C}$.

Proof. We follow the proof of Lemma 3.2 in Wakabayashi[3].

Let $\zeta' \in \mathbf{C}^{n-1}$. We represent the roots of $p_j(\zeta', \lambda) = 0$ as continuous functions of ζ' by

$$\lambda^+_{j1}(\zeta'), \ldots, \lambda^+_{jl^+_j}(\zeta'), \lambda^-_{j1}(\zeta'), \ldots, \lambda^-_{jl^-_j}(\zeta'), \tag{3.5}$$

where

$$\operatorname{Im} \lambda^{\pm}_{jk}(\zeta') \gtrless 0, \quad (\zeta' \in \mathbf{R}^{n-1} - i\Gamma'(P, \vartheta)). \tag{3.6}$$

Of course, l^+_j, l^-_j is invariable when $\zeta' \in \mathbf{R}^{n-1} - i\Gamma'(P, \vartheta)$. It suffices to prove that

$$\{\lambda^+_{jk}(\zeta')\}_{k=1,\ldots,l^+_j} \cap \{\lambda^-_{jk}(\zeta')\}_{k=1,\ldots,l^-_j} = \varnothing, \quad \text{for } \zeta' \in \Lambda. \tag{3.7}$$

Let $1 \leq h \leq l^+_j$ be fixed. If $\lambda^+_{jh}(\xi^{0'})$ is a imaginary root or a simple real root of $p_j(\xi^{0'}, \lambda) = 0$, it follows that

$$\lambda^+_{jh}(\zeta') \neq \lambda^-_{jk}(\zeta'), \quad 1 \leq k \leq l^-_j, \quad \text{for } \zeta' \in \Lambda \tag{3.8}$$

when $t_0 (> 0)$ is small enough. So we investigate the case where $\lambda^+_{jh}(\xi^{0'})$ is a real multiple root of $p_j(\xi^{0'}, \lambda) = 0$.

Let $\tilde{M}$ be a compact set in $\dot{\Gamma}_{\xi^{0'}} \times \mathbf{R}$. Then there exist a conic neighborhood $\tilde{\Delta}$ and a positive number t_0 such that

$$p_j(\xi - it|\xi|\eta) \neq 0 \quad \text{for } \xi \in \tilde{\Delta},\ \eta \in \tilde{M},\ 0 < t \leq t_0 \tag{3.9}$$

by virtue of $\dot{\Gamma}_{\xi^{0'}} \times \mathbf{R} \subset \Gamma_{(\xi^{0'}, \lambda^+_{jh}(\xi^{0'}))}(p_j, \vartheta)$ and Lemma 3.2. If we choose $\tilde{M} = M \times \{0\}$, then we have

$$p_j(\xi' - it|\xi'|\eta', \xi_n) \neq 0 \quad \text{for } \xi \in \tilde{\Delta},\ \eta \in M,\ 0 < t \leq t_0. \tag{3.10}$$

Therefore, it follows that

$$\operatorname{Im} \lambda^{\pm}_{jk} \gtrless 0 \quad \text{when } \zeta' \in \Lambda \tag{3.11}$$

for any k. □

Now we give the definition of the localization of P_+.

Definition 3.4. *Let Γ be an open connected cone in $\mathbf{R}^n$Cand let f be a homogeneous holomorphic function in $\mathbf{R}^n - i\Gamma$. Then the localization f_{ξ^0} of f at $\xi^0 \in \mathbf{R}^n$ is defined by the first non-vanishing homogeneous term in Puiseux expansion*

$$f(\xi^0 + t\zeta) = t^p f_{\xi^0}(\zeta) + o(t^p) \quad \textit{as } t \to +0 \tag{3.12}$$

for $\zeta \in \mathbf{R}^n - i\Gamma$.

When we put $\Gamma = \Gamma'(P, \vartheta) \times \mathbf{R}$ in Definition 3.4, we can localize P_+ and define the local hyperbolic cone $\Gamma_\xi(P_+, \vartheta)$ of P_+. This definition is due to Wakabayashi and so is the proof of localizationability of P_+. Put

$$\Gamma_\xi(P_+, \vartheta) = \text{the connected component of } \{\eta \in \dot{\Gamma}_{\xi'} \times \mathbf{R} \,;\, P_{+\xi}(-i\eta) \neq 0\} \text{ which contains } \vartheta. \tag{3.13}$$

We define the dual cone $\Gamma^\circ_\xi(P_+,\vartheta)$ of $\Gamma_\xi(P_+,\vartheta)$ by

$$\Gamma^\circ_\xi(P_+,\vartheta)=\{x\in\mathbf{R}^n\,;\,x\eta\geq 0 \text{ for all } \eta\in\Gamma_\xi(P_+,\vartheta)\}. \tag{3.14}$$

Proposition 3.5. *Let $F_{k^0}(x)$ be the forward fundamental solution of* (BP). *Then, we have*

$$\operatorname{sing\,supp}_A F_{k^0}\subset\bigcup_{\xi\in\mathbf{R}^n\setminus\{0\}}\Gamma^\circ_\xi(P_+,\vartheta). \tag{3.15}$$

Proof. Let $x^0\notin\cup_{\xi\in\mathbf{R}^n\setminus\{0\}}\Gamma^\circ_\xi(P_+,\vartheta)$, and let $\xi^0\in\mathbf{R}^n\setminus\{0\}$. Then there exist a conic neighborhood Δ of ξ^0, a neighborhood U of x^0, $\eta\in\Gamma_{\xi^0}(P_+,\vartheta)$, and positive numbers δ,t_0 such that

$$x\eta<0\quad\text{when } x\in U, \tag{3.16}$$

$$|P_+(\xi-it|\xi|\eta)|\geq\delta|\xi|^\mu\quad\text{when } \xi\in\Delta,\ 0<t\leq t_0. \tag{3.17}$$

In fact, by the definition of $\Gamma^\circ_\xi(P_+,\vartheta)$ there exist $\eta\in\Gamma_{\xi^0}(P_+,\vartheta)$ and U which satisfies (3.16). Since $P_{+\xi^0}(\eta)\neq 0$ and P_+ is homogeneous of degree μ and continuous, (3.17) holds.

Let Δ_l $(0\leq l\leq N)$ be closed proper convex cones which satisfy $\mathbf{R}^n=\bigcup_{l=0}^N\Delta_l$ and that the measure of $\Delta_k\cap\Delta_l$ $(k\neq l)$ equals to 0. Here Δ_0 is a conic neighborhood of ξ^0 satisfying $\Delta_0\subset\Delta$. Let $\sigma\in\Gamma'(P,\vartheta)\times\mathbf{R}$, and let

$$F_{k^0,l}(x)=(2\pi)^{-n}i^{-1}\sum_{j=1}^{\mu}\int_{\Delta_l-i\sigma}e^{ix\zeta}R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,d\zeta. \tag{3.18}$$

Then $F_{k^0}(x)=\sum_{l=0}^N F_{k^0,l}(x)$, and $F_{k^0,l}(x)$ for $l\geq 1$ can be analytically continued in $\mathbf{R}^n+i\Delta_l^\circ$.

Next, by using (3.17) we can transform the chain of integration of $F_{k^0,0}$ as follows.

$$\begin{aligned}F_{k^0,0}(x)&=(2\pi)^{-n}i^{-1}\sum_{j=1}^{\mu}\int_{\Delta_0-i\sigma}e^{ix\zeta}R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,d\zeta\\&=(2\pi)^{-n}i^{-1}\sum_{j=1}^{\mu}\int_{\Delta_0-i\eta}e^{ix\zeta}R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,d\zeta\\&\quad+(2\pi)^{-n}i^{-1}\sum_{j=1}^{\mu}\int_{C}e^{ix\zeta}R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,d\zeta,\end{aligned} \tag{3.19}$$

where

$$C=\{\zeta\in\mathbf{C}^n\,;\,\zeta=\xi-i(t\sigma+(1-t)\eta),\ \xi\in\partial\Delta_0,\ 0\leq t\leq 1\}, \tag{3.20}$$

η belongs to $\Gamma_{\xi^0}(P_+,\vartheta)$ and satisfies (3.16),(3.17). Dividing C into closed convex cones, we have

$$F_{k^0}(x)=\sum_{j=0}^{N'}G_j(x)+(2\pi)^{-n}i^{-1}\sum_{j=1}^{\mu}\int_{\Delta_0-i\eta}e^{ix\zeta}R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,d\zeta \tag{3.21}$$

by (3.18),(3.19). Here $G_j(x)$ $(0\leq j\leq N')$ are holomorphic functions in $\mathbf{R}^n+iU_j$, and U_j are open convex cone in $\mathbf{R}^n$ satisfying

$$U_j\cap\{y\in\mathbf{R}^n\,;\,\xi^0y<0\}\neq\varnothing. \tag{3.22}$$

Since the second term of right-hand side of (3.21) is holomorphic in U by virtue of (3.16), it follows that

$$(x^0,\xi^0)\notin WF_A(F_{k^0}). \tag{3.23}$$

ξ^0 is arbitrarily fixed. Therefore we have

$$x^0 \notin \operatorname{sing\,supp}_A F_{k^0}. \tag{3.24}$$

□

We can prove the following corollary in the same way as Lemma 6.7 in Atiyah-Bott-Gårding[1] replacing a by P_+.

Corollary 3.6. *When $x \notin \cup_{\xi \in \mathbf{R}^n \setminus \{0\}} \Gamma_\xi^\circ(P_+, \vartheta)$, there exists a C^∞ real vector field $v(\xi)$ which satisfies next conditions.*

- *When $\lambda \in \mathbf{R} \setminus \{0\}$,*

$$v(\lambda\xi) = |\lambda| v(\xi). \tag{3.25}$$

- *For any $\xi \in \mathbf{R}^n \setminus \{0\}$,*

$$v(\xi) \in \Gamma_\xi(P_+, \vartheta) \cap \{\xi \in \mathbf{R}^n\,;\, x\xi = 0\}. \tag{3.26}$$

- *When $\xi \in \mathbf{R}^n \setminus \{0\},\ 0 < t \le 1$,*

$$P_+(\xi - itv(\xi)) \neq 0. \tag{3.27}$$

We denote the family of the C^∞ real vector fields satisfying (3.25),(3.26),(3.27) by $V(P_+, X)$.

4. The Herglotz-Petrovskii-Leray formula.

Now we define the function $\chi_s(z)$ which appeared in Atiyah-Bott-Gårding[1].

Let $z, s \in \mathbf{C}$ $(0 < \arg z < \pi)$ and put

$$\chi_s(z) = \Gamma(-s) e^{-\pi i s} z^s, \quad s \neq 0, 1, \dots. \tag{4.1}$$

When $s = 0, 1, \dots$, we put

$$\begin{aligned} \chi_s(z) &= \frac{d}{dt} \{t \chi_{s+t}(z)\} \Big|_{t=0} \\ &= z^s (\log z^{-1} + c_s + \pi i)/s!, \end{aligned} \tag{4.2}$$

where $c_s = \Gamma'(1) + \sum_{k=1}^s k^{-1}$ $(s \neq 0)$, $c_0 = \Gamma'(1)$. $\chi_s(z)$ satisfies $\frac{d}{dz}\chi_s(z) = \chi_{s-1}(z)$ for all s. When $\operatorname{Re} s < 0$, $\chi_s(z)$ is i^{-s} times the Fourier-Laplace transform of ρ_+^{-s-1}, that is,

$$\chi_s(z) = i^{-s} \int_0^\infty \rho^{-s-1} e^{i\rho z}\, d\rho, \tag{4.3}$$

where we choose $i^{-s}|_{s=-1} = i$. Since $\chi_s(z)$ is holomorphic in $\{\operatorname{Im} z > 0\}$ for all s, the distributions $\lim_{y \to +0} \chi_s(x+iy)$ exist. We denote $\lim_{y \to +0} \chi_s(x+iy)$ by $\chi_s(x+i0)$. When we put

$$\sigma_q(x) = (2\pi i)^{-1} \{\chi_q(x+i0) - (-1)^q \chi_q(-x+i0)\}, \quad q = 0, \pm 1, \pm 2, \dots, \tag{4.4}$$

we have

$$\sigma_q(x) = \begin{cases} 2^{-1}(\operatorname{sgn} x) x^q / q!, & q = 0, 1, \dots, \\ \delta^{(-q-1)}(x), & q = -1, -2, \dots. \end{cases} \tag{4.5}$$

Making use of one of v in $V(P_+, X)$ and χ_s, let us transform F_{k^0} in (2.7).

Theorem 4.1. *Let $x \notin \cup_{\xi\in\mathbf{R}^n\setminus\{0\}}\Gamma^\circ_\xi(P_+,\vartheta)\cup\{x\vartheta<0\}$. Then $F_{k^0}(x)$ is holomorphic and transformed in the form*

$$F_{k^0}(x)=(2\pi)^{1-n}\sum_{j=1}^{\mu} i^q\int_{\gamma(\xi)=1}2^{-1}(\operatorname{sgn} x\xi)\frac{(x\xi)^q}{q!}R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,\omega(\zeta),\qquad \zeta=\xi-iv(\xi) \tag{4.6}$$

when $q=k^0-n\geq 0$ and

$$F_{k^0}(x)=(2\pi)^{1-n}\sum_{j=1}^{\mu} i^q\int_{\gamma(\xi)=1}\delta^{(-q-1)}(x\xi)R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,\omega(\zeta),\qquad \zeta=\xi-iv(\xi) \tag{4.7}$$

when $q=k^0-n<0$. Here $\gamma(\xi)$ is a C^∞ function satisfying $\gamma(\lambda\xi)=|\lambda|\gamma(\xi)$ for $\lambda\in\mathbf{R}$ and $d\,\gamma|_{\gamma=1}\neq 0$. $\{\gamma(\xi)=1\}$ is oriented by $\omega(\xi)>0$. $\omega(\zeta)$ is the Kronecker form, that is,

$$\omega(\zeta)=\sum_{j=1}^{n}(-1)^{j-1}\zeta_j d\zeta_1\wedge\cdots\wedge\widehat{d\zeta_j}\wedge\cdots\wedge d\zeta_n. \tag{4.8}$$

RemarkFThe Herglotz-Petrovskii-Leray formula is represented by integration over certain homology class in projective space. $\{\gamma(\xi)=1\}$ can represent all chains which is homologous to $(n-1)$-dimensional sphere $\{|\xi|=1\}$ in $(\mathbf{R}^n\setminus\{0\})/\mathbf{R}_+$. Therefore, we employ $\gamma(\xi)$.

Proof. When $x\notin\cup_{\xi\in\mathbf{R}^n\setminus\{0\}}\Gamma^\circ_\xi(P_+,\vartheta)$, we can transform the chain $\mathbf{R}^n-i\sigma$ of integration in (2.7) into one of $\{\xi-i(v(\xi)-\varepsilon|\xi|\vartheta)\,;\,v\in V(P_+,X)\}$ by Stokes' formula. So we have

$$F_{k^0}(x)=(2\pi)^{-n}i^{-1}\sum_{j=1}^{\mu}\int_{\mathbf{R}^n}e^{ix\zeta}R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,d\zeta,\qquad \zeta=\xi-i(v(\xi)-\varepsilon|\xi|\vartheta). \tag{4.9}$$

Here $\varepsilon(>0)$ is small enough to satisfy $P_+(\xi-i(v(\xi)-\varepsilon|\xi|\vartheta))\neq 0$ when $\xi\in\mathbf{R}^n\setminus\{0\}$. Take $\rho>0$ and η such that $\xi=\rho\eta,\ \gamma(\eta)=1$, and consider (ρ,η) a system of coordinates of $\mathbf{R}^n\setminus\{0\}$. Since $R_{jk^0}(\zeta')$ and $P_+(\zeta)$ are homogeneous of degree $\mu-k^0-j+1$ and μ respectively, it follows that

$$F_{k^0}(x)=(2\pi)^{-n}i^{-1}\sum_{j=1}^{\mu}\int_0^\infty\!\!\int_{\gamma(\eta)=1}\rho^{-k^0+n-1}e^{i\rho x\zeta}R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,d\rho\wedge\omega(\zeta),\qquad \zeta=\eta-i(v(\eta)-\varepsilon|\eta|\vartheta). \tag{4.10}$$

By using (4.2) and (4.3), and making radial integration, we have

$$F_{k^0}(x)=(2\pi)^{-n}i^{-1}\sum_{j=1}^{\mu}i^{k^0-n}\int_{\gamma(\eta)=1}\chi_{k^0-n}(x\zeta)R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,\omega(\zeta),\qquad \zeta=\eta-i(v(\eta)-\varepsilon|\eta|\vartheta). \tag{4.11}$$

Here $\{\gamma(\eta)=1\}$ has the orientaion $\omega(\eta)>0$ induced by $d\zeta>0$. Since $x\notin\{x\vartheta<0\}$, $F_{k^0}(-x)=0$ by (2.9). Therefore we obtain

$$
\begin{aligned}
F_{k^0}(x) &= F_{k^0}(x)-(-1)^{k^0-n}F_{k^0}(-x)\\
&= (2\pi)^{-n}\sum_{j=1}^{\mu} i^{k^0-n-1}\Big\{\int_{\gamma(\eta)=1}\chi_{k^0-n}(x\zeta)R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,\omega(\zeta)\\
&\qquad -(-1)^{k^0-n}\int_{\gamma(\eta)=1}\chi_{k^0-n}(-x\bar\zeta)R_{jk^0}(\bar\zeta')\bar\zeta_n^{j-1}P_+(\bar\zeta)^{-1}\,\omega(\bar\zeta)\Big\},\\
&\qquad \zeta=\eta-i(v(\eta)-\varepsilon|\eta|\vartheta),\ \bar\zeta=\eta-i(v(\eta)+\varepsilon|\eta|\vartheta).
\end{aligned}
\tag{4.12}
$$

Considering the integral of the right-hand side in the distribution sense and taking the limit as $\varepsilon\to+0$, by using (4.4) we obtain

$$
F_{k^0}(x)=(2\pi)^{1-n}\sum_{j=1}^{\mu} i^{k^0-n}\int_{\gamma(\eta)=1}\sigma_{k^0-n}(x\xi)R_{jk^0}(\zeta')\zeta_n^{j-1}P_+(\zeta)^{-1}\,\omega(\zeta),\qquad \zeta=\eta-iv(\eta).
\tag{4.13}
$$

Thus we have (4.6),(4.7) by (4.5). $\square$

The Herglotz-Petrovskii-Leray formula given in [1] is transformed into the integration over certain homology class. But in this paper, we can not investigate to which homology group the chain $\{\eta-iv(\eta)\,;\,\gamma(\eta)=1\}$ should belong. Probably, the chain $\{\eta-iv(\eta)\,;\,\gamma(\eta)=1\}$ is a cycle of the $(\prod_{j=1}^{q}\deg p_j)$-sheeted covering of

$$
\mathbf{C}^n\setminus\left\{\zeta\in\mathbf{C}^n\,;\,\prod_{j=1}^{q}\Delta_j(\zeta)=0\right\},
\tag{4.14}
$$

where $\Delta_j(\xi)$ is the resultant of $p_j(\xi)$ and $\frac{\partial p_j}{\partial\xi_n}(\xi)$ with respect to ξ_n.

References

[1] M. F. Atiyah, R. Bott and L. Gårding. Lacunas for hyperbolic differential operators with constant coefficients I. *Acta Math.*, Vol. 124, 109–189, 1970.
[2] S. Wakabayashi. Singularities of the Riemann functions of hyperbolic mixed problems in a quarter-space. *Publ. Res. Inst. Math. Sci.*, Vol. 11, 417–440, 1976.
[3] S. Wakabayashi. Analytic wave front sets of the Riemann functions of hyperbolic mixed problems in a quarter-space. *Publ. Res. Inst. Math. Sci.*, Vol. 11, 785–807, 1976.

Koichi Yura
Address: 1-2-204-304 Teraikedai,
Tondabayashi, Osaka 584-0073, JAPAN
Tel: 0721-28-2024
Fax: 0721-28-2024
E-mail: ko-1-yura@max.hi-ho.ne.jp