Scientiae Mathematicae Japonicae Online, **Vol. 7**, (2002), 513–522

MULTI-SAMPLE PROBLEM FOR ARCH RESIDUAL EMPIRICAL PROCESSES

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Received June 6, 2002

Abstract. This paper gives the asymptotic theory of a class of rank order statistics $\{T_{N,j}, j = 1 \ldots, c\}$ for c-sample problem pertaining to empirical processes based on the squared residuals from a class of ARCH models. An important aspect is that, unlike the residuals of ARMA models, the asymptotic distribution depends on those of ARCH volatility estimators. By an application of the asymptotic results, we propose the c-sample analogues of Mood's two-sample and Klotz's two-sample normal scores tests. These studies help to highlight some important features of ARCH residuals in comparison with the i.i.d. or ARMA settings.

1 Introduction. Traditional time series models assume a constant one-period forecast variance. In order to overcome this implausible assumption, Engle (1982) introduced a class of ARCH(p) models, which proved to be extremely useful in analyzing economic time series. Since then, ARCH related models have become perhaps the most popular and extensively studied financial econometric models (Engle (1995), Chandra and Taniguchi (2002), Tsay (2002)). Moreover, Giraitis *et. al* (2000) discussed a class of ARCH(∞) models, which includes that of ARCH(p) models as a special case, and established sufficient conditions for the existence of a stationary solution and its explicit representation.

For an ARCH(p) model, Horváth *et. al* (2001) derived the asymptotic distribution of the empirical process based on the squared residuals which is considered of fundamental importance for statistical analysis. Then they showed that, unlike the residuals of ARMA models, these residuals do not behave in this context like asymptotically independent random variables, and the asymptotic distribution involves a term depending on estimators of the volatility parameters of the model. Also Lee and Taniguchi (2000) proved the local asymptotic normality for ARCH(∞) models, and discussed the residual empirical process for an ARCH(1) model with stochastic mean.

In the i.i.d. settings, the study of the asymptotic properties based on two-sample rank order statistics is fundamental and essential part of nonparametric statistics. The classical limit theorem which generated much interest in this area is the celebrated Chernoff-Savage (1958) theorem. It is well known that the theorem is widely used to study the asymptotic power and power efficiency of a class of two-sample tests. Puri (1964) generalized the situation covered by this theorem to the c-sample problem. Later, under less stringent conditions on the score generating functions, Puri and Sen (1993) formulated the Chernoff-Savage theorem for the c-sample problem. Regarding the two-sample problem for ARCH residual empirical processes, Chandra and Taniguchi (2002) developed some test procedures.

The present paper discusses the asymptotic theory of the c-sample rank order statistics $\{T_{N,j}, j = 1 \ldots, c\}$ for ARCH residual empirical processes based on the techniques of Puri and Sen (1993), Horváth *et. al* (2001), and Chandra and Taniguchi (2002). Since the asymptotics of the residual empirical processes are different from those for the usual ARMA case, the limiting

2000 *Mathematics Subject Classification.* 62G10, 62G30, 62H15.
Key words and phrases. ARCH model; c-sample problem; rank order statistics; empirical processes; squared residuals; nonparametric tests.

distribution of $\{T_{N,j}, j = 1\dots,c\}$ is greatly different from that of ARMA case (of course i.i.d. case). Section 2 gives the setting of $\{T_{N,j}, j = 1\dots,c\}$ pertaining to empirical processes based on the squared residuals from a class of ARCH(p) models and establishes its asymptotic distribution. In Section 3, we use this result to propose the c-sample analogues of Klotz's two-sample normal scores and Mood's two-sample tests. These studies illuminate some interesting characteristics of ARCH residuals in comparison with the i.i.d. settings.

2 c-sample rank order statistics and results. In this section, we study a c-sample problem pertaining to a class of rank order statistics based on ARCH residual empirical processes.

Let us consider the c independent random samples from the following ARCH(p) models

$$X_{i,t} = \begin{cases} \sigma_{i,t}\varepsilon_{i,t}, \quad \sigma_{i,t}^2 = \theta_i^0 + \sum\limits_{l=1}^{p_i} \theta_i^l X_{i,t-l}^2 & \text{for} \quad t = 1,\dots,n_i, \\ 0 & \text{for} \quad t = -p_i+1,\dots,0, \quad i = 1,\dots,c, \end{cases}$$

where the $\varepsilon_{i,t}$ are i.i.d.(0,1) random variables with corresponding fourth-order cumulants $\kappa_4^{(i)}$, $\theta_i = (\theta_i^0, \theta_i^1, \dots, \theta_i^{p_i})^T \in \Theta \subset \mathbb{R}^{p_i+1}$, and $\varepsilon_{i,t}$ are independent of $X_{i,s}, s < t$. It is assumed that $\theta_i^0 > 0$, $\theta_i^l \geq 0$ and $\theta_i^1 + \dots + \theta_i^{p_i} < 1$ for stationarity. Denote by $F_i(x)$ the distribution function of $\varepsilon_{i,t}^2$ and we assume that $f_i(x) = F_i'(x)$ exists and is continuous on $(0,\infty)$.

In the following, we are concerned with the c-sample problem of testing

(1)
$$H_0 : F_1(x) = \cdots = F_c(x) \text{ for all } x \quad \text{against} \quad H_A : F_i(x) \neq F_j(x) \text{ for some } x, \text{ and } i \neq j.$$

Write $Y_{i,t} = X_{i,t}^2$, $\zeta_{i,t} = (\varepsilon_{i,t}^2 - 1)\sigma_{i,t}^2$ and $Z_{i,t} = (1, Y_{i,t}, \dots, Y_{i,t-p_i+1})^T$. Then the autoregressive representation is given by

$$Y_{i,t} = \theta_i^T Z_{i,t-1} + \zeta_{i,t}.$$

Note that $E[\zeta_{i,t}|\mathcal{B}_{i,t-1}] = 0$, where $\mathcal{B}_{i,t}$ is the σ-field generated by $\{X_{i,t}, X_{i,t-1},\dots\}$. Let us first consider the estimation of θ_i. Suppose that observed stretches $(Y_{i,1},\dots,Y_{i,n_i})$ are available. Let

$$\mathcal{Q}_{n_i}(\theta_i) = \sum_{t=2}^{n_i} (Y_{i,t} - \theta_i^T Z_{i,t-1})^2, \quad i = 1,\dots,c, \tag{2}$$

be the penalty functions. Then the conditional least squares estimators (see Tjøstheim (1986)) $\hat{\theta}_{i,n_i}$ of θ_i; $i = 1,\dots,c$, are obtained by minimizing (2) with respect to θ_i, $i = 1,\dots,c$, respectively. For $\hat{\theta}_{i,n_i}$, we assume that

$$\|\hat{\theta}_{i,n_i} - \theta_i\| = \mathcal{O}_p(n_i^{-1/2}), \quad i = 1,\dots,c, \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm. Tjøstheim (1986, p. 254-256) gave a set of sufficient conditions to validate (3), and it is also satisfied by the pseudo maximum likelihood and conditional likelihood estimators (see e.g., Gouriéroux (1997)). The empirical squared residuals are given by

$$\hat{\varepsilon}_{i,t}^2 = X_{i,t}^2 / \hat{\sigma}_{i,t}^2, \tag{4}$$

where $\hat{\sigma}_{i,t}^2 = \hat{\theta}_{i,n_i}^0 + \sum_{l=1}^{p_i} \hat{\theta}_{i,n_i}^l X_{i,t-l}^2$, $i = 1,\dots,c$.

We begin by setting up our notation and describing our approach in line with Puri and Sen (1993). Let $N = \sum_{i=1}^c n_i$ and $\lambda_{i,N} = n_i/N$, $i = 1,\dots,c$. For (4), the size N is assumed to be such that $0 < \lambda_0 \leq \lambda_{1,N},\dots,\lambda_{c,N} \leq 1 - \lambda_0 < 1$ hold for some $\lambda_0 \leq 1/c$. Then the combined distribution is defined by

$$H_N(x) = \sum_{i=1}^c \lambda_{i,N} F_i(x).$$

Likewise, if $\hat{F}_{n_i}^{(i)}(x)$ denotes the empirical distribution function of $\hat{\varepsilon}_{i,t}^2$, the corresponding empirical distribution is

$$\hat{\mathscr{H}}_N(x) = \sum_{i=1}^{c} \lambda_{i,N} \hat{F}_{n_i}^{(i)}(x). \tag{5}$$

Write $\hat{B}_{n_i}^{(i)}(x) = n_i^{1/2}(\hat{F}_{n_i}^{(i)}(x) - F_i(x))$. Then

$$\hat{B}_{n_i}^{(i)}(x) \quad = \quad n_i^{-1/2} \sum_{t=1}^{n_i} [I(\hat{\varepsilon}_{i,t}^2 \le x) - F_i(x)], \quad i = 1, \dots, c,$$

where $I(A)$ is the indicator function of the event A. From the result by Horváth *et. al* (2001) in the case of $c = 1$, we observe that

$$\hat{B}_{n_i}^{(i)}(x) = \mathscr{E}_{n_i}^{(i)}(x) + \mathcal{A}_i x f_i(x) + \eta_{n_i}^{(i)}(x), \quad i = 1, \dots, c, \tag{6}$$

where

$$\mathscr{E}_{n_i}^{(i)}(x) = n_i^{-1/2} \sum_{t=1}^{n_i} [I(\varepsilon_{i,t}^2 \le x) - F_i(x)], \quad \mathcal{A}_i = \sum_{l=0}^{p_i} n_i^{1/2} (\hat{\theta}_{i,n_i}^l - \theta_i^l) \tau_{i,l} \tag{7}$$

and $\sup_x |\eta_{n_i}^{(i)}(x)| = o_p(1)$ with $\tau_{i,0} = E[1/\sigma_{i,t}^2]$ and $\tau_{i,l} = E[\sigma_{i,t-l}^2 \varepsilon_{i,t-l}^2 / \sigma_{i,t}^2]$, $1 \le l \le p_i$. Denote by $F_{n_i}^{(i)}(x) = n_i^{-1} \sum_{t=1}^{n_i} I[\varepsilon_{i,t}^2 \le x]$ the usual empirical distribution function of $\varepsilon_{i,t}^2$. Then, from (6), the preceding (5) becomes

$$\hat{\mathscr{H}}_N(x) \quad = \quad \mathcal{H}_N(x) + \sum_{i=1}^{c} n_i^{-1/2} \lambda_{i,N} \mathcal{A}_i x f_i(x) + \xi_N(x), \tag{8}$$

where $\mathcal{H}_N(x) = \sum_{i=1}^{c} \lambda_{i,N} F_{n_i}^{(i)}(x)$ and $\xi_N(x) = \sum_{i=1}^{c} n_i^{-1/2} \lambda_{i,N} \eta_{n_i}^{(i)}(x)$. The decomposition (8) is basic and will be used repeatedly in the sequel.

Let $S_{N,i}^{(j)} = 1$, if the ith smallest one in the combined residuals $\hat{\varepsilon}_{1,1}^2, \dots, \hat{\varepsilon}_{1,n_1}^2, \dots, \hat{\varepsilon}_{c,1}^2, \dots,$ $\hat{\varepsilon}_{c,n_c}^2$ is from $\hat{\varepsilon}_{j,1}^2, \dots, \hat{\varepsilon}_{j,n_j}^2$, and otherwise let $S_{N,i}^{(j)} = 0$, $i = 1, \dots, N$, $j = 1, \dots, c$. Then, for the testing problem (1), let us consider the rank order statistics of the form

$$T_{N,j} = \frac{1}{n_j} \sum_{i=1}^{N} \mathscr{S}_{N,i} S_{N,i}^{(j)}, \quad j = 1, \dots, c,$$

where the $\mathscr{S}_{N,i}$ are given constants called weights or scores. The definition of $T_{N,j}$ is the one conventionally used. We shall, however, use its equivalent representation given by

$$T_{N,j} = \int J\left[\frac{N}{N+1} \hat{\mathscr{H}}_N(x)\right] d\hat{F}_{n_j}^{(j)}(x), \quad j = 1, \dots, c, \tag{9}$$

where $\mathscr{S}_{N,i} = J(i/(N+1))$, and $J(u)$, $0 < u < 1$, is a continuous function.

We now give typical examples of J, which have been reported in Puri and Sen (1993):

(i) Wilcoxon's two-sample test with $J(u) = u$, $0 < u < 1$,

(ii) Van der Waerden's two-sample test with $J(u) = \Phi^{-1}(u)$, $0 < u < 1$, where $\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2} dt$,

(iii) Mood's two-sample test with $J(u) = (u - \frac{1}{2})^2$, $0 < u < 1$,

(iv) Klotz's normal score test with $J(u) = (\Phi^{-1}(u))^2$, $0 < u < 1$.

Examples (i) to (ii) are the tests for location, (iii) and (iv) are tests for scale.

In the following, K will denote a generic constant which may depend on J but will not depend on $F_1(x), \dots, F_c(x)$, $n_1, \dots, n_c$ and N.

We now impose the following regularity conditions.

Assumption 1.

(A.1) $J(u)$

is not constant and has a continuous derivative $J'(u)$ on (0,1).

(A.2) $|J| \le K[u(1-u)]^{-\frac{1}{2}+\delta}$ and $|J'| \le K[u(1-u)]^{-\frac{3}{2}+\delta}$ for some $\delta > 0$.

(A.3) $xf_j(x)$ and $xf_j'(x)$ are uniformly bounded continuous, and integrable functions on $(0, \infty)$ for all $j = 1, \dots, c$.

(A.4) There exist constants $d_j > 0$ such that $F_j(x) \ge d_j\{xf_j(x)\}$ for all $x > 0$, $j = 1, \dots, c$.

We also require the following regularity condition.

Assumption 2.

$$E(Y_{i,t}^4) < \infty, \quad i = 1, \dots, c.$$

Sufficient conditions to validate this assumption are given by Chen and An (1998) in the case of $c = 1$.

In order to elucidate the asymptotics of (9), we require further settings. Recalling (2) and using the notation $\sigma_{i,t}^2(\theta_i) = \theta_i^0 + \theta_i^1 Y_{i,t-1} + \cdots + \theta_i^{p_i} Y_{i,t-p_i}$, $i = 1, \dots, c$, we notice that

$$\begin{aligned}
\frac{\partial \mathcal{Q}_{n_i}}{\partial \theta_i^0} &= -2\sum_{t=2}^{n_i} (\varepsilon_{i,t}^2 - 1)\sigma_{i,t}^2(\theta_i) \equiv -2\sum_{t=2}^{n_i} \phi_i(\varepsilon_{i,t}^2)\vartheta_i^0, \\
\frac{\partial \mathcal{Q}_{n_i}}{\partial \theta_i^l} &= -2\sum_{t=2}^{n_i} (\varepsilon_{i,t}^2 - 1)\sigma_{i,t}^2(\theta_i)Y_{i,t-l} \equiv -2\sum_{t=2}^{n_i} \phi_i(\varepsilon_{i,t}^2)\vartheta_i^l, \quad 1 \le l \le p_i, \quad i = 1, \dots, c,
\end{aligned}$$

where $\phi_i(u) = u - 1$. Write

$$\mathcal{R}_i = 2E[\sigma_{i,t}^4(\theta_i)Z_{i,t-1}Z_{i,t-1}^T], \quad \text{and} \quad \mathcal{U}_i = E[Z_{i,t-1}Z_{i,t-1}^T],$$

and $\vartheta_i = (\vartheta_i^0, \dots, \vartheta_i^{p_i})^T$, $i = 1, \dots, c$. Then, using standard arguments, it seen that the lth component of each $\hat{\theta}_{i,n_i}$, $i = 1, \dots, c$ admits the representation

$$\hat{\theta}_{i,n_i}^l - \theta_i^l = \frac{1}{n_i}\sum_{t=1}^{n_i} U_i^l \phi_i(\varepsilon_{i,t}^2) + o_p(n_i^{-1/2}), \quad 0 \le l \le p_i, \tag{10}$$

where U_i^l is the lth component of each $\mathcal{U}_i^{-1}\vartheta_i$, $i = 1, \dots, c$. Write $\alpha_i^l = E(U_i^l)$, $0 \le l \le p_i$ and $\tau_i = (\tau_{i,0}, \dots, \tau_{i,p_i})^T$, $i = 1, \dots, c$ (recall (7)). Then, we have the following theorem, whose proof is given in Section 4.

Theorem 1. *Suppose that Assumptions 1 and 2 hold and that, in addition,* $\hat{\theta}_{i,n_i}$, $i = 1, \ldots, c$, *are the respective conditional least squares estimators of* θ_i, $i = 1, \ldots, c$, *satisfying (3). If* $\mathcal{U}_i$ *and* $\mathcal{R}_i$; $i = 1, \ldots, c$, *are positive definite matrices with bounded elements, then*

$$N^{1/2}\Sigma_N^{-1/2}(T_{N,1} - \mu_{N,1}, \ldots, T_{N,c} - \mu_{N,c})^T \xrightarrow{d} \mathscr{N}(0, I) \quad \text{as } N \to \infty,$$

where

$$\mu_{N,j} = \int J[H_N(x)]dF_j(x) \quad \text{and} \quad \Sigma_N = ((\sigma_{N,jk}))$$

with $\sigma_{N,jj} = \sigma_{1N,jj} + \sigma_{2N,jj} + \sigma_{3N,jj} + \gamma_{N,jj}$, *where*

$$\begin{aligned}
\sigma_{1N,jj} &= 2\Bigg[\sum_{\substack{i=1\\i\neq j}}^{c} \lambda_{i,N} \iint_{x<y} A_{i,N}(x,y)dF_j(x)dF_j(y) + \frac{1}{\lambda_{N,j}} \sum_{\substack{i=1\\i\neq j}}^{c} \lambda_{i,N}^2 \iint_{x<y} A_{j,N}(x,y)dF_i(x)dF_i(y)\Bigg] \\
&\quad + \frac{1}{\lambda_{N,j}} \sum_{\substack{i,k=1\\i\neq k, i\neq j, k\neq j}}^{c} \lambda_{i,N}\lambda_{k,N}\Bigg[\iint_{x<y} A_{j,N}(x,y)dF_i(x)dF_k(y) + \iint_{y<x} A_{j,N}(y,x)dF_i(x)dF_k(y)\Bigg], \\
\sigma_{2N,jj} &= \omega_{j,N}^T \mathcal{U}_j^{-1}\mathcal{R}_j\mathcal{U}_j^{-1}\omega_{j,N}, \quad \sigma_{3N,jj} = \sum_{\substack{i=1\\i\neq j}}^{c} \nu_{i,N}^T \mathcal{U}_i^{-1}\mathcal{R}_i\mathcal{U}_i^{-1}\nu_i, \quad \text{and} \\
\gamma_{N,jj} &= 2\Bigg[\sum_{\substack{i=1\\i\neq j}}^{c} \sum_{l=0}^{p_i} \lambda_{i,N}\tau_{i,l} \iint h_i^l(x)\psi_{i,N}(x,y)dF_j(x)dF_j(y) \\
&\quad + \frac{1}{\lambda_{N,j}} \sum_{\substack{i=1\\i\neq j}}^{c} \sum_{l=0}^{p_j} \lambda_{i,N}^2\tau_{j,l} \iint h_j^l(x)\psi_{j,N}(x,y)dF_i(x)dF_i(y)\Bigg],
\end{aligned}$$

and $\sigma_{N,jj'} = \sigma_{1N,jj'} + \sigma_{2N,jj'}$, $j \neq j'$ *with*

$$\begin{aligned}
\sigma_{1N,jj'} &= -\sum_{i=1}^{c} \lambda_{i,N}\Bigg[\iint_{x<y} A_{j,N}(x,y)dF_i(x)dF_{j'}(y) + \iint_{y<x} A_{j,N}(y,x)dF_i(x)dF_{j'}(y)\Bigg] \\
&\quad - \sum_{i=1}^{c} \lambda_{i,N}\Bigg[\iint_{x<y} A_{j',N}(x,y)dF_i(x)dF_j(y) + \iint_{y<x} A_{j',N}(y,x)dF_i(x)dF_j(y)\Bigg] \\
&\quad + \sum_{i=1}^{c} \lambda_{i,N}\Bigg[\iint_{x<y} A_{i,N}(x,y)dF_j(x)dF_{j'}(y) + \iint_{y<x} A_{i,N}(y,x)dF_j(x)dF_{j'}(y)\Bigg], \quad \text{and} \\
\sigma_{2N,jj'} &= -\sum_{i=1}^{c} \lambda_{i,N}\Bigg\{\sum_{l=0}^{p_j} \tau_{j,l} \iint h_j^l(x)\psi_{j,N}(x,y)dF_i(x)dF_{j'}(y) \\
&\quad + \sum_{l=0}^{p_{j'}} \tau_{j',l} \iint h_{j'}^l(x)\psi_{j',N}(x,y)dF_i(x)dF_j(y) \\
&\quad + \sum_{l=0}^{p_j} \tau_{j,l} \iint h_j^l(y)\psi_{j,N}(y,x)dF_i(x)dF_{j'}(y) \\
&\quad + \sum_{l=0}^{p_{j'}} \tau_{j',l} \iint h_{j'}^l(y)\psi_{j',N}(y,x)dF_i(x)dF_j(y)\Bigg\},
\end{aligned} \tag{11}$$

where

$$
\begin{aligned}
A_{j,N}(u,v) &= F_j(u)[1-F_j(v)]J'[H_N(u)]J'[H_N(v)],\\
\omega_{j,N} &= -\lambda_{N,j}^{-1/2}\sum_{\substack{i=1\\ i\neq j}}^{c}\lambda_{i,N}\int xf_j(x)J'[H_N(x)]dF_i(x)\times\tau_j,\\
\nu_{i,N} &= \lambda_{i,N}^{1/2}\int xf_i(x)J'[H_N(x)]dF_j(x)\times\tau_i,\\
\psi_{j,N}(u,v) &= vf_j(v)J'[H_N(u)]J'[H_N(v)],\\
h_j^l(v) &= \alpha_j^l\int_0^v\phi_j(u)f_j(u)du.
\end{aligned}
$$

Remark 1. Observe that the terms $\sigma_{2N,jj}$, $\sigma_{3N,jj}$, $\gamma_{N,jj}$ and $\sigma_{2N,jj'}$ depend on the volatility estimators $\hat{\theta}_{i,n_i}$, $i=1,\dots,c$. Hence, the asymptotics of $\{T_{N,j}, i=1,\dots,c\}$ are greatly different from those for i.i.d. case.

Remark 2. As an application of Theorem 1 in the two-sample testing problem for scale, Chandra and Taniguchi (2002) studied the asymptotic performance of $\{T_{N,j}, j=1, c=2\}$, like the confidence intervals, asymptotic relative efficiency and ARCH affection for some ARCH residual distributions via numerical illustrations. The results for the score function $J(u)=u$, $0<u<1$ show that the Wilcoxon test $\{T^*_{N,j}, j=1, c=2\}$ is preferable if the underlying distribution is logistic.

3 Some applications.

The c-sample testing problem considered in Section 2 can be applied to many situations in time series analysis. In the following, we propose the c-sample analogues of Mood's two-sample and Klotz's two-sample normal scores tests.

Recall that $F_i(x)$, $i=1,\dots,c$, are the distribution functions of $\varepsilon_{i,t}^2$, $t=1,\dots,n_i$, $i=1,\dots,c$, respectively. Let us now consider the scale problem in the case of $F_i(x)=F(\delta_i x)$, for all $i=1,\dots,c$, where the δ_i are real constants. Henceforth, it is assumed that F is arbitrary and has finite variance σ_F^2. The c-sample testing problem for scale can be described as follows;

$$H_0: \delta_1=\cdots=\delta_c=1 \quad \text{against} \quad H_A: \delta_i\neq\delta_j \text{ for at least one } i\neq j.$$

By virtue of Theorem 1, we propose the following test statistic defined as

$$\mathcal{L}_N = N\mathscr{T}_N^T\Sigma_N^{-1}\mathscr{T}_N,$$

where $\mathscr{T}_N=(T_{N,1}-\mu_{N,1},\dots,T_{N,c}-\mu_{N,c})^T$. For $\mathcal{L}_N$ to be practically useful, it is necessary to replace Σ_N which depends on several unknown parameters and functions by a consistent estimator $\widehat{\Sigma}_N$. Observe that α_i^l, $h_i^l(v)$ and τ_i; $i=1,\dots,c$, are expected values and can be consistently estimated by the corresponding averages. Note also that $\mathcal{U}_i^{-1}\mathcal{R}_i\mathcal{U}_i^{-1}$ is the asymptotic covariance matrix of $n_i^{1/2}(\hat{\theta}_{i,n_i}-\theta_i)$ and its estimation is discussed in Gouriéroux (1997). Then, we can propose

$$\hat{\mathcal{L}}_N = N\mathscr{T}_N^T\widehat{\Sigma}_N^{-1}\mathscr{T}_N,$$

for the testing problem H_0 against H_A. Here, we may take the following J:

(a) Mood's c-sample test with $J(u)=(u-\frac{1}{2})^2$, $0<u<1$.

(b) Klotz's normal score c-sample test with $J(u)=[\Phi^{-1}(u)]^2$, $0<u<1$.

4 Proof. In this section we provide the proof of Theorem 1. Write $d\hat{F}_{n_j}^{(j)} = d(\hat{F}_{n_j}^{(j)} - F_j + F_j)$ and

$$\begin{aligned} J\left[\frac{N}{N+1}\hat{\mathscr{H}}_N\right] &= J[H_N] + (\hat{\mathscr{H}}_N - H)J'[H_N] - \frac{\hat{\mathscr{H}}_N}{N+1}J'[H_N] \\ &\quad + \left\{ J\left[\frac{N}{N+1}\hat{\mathscr{H}}_N\right] - J[H_N] - \left(\frac{N}{N+1}\hat{\mathscr{H}}_N - H\right)J'[H_N]\right\}, \end{aligned}$$

Then the statistics (9) after a little simplification becomes

$$T_{N,j} = \mu_{N,j} + B_{1N,j} + B_{2N,j} + \sum_{i=1}^{3} C_{iN,j},$$

where

$$\begin{aligned} B_{1N,j} &= \int J[H_N]d(\hat{F}_{n_j}^{(j)} - F_j)(x), \\ B_{2N,j} &= \int (\hat{\mathscr{H}}_N(x) - H_N(x))J'[H_N]dF_j(x), \\ C_{1N,j} &= -\frac{1}{N+1}\int \hat{\mathscr{H}}_N(x)J'[H_N]d\hat{F}_{n_j}^{(j)}(x), \\ C_{2N,j} &= \int (\hat{\mathscr{H}}_N(x) - H_N(x))J'[H_N]d(\hat{F}_{n_j}^{(j)} - F_j)(x), \\ C_{3N,j} &= \int \left\{ J\left[\frac{N}{N+1}\hat{\mathscr{H}}_N(x)\right] - J[H_N] \right. \\ &\quad \left. - \left(\frac{N}{N+1}\hat{\mathscr{H}}_N(x) - H_N(x)\right)J'[H_N]\right\} d\hat{F}_{n_j}^{(j)}(x). \end{aligned}$$

To prove this theorem, we are required to show that (i) $B_{1N,j} + B_{2N,j}$ has a limiting Gaussian distribution, and (ii) the C_* terms are of higher order.

First, we show the statement (i). From (6), we observe that

$$B_{1N,j} = \int J[H_N]d(F_{n_j}^{(j)} - F_j)(x) + n_j^{-1/2}\mathcal{A}_j \int J[H_N]d[xf_j(x)] + \text{lower order terms.} \tag{12}$$

Then, integrating $B_{2N,j}$ by parts, and adding it to (12), we obtain

$$\begin{aligned} & N^{1/2}(B_{1N,j} + B_{2N,j}) \\ &= N^{1/2}\Bigg\{ -\sum_{\substack{i=1\\ i\neq j}}^{c} \lambda_{i,N} \int B_j(x)d(F_{n_i}^{(i)} - F_i)(x) \\ &\quad + \int \{J[H_N] - \lambda_j B_j(x)\}d(F_{n_j}^{(j)} - F_j)(x) \\ &\quad - n_j^{-1/2}\mathcal{A}_j \sum_{\substack{i=1\\ i\neq j}}^{c} \lambda_{i,N} \int xf_j(x)J'[H_N]dF_i(x) \\ &\quad + \sum_{\substack{i=1\\ i\neq j}}^{c} \lambda_{i,N} n_i^{-1/2}\mathcal{A}_i \int xf_i(x)J'[H_N]dF_j(x)\Bigg]\Bigg\} \\ &\quad + \text{lower order terms} \\ &= a_{N,j} + b_{N,j} + c_{N,j} + d_{N,j} + \text{lower order terms}, \quad \text{(say)}, \end{aligned} \tag{13}$$

where $B_j(x) = \int_{x_0}^{x} J'[H_N(y)]dF_j(y)$ with x_0 determined somewhat arbitrarily, say by $H_N(x_0) = 1/2$.

Let us first compute the variance of (13). From the result by Puri and Sen (1993), we obtain

$$\sigma_{1N,jj} = Var(a_{N,j} + b_{N,j}). \tag{14}$$

Similarly, we can compute the same for $c_{N,j}$ and $d_{N,j}$ by first noting the result of Tjøstheim (1986) that

$$Var(n_i^{1/2}(\hat{\theta}_{i,n_i} - \theta_i)) = \mathcal{U}_i^{-1}\mathcal{R}_i\mathcal{U}_i^{-1}, \quad i = 1, \dots, c.$$

Thus, from (7) and (13), we obtain

$$Var(c_{N,j}) = \sigma_{2N,jj} \quad \text{and} \quad Var(d_{N,j}) = \sigma_{3N,jj}. \tag{15}$$

As a part of the main diagonal terms, we have only to evaluate

$$K_{1N,j} = 2E[a_{N,j}d_{N,j}] \quad \text{and} \quad K_{2N,j} = 2E[b_{N,j}c_{N,j}],$$

since $X_{i,1}, \dots, X_{i,n_i}$, $i = 1, \dots, c$, are mutually independent samples. From (13), we obtain

$$K_{1N,j} = 2\sum_{\substack{i=1 \\ i\neq j}}^{c} \lambda_{i,N} \iint E[(n_i^{1/2}(F_{n_i}^{(i)} - F_i)(x))\mathcal{A}_i]\psi_{i,N}(x,y)dF_j(x)dF_j(y),$$

for which, it is necessary to find $E[\cdot]$. Using the result by Horváth *et. al* (2001), it follows from (7) and (10) that

$$E[n_i^{1/2}(F_{n_i}^{(i)}(x) - F_i(x))\mathcal{A}_i] = \sum_{l=0}^{p_i} \tau_{i,l} h_i^l(x).$$

Thus,

$$K_{1N,j} = 2\sum_{\substack{i=1 \\ i\neq j}}^{c}\sum_{l=0}^{p_i} \lambda_{i,N}\tau_{i,l} \iint h_i^l(x)\psi_{i,N}(x,y)dF_j(x)dF_j(y), \tag{16}$$

and analogously

$$K_{2N,j} = \frac{2}{\lambda_{N,j}}\sum_{\substack{i=1 \\ i\neq j}}^{c}\sum_{l=0}^{p_j} \lambda_{i,N}^2\tau_{j,l} \iint h_j^l(x)\psi_{j,N}(x,y)dF_i(x)dF_i(y). \tag{17}$$

Adding (16) and (17) yields $\gamma_{N,jj}$.

To compute the covariance terms, let us first rewrite (13) as

$$\begin{aligned}
&N^{1/2}(B_{1N,j} + B_{2N,j}) \\
&= N^{1/2}\sum_{i=1}^{c}\lambda_{i,N}\Big\{-\int [F_{n_j}^{(j)}(x) - F_j(x)]J'[H_N]dF_i(x) \\
&\quad + \int [F_{n_i}^{(i)}(x) - F_i(x)]J'[H_N]dF_j(x) - n_j^{-1/2}\mathcal{A}_j\int xf_j(x)J'[H_N]dF_i(x) \\
&\quad + n_i^{-1/2}\mathcal{A}_i\int xf_i(x)J'[H_N]dF_j(x)\Big\} + \text{ lower order terms} \\
&= a_{1N,j} + b_{1N,j} + c_{1N,j} + d_{1N,j} + \text{ lower order terms}, \quad \text{(say)}.
\end{aligned}$$

Using again, the result by Puri and Sen (1993), it follows that

$$\begin{aligned} & Cov(a_{1N,j}+b_{1N,j},a_{1N,j'}+b_{2N,j'}) \\ (18) \qquad & = E(a_{1N,j}b_{1N,j'})+E(b_{1N,j}a_{1N,j'})+E(b_{1N,j}b_{1N,j'})=\sigma_{1Njj'}, \quad j\neq j'. \end{aligned}$$

Moreover, by independence of $X_{i,1},\ldots,X_{i,n_i}$, $i=1,\ldots,c$, we have only to evaluate, for $j\neq j'$, the followings:

$$L_{1N,jj'}=E(a_{1N,j}d_{1N,j'})+E(d_{1N,j}a_{1N,j'}) \quad \text{and} \quad L_{2N,jj'}=E(b_{1N,j}c_{1N,j'})+E(c_{1N,j}b_{1N,j'}).$$

By the above arguments, we have

$$\begin{aligned} L_{1N,jj'} &= -\sum_{i=1}^{c}\lambda_{i,N}\Big\{\sum_{l=0}^{p_j}\tau_{j,l}\iint h_j^l(x)\psi_{j,N}(x,y)dF_i(x)dF_{j'}(y) \\ (19) \qquad & \quad +\sum_{l=0}^{p_{j'}}\tau_{j',l}\iint h_{j'}^l(x)\psi_{j',N}(x,y)dF_i(x)dF_j(y)\Big\}, \quad \text{and} \\ L_{2N,jj'} &= -\sum_{i=1}^{c}\lambda_{i,N}\Big\{\sum_{l=0}^{p_j}\tau_{j,l}\iint h_j^l(y)\psi_{j,N}(y,x)dF_i(x)dF_{j'}(y) \\ (20) \qquad & \quad +\sum_{l=0}^{p_{j'}}\tau_{j',l}\iint h_{j'}^l(y)\psi_{j',N}(y,x)dF_i(x)dF_j(y)\Big\}. \end{aligned}$$

Adding (19) and (20) yields $\sigma_{2N,jj'}$ defined in (11).

Hence, using (11), (14), (15), (18), the term $\gamma_{N,j}$ and central limit theorems given by Horváth *et. al* (2001), and Tjøstheim (1986), we may conclude that

$$N^{1/2}\Sigma_N^{-1/2}(B_{1N,1}+B_{2N,1},\ldots,B_{1N,c}+B_{2N,c})^T \xrightarrow{d} \mathscr{N}(0,I) \quad \text{as } N\to\infty.$$

Since the statement (ii) can be shown in the same way as in Chandra and Taniguchi (2002), the proof is omitted. The details are given in Chandra (2002), which can be obtained from the author on request. □

Acknowledgement

The author wishes to thank Professors S. Shirahata and M. Taniguchi for their valuable comments and encouragement.

References

[1] Chandra, S. A. (2002). Multi-sample problem for ARCH residual empirical processes. Preprint Series S-64, Department of Mathematical Science, Osaka University.

[2] Chandra, S. A. and Taniguchi, M. (2002). Asymptotics of rank order statistics for ARCH residual empirical processes. Preprint Series S-62, Department of Mathematical Science, Osaka University.

[3] Chen, M. and An, H. Z. (1998). A note on the stationarity and the existence of moments of the GARCH model. *Statist. Sinica*, 8, 505-510.

[4] Chernoff, H. and Savage, I. R. (1958). Asymptotic normality and eficiency of certain nonparametric test statistics. *Ann. Math. Statist.*, 29, 972-994.

[5] Engle, R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of U.K. inflation. *Econometrica*, **50**, 987-1007.

[6] Engle, R. (1995). *ARCH Selected Readings.* New York: Oxford University Press.

[7] Giraitis, L., Kokoszka, P. and Leipus, R. (2000). Stationary ARCH models: dependence structure and central limit theorem. *Econometric Theory*, **16**, 3-22.

[8] Gouriéroux, C. (1997). *ARCH Models and Financial Applications.* New York: Springer.

[9] Horváth, L., Kokoszka, P. and Teyssiére, G. (2001). Empirical process of the squared residuals of an ARCH sequence. *Ann. Statist.*, **29**, 445-469.

[10] Kondo, M. (1994). K-sample problem in time series analysis. *Math. Japonica*, **39**, 537-544.

[11] Lee, S. and Taniguchi, M. (2000). Asymptotic theory for ARCH-SM models: LAN and residual empirical processes. Preprint Series S-51, Department of Mathematical Science, Osaka University.

[12] Puri, M. L. (1964). Asymptotic efficiency of a class of c-sample tests. *Ann. Math. Statist.*, **35**, 102-121.

[13] Puri, M. L. and Sen, P. K. (1993). *Nonparametrics Methods in Multivariate Analysis.* Wiley, New York.

[14] Taniguchi, M. and Kakizawa, Y. (2000). *Asymptotic Theory of Statistical Inference for Time Series.* New York: Springer-Verlag.

[15] Tjøstheim, D. (1986). Estimation in nonlinear time series models. *Stoch. Proc. Appl.*, **21**, 251-273.

[16] Tsay, R. S. (2002). *Analysis of Financial Times Series.* New York: Wiley.

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