INFINITESIMAL PROJECTIVE TRANSFORMATIONS ON TANGENT BUNDLES WITH LIFT CONNECTIONS

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Abstract. Let M be a Riemannian manifold and TM its tangent bundle with (1) horizontal lift connection, (2) complete lift connection, (3) diagonal lift connection or (4) lift connection II+III. We determine the infinitesimal projective transformations on TM. Furthermore, if M is complete and TM admits a non-affine infinitesimal projective transformation, then M and TM are locally flat.

1. Introduction

Let M be a differentiable manifold, and ∇ an affine connection on M. A transformation f of M is called a projective transformation if it preserves the geodesics, where each geodesic should be confounded with a subset of M by neglecting its affine parameter. Furthermore, f is called an affine transformation if it preserves the connection. We then remark that an affine transformation may be characterized as a projective transformation which preserves the affine parameter together with the geodesics. Let V be a vector field on M, and let us consider a 1ocal one-parameter group $\{f_t\}$ of local transformations of M generated by V. Then V is called an infinitesimal projective transformation if each f_t is a local projective transformation. Similarly V is called an infinitesimal affine transformation if each f_t is a local affine transformation. Clearly an infinitesimal affine transformation is an infinitesimal projective transformation. The converse is not true in general. Indeed consider the real projective space $P^n(\mathbf{R})$ with the standard Riemannian metric, which is the standard projectively flat Riemannian manifold and a space of positive constant curvature. It is known that $P^n(\mathbf{R})$ admits a non-affine infinitesimal projective transformation. As a converse problem, the following conjecture is known.

Conjecture. *Let M be a complete Riemannian manifold admitting a non-affine infinitesimal projective transformation. Then M is a space of non-negative constant curvature.*

Let $(M,\ g)$ be a Riemannian manifold and TM its tangent bundle. Then we can consider some connections on TM, for example, the horizontal lift connection, the complete lift connection, the diagonal lift connection, etc. Let $\widetilde{V}$ be a vector field on TM, and $\{\widetilde{f}_t\}$ a 1ocal one-parameter group of local transformations of TM generated by $\widetilde{V}$. Then $\widetilde{V}$ is called an infinitesimal fibre-preserving transformation if each $\widetilde{f}_t$ is a local fibre-preserving transformation of TM (i.e., $\widetilde{f}_t$ sends each fibre of TM into a fibre).

Recently one of the authors proved the following

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Theorem A ([Y3]). *Let $(M,\ g)$ be a complete Riemannian manifold and TM its tangent bundle with the diagonal lift connection. If TM admits a non-affine infinitesimal fibre-preserving projective transformation, then M is locally flat.*

Theorem B ([Y2]). *Let $(M,\ g)$ be a complete Riemannian manifold and TM its tangent bundle with the lift connection* II+III. *If TM admits a non-affine infinitesimal fibre-preserving projective transformation, then M is locally flat.*

We hope to get rid of the assumption "fibre-preserving" in our results. In this paper, we determine the infinitesimal projective transformations on TM (Theorem $1 \sim 4$), and prove the following

Theorem 5. *Let $(M,\ g)$ be a complete Riemannian manifold and TM its tangent bundle. Assume that TM admits a non-affine infinitesimal projective transformation with respect to one of the following lift connections:*

(1) *the horizontal lift connection.*
(2) *the complete lift connection.*
(3) *the diagonal lift connection.*
(4) *the lift connection* II+III.

Then M and TM are locally flat.

Remark. *Recently we proved the case* (1) *in* [H-Y]. *One of the authors stated the case* (2) *in* [Y1].

In the present paper everything will be always discussed in the C^∞-category, and Riemannian manifolds will be assumed to be connected and dimension $n > 1$.

2. Preliminaries

Let M be a differentible manifold with an affine connection ∇, and ${\mathit\Gamma_{ji}}^h$ the coefficients of ∇, i.e., ${\mathit\Gamma_{ji}}^a\partial_a := \nabla_{\partial_j}\partial_i$, where $\partial_h = \frac{\partial}{\partial x^h}$ and (x^h) is the local coordinates of M. We define a local frame $\{E_i,\ E_{\bar{i}}\}$ of TM as follows:

$$E_i := \partial_i - y^b {\mathit\Gamma_{ib}}^a \partial_{\bar{a}} \quad \text{and} \quad E_{\bar{i}} := \partial_{\bar{i}},$$

where $(x^h,\ y^h)$ is the induced coordinates of TM and $\partial_{\bar{i}} := \frac{\partial}{\partial y^i}$, and call this frame $\{E_i,\ E_{\bar{i}}\}$ the adapted frame of TM. Then $\{dx^h,\ \delta y^h\}$ is the dual frame of $\{E_i,\ E_{\bar{i}}\}$, where $\delta y^h := dy^h + y^b {\mathit\Gamma_{ab}}^h dx^a$.

Then, by straightforward calculation, we have the following

Lemma 1. *The Lie brackets of the adapted frame of TM satisfy the following identities:*

(1) $[E_j,\ E_i] = y^b {K_{ijb}}^a E_{\bar{a}}$,
(2) $[E_j,\ E_{\bar{i}}] = {\mathit\Gamma_{ji}}^a E_{\bar{a}}$,
(3) $[E_{\bar{j}},\ E_{\bar{i}}] = 0$,

where $K = ({K_{kji}}^h)$ denotes the curvature tensor of $(M,\ \nabla)$ defined by ${K_{kji}}^h := \partial_k {\mathit\Gamma_{ji}}^h - \partial_j {\mathit\Gamma_{ki}}^h + {\mathit\Gamma_{ji}}^a {\mathit\Gamma_{ka}}^h - {\mathit\Gamma_{ki}}^a {\mathit\Gamma_{ja}}^h$.

Lemma 2. *Let $\widetilde{V}$ be a vector field on TM. Then*

(1) $[\widetilde{V},\ E_i] = -(E_i\widetilde{V}^a)E_a + (\widetilde{V}^c y^b {K_{icb}}^a - \widetilde{V}^b {\mathit\Gamma_{bi}}^a - E_i\widetilde{V}^{\bar{a}})E_{\bar{a}}$,
(2) $[\widetilde{V},\ E_{\bar{i}}] = -(\partial_{\bar{i}}\widetilde{V}^a)E_a + (\widetilde{V}^b {\mathit\Gamma_{bi}}^a - \partial_{\bar{i}}\widetilde{V}^{\bar{a}})E_{\bar{a}}$,

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) = \widetilde{V}^a E_a + \widetilde{V}^{\bar{a}} E_{\bar{a}} := \widetilde{V}$.

We denote by $\mathfrak{T}^r_s(M)$ the set of all tensor fields of class C^∞ and of type $(r,\ s)$ on M. Similarly, we denote by $\mathfrak{T}^r_s(TM)$ the corresponding set on TM.

Let V be a vector field on M. It is well-known that V is an infinitesimal isometry if and only if $L_V g = 0$, where L_V is the Lie derivation with respect to V. V is an infinitesimal conformal transformation if and only if there exists a function f on M satisfying $L_V g = fg$. Especially, if f is constant, then V is called an infinitesimal homothetic transformation. A vector field V on M is an infinitesimal projective transformation if and only if there exists a 1-form Ω such that

$$(L_V\nabla)(X,\ Y) = \Omega(X)Y + \Omega(Y)X$$

for any $X,\ Y \in \mathfrak{T}^1_0(M)$. In this case Ω is called the associated 1-form of V.

3. Connections on TM (cf. [Y-I])

Horizontal lift connection

Let M be a differentiable manifold with affine connection ∇. Let $X = X^a\partial_a$ be a vector field on M. Then the vertical lift X^V and the horizontal lift X^H of X are defined as follows:

$$X^H := X^a E_a \quad \text{and} \quad X^V := X^a E_{\bar{a}}. \tag{3.1}$$

There exists a unique affine connection $\widetilde{\nabla}$ on TM which satisfies

$$\begin{aligned} &\widetilde{\nabla}_{X^H} Y^H = (\nabla_X Y)^H, \\ &\widetilde{\nabla}_{X^H} Y^V = (\nabla_X Y)^V, \\ &\widetilde{\nabla}_{X^V} Y^H = 0, \qquad \widetilde{\nabla}_{X^V} Y^V = 0 \end{aligned} \tag{3.2}$$

for any $X,\ Y \in \mathfrak{T}^1_0(M)$. This affine connection is called the *horizontal lift connection* of ∇ to TM. Then we have

$$\begin{aligned} &\widetilde{\nabla}_{E_j} E_i = \Gamma_{ji}{}^a E_a, \\ &\widetilde{\nabla}_{E_j} E_{\bar{i}} = \Gamma_{ji}{}^a E_{\bar{a}}, \\ &\widetilde{\nabla}_{E_{\bar{j}}} E_i = 0, \qquad \widetilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}} = 0. \end{aligned} \tag{3.3}$$

If M is a Riemannian manifold with metric g, then this lift connection is the metric connection of the complete lift metric $\widetilde{g} = 2g_{ba}dx^b\delta y^a$ or the lift metric I+II: $\widetilde{g} = g_{ba}dx^b dx^a + 2g_{ba}dx^b\delta y^a$.

Complete lift connection

Let $X = X^a\partial_a$ be a vector field on M. Then the complete lift X^C of X is defined as follows:

$$X^C := X^a E_a + y^b\nabla_b X^a E_{\bar{a}}. \tag{3.4}$$

There exists a unique affine connection $\widetilde{\nabla}$ on TM which satisfies

$$\widetilde{\nabla}_{X^C} Y^C = (\nabla_X Y)^C \tag{3.5}$$

for any $X,\ Y \in \mathfrak{T}^1_0(M)$. This affine connection $\tilde{\tilde{\nabla}}$ is called the *complete lift connection* of ∇ to TM. Then we have

$$\begin{aligned} &\tilde{\tilde{\nabla}}_{E_j} E_i = \Gamma_{ji}{}^a E_a + y^b K_{bji}{}^a E_{\bar{a}}, \\ &\tilde{\tilde{\nabla}}_{E_j} E_{\bar{i}} = \Gamma_{ji}{}^a E_{\bar{a}}, \\ &\tilde{\tilde{\nabla}}_{E_{\bar{j}}} E_i = 0, \qquad \tilde{\tilde{\nabla}}_{E_{\bar{j}}} E_{\bar{i}} = 0. \end{aligned} \tag{3.6}$$

If M is a Riemannian manifold with metric g, then this connection is the the Levi-Civita connection of the complete lift metric $\tilde{g} = 2g_{ba} dx^b \delta y^a$ or the lift metric I+II: $\tilde{g} = g_{ba} dx^b dx^a + 2g_{ba} dx^b \delta y^a$.

Diagonal lift connection

Let $\tilde{\tilde{\nabla}}$ be the *diagonal lift connection* on TM defined as follows:

$$\begin{aligned} &\tilde{\tilde{\nabla}}_{E_j} E_i = \Gamma_{ji}{}^a E_a - \frac{1}{2} y^b K_{jib}{}^a E_{\bar{a}}, \\ &\tilde{\tilde{\nabla}}_{E_j} E_{\bar{i}} = \frac{1}{2} y^b K_{bij}{}^a E_a + \Gamma_{ji}{}^a E_{\bar{a}}, \\ &\tilde{\tilde{\nabla}}_{E_{\bar{j}}} E_i = \frac{1}{2} y^b K_{bji}{}^a E_a, \qquad \tilde{\tilde{\nabla}}_{E_{\bar{j}}} E_{\bar{i}} = 0. \end{aligned} \tag{3.7}$$

If M is a Riemannian manifold with metric g, then this connection is the the Levi-Civita connection of the diagonal lift metric $\tilde{g} = g_{ba} dx^b dx^a + g_{ba} \delta y^b \delta y^a$ which was originally defined by S. Sasaki [S].

Lift connection II+III

Let $\tilde{\tilde{\nabla}}$ be a lift connection on TM defined as follows:

$$\begin{aligned} &\tilde{\tilde{\nabla}}_{E_j} E_i = \{\Gamma_{ji}{}^a - \frac{1}{2} y^b (K_{bji}{}^a + K_{bij}{}^a)\} E_a + y^b K_{bji}{}^a E_{\bar{a}}, \\ &\tilde{\tilde{\nabla}}_{E_j} E_{\bar{i}} = -\frac{1}{2} y^b K_{bij}{}^a E_a + (\Gamma_{ji}{}^a + \frac{1}{2} y^b K_{bij}{}^a) E_{\bar{a}}, \\ &\tilde{\tilde{\nabla}}_{E_{\bar{j}}} E_i = -\frac{1}{2} y^b K_{bji}{}^a E_a + \frac{1}{2} y^b K_{bji}{}^a E_{\bar{a}}, \\ &\tilde{\tilde{\nabla}}_{E_{\bar{j}}} E_{\bar{i}} = 0. \end{aligned} \tag{3.8}$$

This affine connection $\tilde{\tilde{\nabla}}$ is called the *lift connection* II+III. If M is a Riemannian manifold with metric g, then this connection is the Levi-Civita connection of lift metric II+III : $\tilde{g} = 2g_{ba} dx^b \delta y^a + g_{ba} \delta y^b \delta y^a$.

4. Infinitesimal projective transformations on TM

Let M be a differentiable manifold and TM its tangent bundle with affine connection $\tilde{\nabla}$. A vector field $\tilde{V}$ on TM is an infinitesimal projective transformation if and only if there exists a 1-form $\tilde{\Omega}$ such that

$$(L_{\tilde{V}} \tilde{\nabla})(\tilde{X},\ \tilde{Y}) = \tilde{\Omega}(\tilde{X}) \tilde{Y} + \tilde{\Omega}(\tilde{Y}) \tilde{X}$$

for any $\tilde{X},\ \tilde{Y} \in \mathfrak{T}^1_0(TM)$. Then we have the following theorems.

Theorem 1 [H-Y]. *Let $(M,\ g)$ be a Riemannian manifold and TM its tangent bundle with the horizontal lift connection. $\widetilde{V}$ is an infinitesimal projective transformation with the associated 1-form $\widetilde{\Omega}$ on TM if and only if there exist $\varphi,\ \psi\in\mathfrak{T}^0_0(M)$, $B=(B^h)$, $D=(D^h)\in\mathfrak{T}^1_0(M)$, $A=(A_i{}^h)$, $C=(C_i{}^h)\in\mathfrak{T}^1_1(M)$ satisfying*

(1) $(\widetilde{V}^h,\ \widetilde{V}^{\bar h})=(B^h+y^aA_a{}^h,\ D^h+y^aC_a{}^h+y^ay^h\Phi_a)$,

(2) $(\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar i})=(\partial_i\psi,\ \partial_i\varphi)=(\Psi_i,\ \Phi_i)$,

(3) $\nabla_j\Phi_i=0, \qquad \nabla_j\Psi_i=0$,

(4) $\nabla_jA_i{}^h=\Phi_i\delta^h_j$,

(5) $\nabla_jC_i{}^h=\Psi_j\delta^h_i-K_{aji}{}^hB^a$,

(6) $L_B\Gamma_{ji}{}^h=\nabla_j\nabla_iB^h+K_{aji}{}^hB^a=\Psi_j\delta^h_i+\Psi_i\delta^h_j$,

(7) $\nabla_j\nabla_iD^h=0$,

(8) $K_{kja}{}^hA_i{}^a=0$,

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar h}):=\widetilde{V}^aE_a+\widetilde{V}^{\bar a}E_{\bar a}=\widetilde{V}$, $(\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar i}):=\widetilde{\Omega}_adx^a+\widetilde{\Omega}_{\bar a}\delta y^a=\widetilde{\Omega}$.

Theorem 2 (cf. [Y1]). *Let $(M,\ g)$ be a Riemannian manifold and TM its tangent bundle with the complete lift connection. Then $\widetilde{V}$ is an infinitesimal projective transformation with the associated 1-form $\widetilde{\Omega}$ on TM if and only if there exist $\varphi,\ \psi\in\mathfrak{T}^0_0(M)$, $B=(B^h)$, $D=(D^h)\in\mathfrak{T}^1_0(M)$, $A=(A_i{}^h)$, $C=(C_i{}^h)\in\mathfrak{T}^1_1(M)$ satisfying*

(1) $(\widetilde{V}^h,\ \widetilde{V}^{\bar h})=(B^h+y^aA_a{}^h,\ D^h+y^aC_a{}^h+y^ay^h\Phi_a)$,

(2) $(\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar i})=(\partial_i\psi,\ \partial_i\varphi)=(\Psi_i,\ \Phi_i)$,

(3) $\nabla_j\Phi_i=0, \qquad \nabla_j\Psi_i=0$,

(4) $\nabla_jA_i{}^h=\Phi_i\delta^h_j$,

(5) $\nabla_jC_i{}^h=\Psi_j\delta^h_i-K_{aji}{}^hB^a$,

(6) $L_B\Gamma_{ji}{}^h=\nabla_j\nabla_iB^h+K_{aji}{}^hB^a=\Psi_j\delta^h_i+\Psi_i\delta^h_j$,

(7) $L_D\Gamma_{ji}{}^h=\nabla_j\nabla_iD^h+K_{aji}{}^hD^a=0$,

(8) $K_{kja}{}^hA_i{}^a=0$,

(9) $B^a\nabla_aK_{kji}{}^h=-K_{aji}{}^h\nabla_kB^a-K_{kja}{}^h\nabla_iB^a+K_{kji}{}^aC_a{}^h-K_{kai}{}^hC_j{}^a$,

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar h}):=\widetilde{V}^aE_a+\widetilde{V}^{\bar a}E_{\bar a}=\widetilde{V}$ *and* $(\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar i}):=\widetilde{\Omega}_adx^a+\widetilde{\Omega}_{\bar a}\delta y^a=\widetilde{\Omega}$.

Theorem 3. *Let $(M,\ g)$ be a Riemannian manifold and TM its tangent bundle with the diagonal lift connection. Then $\widetilde{V}$ is an infinitesimal projective transformation with the associated 1-form $\widetilde{\Omega}$ on TM if and only if there exist $\varphi,\ \psi\in\mathfrak{T}^0_0(M)$, $B=(B^h)$, $D=(D^h)\in\mathfrak{T}^1_0(M)$, $A=(A_i{}^h)$, $C=(C_i{}^h)\in\mathfrak{T}^1_1(M)$ satisfying*

(1) $(\widetilde{V}^h,\ \widetilde{V}^{\bar h})=(B^h+y^aA_a{}^h,\ D^h+y^aC_a{}^h+y^ay^h\Phi_a)$,

(2) $(\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar i})=(\partial_i\psi,\ \partial_i\varphi)=(\Psi_i,\ \Phi_i)$,

(3) $\nabla_j\Phi_i=0, \qquad \nabla_j\Psi_i=0$,

(4) $\nabla_jA_i{}^h=\Phi_i\delta^h_j-\frac{1}{2}K_{aij}{}^hD^a$,

(5) $\nabla_j C_i{}^h = \Psi_j \delta_i^h - K_{aji}{}^h B^a,$

(6) $L_B \Gamma_{ji}{}^h = \nabla_j \nabla_i B^h + K_{aji}{}^h B^a = \Psi_j \delta_i^h + \Psi_i \delta_j^h,$

(7) $\nabla_j \nabla_i D^h = \frac{1}{2} K_{jia}{}^h D^a,$

(8) $K_{kja}{}^h A_i{}^a = 0,$

(9) $K_{kji}{}^a \nabla_a D^h = 0,$

(10) $B^a \nabla_a K_{kji}{}^h = K_{kji}{}^a \nabla_a B^h - K_{kja}{}^h \nabla_i B^a - K_{kai}{}^h C_j{}^a - K_{aji}{}^h C_k{}^a,$

(11) $D^a \nabla_k K_{aji}{}^h = K_{kij}{}^a A_a{}^h - K_{ajk}{}^h \nabla_i D^a - 2K_{aji}{}^h \nabla_k D^a,$

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) := \widetilde{V}^a E_a + \widetilde{V}^{\bar{a}} E_{\bar{a}} = \widetilde{V}$ *and* $(\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar{i}}) := \widetilde{\Omega}_a dx^a + \widetilde{\Omega}_{\bar{a}} \delta y^a = \widetilde{\Omega}.$

Theorem 4. *Let $(M,\ g)$ be a Riemannian manifold and TM its tangent bundle with the lift connection* II+III. *Then $\widetilde{V}$ is an infinitesimal projective transformation with the associated 1-form $\widetilde{\Omega}$ on TM if and only if there exist $\varphi,\ \psi \in \mathfrak{T}_0^0(M),\ B = (B^h),\ D = (D^h) \in \mathfrak{T}_0^1(M),\ A = (A_i{}^h),\ C = (C_i{}^h) \in \mathfrak{T}_1^1(M)$ satisfying*

(1) $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) = (B^h + y^a A_a{}^h,\ D^h + y^a C_a{}^h + y^a y^h \Phi_a),$

(2) $(\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar{i}}) = (\partial_i \psi,\ \partial_i \varphi) = (\Psi_i,\ \Phi_i),$

(3) $\nabla_j \Phi_i = 0, \qquad \nabla_j \Psi_i = 0,$

(4) $\nabla_j A_i{}^h = \Phi_i \delta_j^h + \frac{1}{2} K_{aij}{}^h D^a,$

(5) $\nabla_j C_i{}^h = \Psi_j \delta_i^h - K_{aji}{}^h B^a - \frac{1}{2} K_{aij}{}^h D^a,$

(6) $L_B \Gamma_{ji}{}^h = \nabla_j \nabla_i B^h + K_{aji}{}^h B^a = \Psi_j \delta_i^h + \Psi_i \delta_j^h + \frac{1}{2}(K_{aji}{}^h + K_{aij}{}^h) D^a,$

(7) $L_D \Gamma_{ji}{}^h = \nabla_j \nabla_i D^h + K_{aji}{}^h D^a = 0,$

(8) $K_{aji}{}^h A_k{}^a = 0,$

(9) $K_{kji}{}^a (A_a{}^h - \nabla_a B^h + C_a{}^h - \nabla_a D^h) = 0,$

(10) $B^a \nabla_a K_{kji}{}^h = K_{kji}{}^a (\nabla_a B^h - A_a{}^h) - K_{kja}{}^h \nabla_i B^a - K_{kai}{}^h C_j{}^a - K_{aji}{}^h C_k{}^a,$

(11) $D^a \nabla_k K_{aji}{}^h = -2K_{aji}{}^h (\nabla_k B^a - C_k{}^a + \nabla_k D^a) + K_{kij}{}^a \nabla_a D^h - K_{ajk}{}^h \nabla_i D^a,$

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) := \widetilde{V}^a E_a + \widetilde{V}^{\bar{a}} E_{\bar{a}} = \widetilde{V}$ *and* $(\widetilde{\Omega}_i,\ \widetilde{\Omega}_{\bar{i}}) := \widetilde{\Omega}_a dx^a + \widetilde{\Omega}_{\bar{a}} \delta y^a = \widetilde{\Omega}.$

Theorem 3 and 4 are proved with similar technics, so we prove only Theorem 4.

Proof of Theorem 4. Here we prove only the necessary condition because it is easy to prove the sufficient condition.

Let $\widetilde{V}$ be an infinitesimal projective transformation with the associated 1-form $\widetilde{\Omega}$ on TM.

Step 1: By virtue of Lemma 2 and (3.8), we have

$$\begin{aligned}(L_{\widetilde{V}} \widetilde{\nabla})(E_{\bar{j}},\ E_{\bar{i}}) &= [\widetilde{V},\ \widetilde{\nabla}_{E_{\bar{j}}} E_{\bar{i}}] - \widetilde{\nabla}_{[\widetilde{V},\ E_{\bar{j}}]} E_{\bar{i}} - \widetilde{\nabla}_{E_{\bar{j}}} [\widetilde{V},\ E_{\bar{i}}] \\ &= \{\partial_{\bar{j}} \partial_{\bar{i}} \widetilde{V}^a - \frac{1}{2} y^c (K_{cib}{}^a \partial_{\bar{j}} \widetilde{V}^b + K_{cjb}{}^a \partial_{\bar{i}} \widetilde{V}^b)\} E_a + \{\cdots\} E_{\bar{a}}.\end{aligned}$$

From $(L_{\widetilde{V}}\widetilde{\nabla})(E_{\bar{j}},\ E_{\bar{i}}) = \widetilde{\Omega}_{\bar{j}}E_{\bar{i}} + \widetilde{\Omega}_{\bar{i}}E_{\bar{j}}$, we obtain

$$\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h - \frac{1}{2}y^a(K_{aib}{}^h\partial_{\bar{j}}\widetilde{V}^b + K_{ajb}{}^h\partial_{\bar{i}}\widetilde{V}^b) = 0. \tag{4.1}$$

(4.1) is rewritten as follows:

$$2\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h = \partial_{\bar{j}}(y^bK_{bia}{}^h\widetilde{V}^a) + \partial_{\bar{i}}(y^bK_{bja}{}^h\widetilde{V}^a), \tag{4.2}$$

from which we have

$$\begin{aligned} 2\partial_{\bar{k}}\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h &= \partial_{\bar{k}}\partial_{\bar{j}}(y^bK_{bia}{}^h\widetilde{V}^a) + \partial_{\bar{k}}\partial_{\bar{i}}(y^bK_{bja}{}^h\widetilde{V}^a) \\ &= \partial_{\bar{j}}\partial_{\bar{i}}(y^bK_{bka}{}^h\widetilde{V}^a) + \partial_{\bar{j}}\partial_{\bar{k}}(y^bK_{bia}{}^h\widetilde{V}^a) \\ &= \partial_{\bar{i}}\partial_{\bar{k}}(y^bK_{bja}{}^h\widetilde{V}^a) + \partial_{\bar{i}}\partial_{\bar{j}}(y^bK_{bka}{}^h\widetilde{V}^a). \end{aligned} \tag{4.3}$$

Therefore we obtain $\partial_{\bar{k}}\partial_{\bar{j}}(\partial_{\bar{i}}\widetilde{V}^h - y^bK_{bia}{}^h\widetilde{V}^a) = 0$, hence we can put

$$\partial_{\bar{j}}(\partial_{\bar{i}}\widetilde{V}^h - y^bK_{bia}{}^h\widetilde{V}^a) =: P_{ji}{}^h \tag{4.4}$$

and

$$\partial_{\bar{i}}\widetilde{V}^h - y^bK_{bia}{}^h\widetilde{V}^a =: A_i{}^h + y^aP_{ai}{}^h, \tag{4.5}$$

where $A_i{}^h$ and $P_{ji}{}^h$ are certain functions which depend only on the variables x^h. The coordinate transformation rule implies that $A = (A_i{}^h) \in \mathfrak{T}^1_1(M)$ and $P = (P_{ji}{}^h) \in \mathfrak{T}^1_2(M)$.

From (4.1), we have

$$P_{ji}{}^h + P_{ij}{}^h = 2\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h - y^a(K_{aib}{}^h\partial_{\bar{j}}\widetilde{V}^b + K_{ajb}{}^h\partial_{\bar{i}}\widetilde{V}^b) = 0,$$

from which

$$-\partial_{\bar{j}}(y^bK_{bia}{}^h\widetilde{V}^a) + \partial_{\bar{i}}(y^bK_{bja}{}^h\widetilde{V}^a) = P_{ji}{}^h - P_{ij}{}^h = 2P_{ji}{}^h.$$

Thus we have

$$\begin{aligned} 2y^aP_{ai}{}^h &= -y^a\partial_{\bar{a}}(y^bK_{bic}{}^h\widetilde{V}^c) + y^a\partial_{\bar{i}}(y^bK_{bac}{}^h\widetilde{V}^c) \\ &= -2y^aK_{aib}{}^h\widetilde{V}^b - y^by^aK_{aic}{}^h\partial_{\bar{b}}\widetilde{V}^c. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we have

$$\partial_{\bar{i}}\widetilde{V}^h + \frac{1}{2}y^by^aK_{aic}{}^h\partial_{\bar{b}}\widetilde{V}^c = A_i{}^h, \tag{4.7}$$

from which

$$y^a\partial_{\bar{a}}\widetilde{V}^h = y^aA_a{}^h. \tag{4.8}$$

Therefore, substituting (4.8) into (4.7), we have

$$\partial_{\bar{i}}\widetilde{V}^h = A_i{}^h - \frac{1}{2}y^by^aK_{aic}{}^hA_b{}^c, \tag{4.9}$$

from which

$$\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h = -\frac{1}{2}y^a(K_{aib}{}^h A_j{}^b + K_{jib}{}^h A_a{}^b). \tag{4.10}$$

On the other hand, substituting (4.9) into (4.1), we have

$$\begin{aligned}\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^h = &\frac{1}{2}y^a(K_{aib}{}^h A_j{}^b + K_{ajb}{}^h A_i{}^b)\\ &-\frac{1}{4}y^c y^b y^a(K_{cid}{}^h K_{bje}{}^d A_a{}^e + K_{cjd}{}^h K_{bie}{}^d A_a{}^e).\end{aligned} \tag{4.11}$$

Comparing (4.10) with (4.11), we get

$$2K_{kia}{}^h A_j{}^a + K_{jia}{}^h A_k{}^a + K_{kja}{}^h A_i{}^a = 0,$$

from which, changing the roles of j and i, and adding together, we obtain

$$K_{kja}{}^h A_i{}^a + K_{kia}{}^h A_j{}^a = 0.$$

Furthermore we obtain

$$K_{kja}{}^h A_i{}^a = 0. \tag{4.12}$$

In fact, by virtue of the first Bianchi identity,

$$\begin{aligned}0 &= (K_{aijk} + K_{akij} + K_{ajki})A^{ha}\\ &= -K_{aij}{}^h A_k{}^a - K_{aki}{}^h A_j{}^a - K_{ajk}{}^h A_i{}^a\\ &\left(\begin{aligned}&= K_{ija}{}^h A_k{}^a + K_{jai}{}^h A_k{}^a + K_{akj}{}^h A_i{}^a - K_{ajk}{}^h A_i{}^a\\ &= -K_{kja}{}^h A_i{}^a + K_{akj}{}^h A_i{}^a\end{aligned}\right)\\ &= -K_{ija}{}^h A_k{}^a - K_{kia}{}^h A_j{}^a - K_{jka}{}^h A_i{}^a\\ &= 3K_{kja}{}^h A_i{}^a,\end{aligned}$$

where $A^{ih} := g^{ia}A_a{}^h$ and $K_{kjih} := K_{kji}{}^a g_{ah}$.

From (4.9) and (4.12), we have

$$\partial_{\bar{i}}\widetilde{V}^h = A_i{}^h.$$

Hence we can put

$$\widetilde{V}^h = B^h + y^a A_a{}^h, \tag{4.13}$$

where B^h are certain functions which depend only on x^h. We can see that $B = (B^h) \in \mathfrak{T}_0^1(M)$. Here, substituting (4.13) into (4.4) and using (4.12),

$$P_{ji}{}^h = -K_{jia}{}^h B^a. \tag{4.14}$$

Substituting (4.12) and (4.13) into $(L_{\widetilde{V}}\widetilde{\nabla})(E_{\bar{j}},\ E_{\bar{i}}) = \widetilde{\Omega}_{\bar{j}}E_{\bar{i}} + \widetilde{\Omega}_{\bar{i}}E_{\bar{j}}$,

$$\partial_{\bar{j}}\partial_{\bar{i}}\widetilde{V}^{\bar{h}} = \widetilde{\Omega}_{\bar{j}}\delta_i^h + \widetilde{\Omega}_{\bar{i}}\delta_j^h. \tag{4.15}$$

Step 2: From $(L_{\widetilde{V}}\widetilde{\nabla})(E_{\bar{j}},\ E_i) = \widetilde{\Omega}_{\bar{j}}E_i + \widetilde{\Omega}_i E_{\bar{j}}$, using (4.12), (4.13) and (4.14), we obtain

$$\begin{aligned}\widetilde{\Omega}_{\bar{j}}\delta_i^h = {} & \nabla_i A_j{}^h - \frac{1}{2}\widetilde{V}^{\bar{a}} K_{aji}{}^h - \frac{1}{2}y^a K_{abi}{}^h \partial_{\bar{j}}\widetilde{V}^{\bar{b}} \\ & - \frac{1}{2}y^a (K_{aji}{}^b A_b{}^h + B^b \nabla_b K_{aji}{}^h \\ & \qquad - K_{aji}{}^b \nabla_b B^h + K_{ajb}{}^h \nabla_i B^b) \\ & - \frac{1}{2}y^b y^a (A_b{}^c \nabla_c K_{aji}{}^h - K_{aji}{}^c \nabla_c A_b{}^h + K_{ajc}{}^h \nabla_i A_b{}^c).\end{aligned} \tag{4.16}$$

Contracting i and h in (4.15), we have $n\widetilde{\Omega}_{\bar{j}} = \nabla_a A_j{}^a$. Therefore we have

$$\widetilde{\Omega}_{\bar{i}} = \Phi_i, \tag{4.17}$$

where $\Phi_i := \dfrac{1}{n}\nabla_a A_i{}^a$.

From (4.15) and (4.17), we have

$$\partial_{\bar{i}}\widetilde{V}^{\bar{h}} = C_i{}^h + y^h \Phi_i + y^a \Phi_a \delta_i^h$$

and

$$\widetilde{V}^{\bar{h}} = D^h + y^a C_a{}^h + y^a y^h \Phi_a, \tag{4.18}$$

where D^h and $C_i{}^h$ are certain functions which depend only on x^h. We can see that $D = (D^h) \in \mathfrak{T}_0^1(M)$ and $C = (C_i{}^h) \in \mathfrak{T}_1^1(M)$.

From (4.16), (4.17) and (4.18), we get

$$\nabla_j A_i{}^h = \Phi_i \delta_j^h + \frac{1}{2}K_{aij}{}^h D^a, \tag{4.19}$$

$$\begin{aligned} B^a \nabla_a K_{kji}{}^h = {} & K_{kji}{}^a (\nabla_a B^h - A_a{}^h) \\ & - K_{kja}{}^h \nabla_i B^a - K_{kai}{}^h C_j{}^a - K_{aji}{}^h C_k{}^a \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} A_l{}^a \nabla_a K_{kji}{}^h + A_k{}^a \nabla_a K_{lji}{}^h = {} & K_{kji}{}^a \nabla_a A_l{}^h + K_{lji}{}^a \nabla_a A_k{}^h \\ & - K_{kja}{}^h \nabla_i A_l{}^a - K_{lja}{}^h \nabla_i A_k{}^a \\ & - 2\Phi_l K_{kji}{}^h - 2\Phi_k K_{lji}{}^h. \end{aligned}$$

The last one of these equations is an identity equation. From (4.19), we have

$$\Phi_i = \frac{1}{n}\nabla_a A_i{}^a = \nabla_i A_a{}^a + \frac{1}{2}R_{ai}D^a (= \nabla_a A^a{}_i - \frac{1}{2}R_{ai}D^a). \tag{4.21}$$

Step 3: From $(L_{\widetilde{V}}\widetilde{\nabla})(E_{\bar{j}},\ E_i) = \widetilde{\Omega}_{\bar{j}}E_i + \widetilde{\Omega}_i E_{\bar{j}}$, using (4.12), (4.13), (4.14) and (4.18), we obtain

$$\begin{aligned}\widetilde{\Omega}_i \delta_j^h = {} & \nabla_i C_j{}^h + K_{aij}{}^h B^a + \frac{1}{2}K_{aji}{}^h D^a \\ & + \frac{1}{2}y^a (B^b \nabla_b K_{aji}{}^h + K_{ajb}{}^h \nabla_i B^b \\ & \qquad + K_{bji}{}^h C_a{}^b + K_{abi}{}^h C_j{}^b - K_{aji}{}^b C_b{}^h \\ & \qquad + K_{aji}{}^b \nabla_b D^h + 2\delta_a^h \nabla_i \Phi_j + 2\delta_j^h \nabla_i \Phi_a) \\ & + \frac{1}{2}y^b y^a (A_b{}^c \nabla_c K_{aji}{}^h + K_{bjc}{}^h \nabla_i A_a{}^c - K_{aji}{}^d K_{dcb}{}^h B^c \\ & \qquad + K_{aji}{}^c \nabla_c C_b{}^h + K_{bji}{}^h \Phi_a - \delta_b^h K_{aji}{}^c \Phi_c) \\ & + \frac{1}{2}y^c y^b y^a \delta_c^h K_{bji}{}^d \nabla_d \Phi_a.\end{aligned} \tag{4.22}$$

Contracting j and h in (4.22), we get

$$\begin{aligned}\widetilde{\Omega}_i = \Psi_i + \frac{1}{2n}y^a\{&-B^b\nabla_b R_{ai} - R_{ba}\nabla_i B^b - R_{bi}C_a{}^b \\ &+ K_{abi}{}^c\nabla_c D^b + 2(n+1)\nabla_i\Phi_a\} \\ - \frac{1}{2n}y^b y^a(&A_b{}^c\nabla_c R_{ai} + R_{cb}\nabla_i A_a{}^c + K_{aci}{}^d K_{deb}{}^c B^e \\ &- K_{aci}{}^d\nabla_d C_b{}^c + R_{bi}\Phi_a)\end{aligned} \tag{4.23}$$

where $\Psi_i := \dfrac{1}{n}(\nabla_i C_a{}^a - \dfrac{1}{2}R_{ai}D^a)$ and $R = (R_{ji})$ is the Ricci tensor of M defined by $R_{ji} := K_{aji}{}^a$. Then, from (4.23), we obtain

$$\nabla_j C_i{}^h = \Psi_j\delta_i^h - K_{aji}{}^h B^a - \frac{1}{2}K_{aij}{}^h D^a, \tag{4.24}$$

and

$$\begin{aligned}&2n\delta_k^h\nabla_i\Phi_j - 2\delta_j^h\nabla_i\Phi_k \\ &= n(-B^a\nabla_a K_{kji}{}^h - K_{kja}{}^h\nabla_i B^a \\ &\qquad - K_{aji}{}^h C_k{}^a - K_{kai}{}^h C_j{}^a + K_{kji}{}^a C_a{}^h - K_{kji}{}^a\nabla_a D^h) \\ &\quad - \delta_j^h(B^a\nabla_a R_{ki} + R_{ak}\nabla_i B^a + R_{ai}C_k{}^a + K_{aki}{}^b\nabla_b D^a).\end{aligned} \tag{4.25}$$

The last part of right hand side in (4.23) vanishes by means of (4.12), (4.19), (4.24) and the second Bianchi identity. In fact

$$\begin{aligned}&-\frac{1}{2n}y^b y^a(A_b{}^c\nabla_c R_{ai} + R_{cb}\nabla_i A_a{}^c + K_{aci}{}^d K_{deb}{}^c B^e - K_{aci}{}^d\nabla_d C_b{}^c + R_{bi}\Phi_a) \\ &= -\frac{1}{2n}y^b y^a\{A_b{}^c(\nabla_i R_{ca} + \nabla_d K_{cia}{}^d) + R_{ca}\nabla_i A_b{}^c + \frac{1}{2}K_{aei}{}^d K_{cbd}{}^e D^c + R_{bi}\Phi_a\} \\ &= \frac{1}{4n}y^b y^a(K_{eai}{}^d + K_{eia}{}^d)K_{cbd}{}^e D^c \\ &= 0.\end{aligned}$$

Contracting k and h in (4.25), we have

$$-2(n-1)\nabla_i\Phi_j = B^a\nabla_a R_{ji} + R_{aj}\nabla_i B^a + R_{ai}C_j{}^a + K_{aji}{}^b\nabla_b D^a, \tag{4.26}$$

from which (4.23) and (4.25) are rewritten as follows:

$$\widetilde{\Omega}_i = \Psi_i + 2y^a\nabla_i\Phi_a \tag{4.27}$$

and

$$\begin{aligned}&2(\delta_j^h\nabla_i\Phi_k - \delta_k^h\nabla_i\Phi_j) \\ &= B^a\nabla_a K_{kji}{}^h + K_{kja}{}^h\nabla_i B^a + K_{aji}{}^h C_k{}^a \\ &\quad + K_{kai}{}^h C_j{}^a - K_{kji}{}^a C_a{}^h + K_{kji}{}^a\nabla_a D^h.\end{aligned} \tag{4.28}$$

Step 4: From $(L_{\widetilde{V}}\widetilde{\nabla})(E_j,\ E_i) = \widetilde{\Omega}_j E_i + \widetilde{\Omega}_i E_j$, using (4.12), (4.13) and (4.18), we obtain

$$\begin{aligned}
&\widetilde{\Omega}_j\delta_i^h + \widetilde{\Omega}_i\delta_j^h \\
&= L_B\Gamma_{ji}{}^h - \frac{1}{2}(K_{aji}{}^h + K_{aij}{}^h)D^a \\
&\quad + \frac{1}{2}y^a\{2\nabla_j\nabla_i A_a{}^h - 2K_{aji}{}^b A_b{}^h \\
&\qquad - B^b(\nabla_b K_{aji}{}^h + \nabla_b K_{aij}{}^h) + (K_{aji}{}^b + K_{aij}{}^b)\nabla_b B^h \\
&\qquad - (K_{aib}{}^h + K_{abi}{}^h)\nabla_j B^b - (K_{ajb}{}^h + K_{abj}{}^h)\nabla_i B^b \\
&\qquad - (K_{bji}{}^h + K_{bij}{}^h)C_a{}^b - K_{abi}{}^h\nabla_j D^b - K_{abj}{}^h\nabla_i D^b\} \\
&\quad - \frac{1}{2}y^b y^a\{(K_{bji}{}^h + K_{bij}{}^h)\Phi_a + A_b{}^c(\nabla_c K_{aji}{}^h + \nabla_c K_{aij}{}^h) \\
&\qquad - (K_{bji}{}^c + K_{bij}{}^c)\nabla_c A_a{}^h + (K_{bic}{}^h + K_{bci}{}^h)\nabla_j A_a{}^c \\
&\qquad + (K_{bjc}{}^h + K_{bcj}{}^h)\nabla_i A_a{}^c + K_{bci}{}^h K_{dja}{}^c B^d \\
&\qquad + K_{bcj}{}^h K_{dia}{}^c B^d + K_{bci}{}^h\nabla_j C_a{}^c + K_{bcj}{}^h\nabla_i C_a{}^c\}
\end{aligned} \tag{4.29}$$

and

$$\begin{aligned}
0 &= L_D\Gamma_{ji}{}^h \\
&\quad + y^a\{\nabla_j\nabla_i C_a{}^h + K_{bji}{}^h C_a{}^b - K_{aji}{}^b C_b{}^h \\
&\qquad + \nabla_j(K_{bia}{}^h B^b) + B^b\nabla_b K_{aji}{}^h + K_{abi}{}^h\nabla_j B^b \\
&\qquad + K_{ajb}{}^h\nabla_i B^b + \frac{1}{2}K_{abi}{}^h\nabla_j D^b \\
&\qquad + \frac{1}{2}K_{abj}{}^h\nabla_i D^b + \frac{1}{2}(K_{aji}{}^b + K_{aij}{}^b)\nabla_b D^h\} \\
&\quad + \frac{1}{2}y^b y^a\{2\delta_b^h(\nabla_j\nabla_i\Phi_a - K_{aji}{}^c\Phi_c) + 2A_b{}^c\nabla_c K_{aji}{}^h \\
&\qquad + 2K_{bci}{}^h\nabla_j A_a{}^c + 2K_{bjc}{}^h\nabla_i A_a{}^c \\
&\qquad - (K_{bji}{}^d + K_{bij}{}^d)K_{dca}{}^h B^c \\
&\qquad - K_{cjb}{}^d K_{dai}{}^h B^c - K_{cib}{}^d K_{daj}{}^h B^c \\
&\qquad + (K_{bji}{}^c + K_{bij}{}^c)\nabla_c C_a{}^h \\
&\qquad - K_{cbi}{}^h\nabla_j C_a{}^c - K_{cbj}{}^h\nabla_i C_a{}^c\} \\
&\quad + \frac{1}{2}y^c y^b y^a(K_{cji}{}^d + K_{cij}{}^d)\delta_b^h\nabla_d\Phi_a.
\end{aligned} \tag{4.30}$$

From (4.27) and (4.29), we obtain

$$L_B\Gamma_{ji}{}^h = \Psi_j\delta_i^h + \Psi_i\delta_j^h + \frac{1}{2}(K_{aji}{}^h + K_{aij}{}^h)D^a \tag{4.31}$$

and

$$\begin{aligned}
2\nabla_k\nabla_j A_i{}^h &= 2K_{ikj}{}^a A_a{}^h + B^a(\nabla_a K_{ikj} + \nabla_a K_{ijk}{}^h) \\
&\quad + (K_{ija}{}^h + K_{iaj}{}^h)\nabla_k B^a + +(K_{ika}{}^h + K_{iak}{}^h)\nabla_j B^a \\
&\quad - (K_{ikj}{}^a + K_{ijk}{}^a)\nabla_a B^h + (K_{akj}{}^h + K_{ajk}{}^h)C_i{}^a \\
&\quad + K_{iaj}{}^h\nabla_k D^a + K_{iak}{}^h\nabla_j D^a \\
&\quad + 4\delta_k^h\nabla_j\Phi_i + 4\delta_j^h\nabla_k\Phi_i.
\end{aligned} \tag{4.32}$$

Substituting (4.19) into (4.32), we have

$$
\begin{aligned}
&4\delta_k^h \nabla_j \Phi_i + 2\delta_j^h \nabla_k \Phi_i \\
&= 2K_{kij}{}^a A_a{}^h - B^a(\nabla_a K_{ikj}{}^h + \nabla_a K_{ijk}{}^h) \\
&\quad - (K_{ija}{}^h + K_{iaj}{}^h)\nabla_k B^a - (K_{ika}{}^h + K_{iak}{}^h)\nabla_j B^a \\
&\quad + (K_{ikj}{}^a + K_{ijk}{}^a)\nabla_a B^h - (K_{akj}{}^h + K_{ajk}{}^h)C_i{}^a \\
&\quad + D^a \nabla_k K_{aij}{}^h + K_{aik}{}^h \nabla_j D^a + 2K_{aij}{}^h \nabla_k D^a.
\end{aligned} \tag{4.33}
$$

Contracting j and h in (4.33), and comparing this with (4.26),

$$\nabla_j \Phi_i = 0. \tag{4.34}$$

Substituting (4.20) and (4.34) in (4.33),

$$
\begin{aligned}
&\nabla_k (K_{aji}{}^h D^a) \\
&= -K_{kij}{}^a A_a{}^h - K_{aji}{}^h (\nabla_k B^a - C_k{}^a + \nabla_k D^a) \\
&\quad - K_{ajk}{}^h (\nabla_i B^a - C_i{}^a + \nabla_i D^a).
\end{aligned} \tag{4.35}
$$

From (4.27) and (4.34),

$$\widetilde{\Omega}_i = \Psi_i. \tag{4.36}$$

From (4.20), (4.28) and (4.34),

$$K_{kji}{}^a (A_a{}^h - \nabla_a B^h + C_a{}^h - \nabla_a D^h) = 0. \tag{4.37}$$

Contracting j and h in (4.29), and using (4.12) and (4.18),

$$
\begin{aligned}
(n+1)\widetilde{\Omega}_i = {} & \nabla_i \nabla_a B^a + \frac{1}{2} R_{ai} D^a \\
& + \frac{1}{2} y^a (2n \nabla_i \Phi_a + B^b \nabla_b R_{ai} + R_{ba} \nabla_i B^b + R_{bi} C_a{}^b + K_{bai}{}^c \nabla_c D^b) \\
& + \frac{1}{2} y^b y^a (2R_{bi} \Phi_a + A_b{}^c \nabla_c R_{ai} - K_{bci}{}^e K_{dea}{}^c B^d \\
& \qquad - K_{bci}{}^d \nabla_d C_a{}^c + \frac{1}{2} R_{cb} K_{dai}{}^c D^d).
\end{aligned}
$$

Comaring this with (4.23),

$$\Psi_i = \frac{1}{n}(\nabla_i C_a{}^a - \frac{1}{2} R_{ai} D^a) = \frac{1}{n+1}(\nabla_i \nabla_a B^a + \frac{1}{2} R_{ai} D^a). \tag{4.38}$$

We put $\varphi := A_a{}^a - \dfrac{n}{2n+1} \nabla_a B^a + \dfrac{n+1}{2n+1} C_a{}^a$ and $\psi := \dfrac{1}{2n+1}(\nabla_a B^a + C_a{}^a)$. Then we get

$$\Phi_i = \partial_i \varphi \qquad \text{and} \qquad \Psi_i = \partial_i \psi, \tag{4.39}$$

by virtue of (4.21) and (4.38).

From (4.30),

$$L_D \Gamma_{ji}{}^h = \nabla_j \nabla_i D^h + K_{aji}{}^h D^a = 0 \tag{4.40}$$

and

$$\begin{aligned} 2\nabla_k \nabla_j C_i{}^h = & 2K_{kaj}{}^h C_i{}^a - 2K_{kij}{}^a C_a{}^h \\ & - 2\nabla_k (K_{aji}{}^h B^a) + 2B^a \nabla_a K_{kij}{}^h \\ & + 2K_{aij}{}^h \nabla_k B^a + 2K_{kia}{}^h \nabla_j B^a \\ & + K_{aij}{}^h \nabla_k D^a + K_{aik}{}^h \nabla_j D^a \\ & - (K_{ikj}{}^a + K_{ijk}{}^a) \nabla_a D^h. \end{aligned} \tag{4.41}$$

Substituting (4.20), (4.24) and (4.37) into (4.41),

$$\begin{aligned} 2\delta_i^h \nabla_k \Psi_j = & 2K_{aij}{}^h \nabla_k B^a - 2K_{aij}{}^h C_k{}^a \\ & + \nabla_k (K_{aij}{}^h D^a) + K_{aij}{}^h \nabla_k D^a \\ & + K_{aik}{}^h \nabla_j D^a - K_{kji}{}^a \nabla_a D^h. \end{aligned} \tag{4.42}$$

Contracting j and h in (4.42),

$$2\nabla_k \Psi_i = (K_{aki}{}^b + K_{aik}{}^b) \nabla_b D^a. \tag{4.43}$$

Using (4.19), (4.34) and the Ricci identity,

$$\begin{aligned} & \nabla_b (K_{aij}{}^b D^a) \\ & = \nabla_a (2\Phi_i \delta_j^a + K_{bij}{}^a D^b) - 2n \nabla_j \Phi_i \\ & = 2(\nabla_a \nabla_j A_i{}^a - \nabla_j \nabla_a A_i{}^a) \\ & = 0. \end{aligned} \tag{4.44}$$

Contracting k and h in (4.42), and using (4.12), (4.37), (4.43) and (4.44),

$$\begin{aligned} 2\nabla_i \Psi_j & = 2K_{aij}{}^b (\nabla_b B^a - C_b{}^a + \nabla_b D^a) \\ & \quad + \nabla_b (K_{aij}{}^b D^a) - (K_{aji}{}^b + K_{aij}{}^b) \nabla_b D^a \\ & = -2\nabla_i \Psi_j, \end{aligned}$$

from which

$$\nabla_j \Psi_i = 0. \tag{4.45}$$

Substituting (4.45) into (4.42), we get

$$\begin{aligned} D^a \nabla_k K_{aji}{}^h = & -2K_{aji}{}^h (\nabla_k B^a - C_k{}^a + \nabla_k D^a) \\ & - K_{ajk}{}^h \nabla_i D^a + K_{kij}{}^a \nabla_a D^h. \end{aligned} \tag{4.46}$$

Q.E.D.

Therefore we now have the following

Corollary (cf. [Y2, Y3]). *Let $(M,\ g)$ be a Riemannian manifold and TM its tangent bundle with* (1) *horizontal lift connection,* (2) *complete lift connection,* (3) *diagonal lift connection or* (4) *lift connection* II+III. *If TM admits an infinitesimal fibre-preserving projective transformation $\widetilde{V}$ with the associated* 1*-form $\widetilde{\Omega}$, then there exist $\psi \in \mathfrak{T}^0_0(M)$, $B = (B^h)$, $D = (D^h) \in \mathfrak{T}^1_0(M)$and $C = (C_i{}^h) \in \mathfrak{T}^1_1(M)$ satisfying*

(1) $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) = (B^h,\ D^h + y^a C_a{}^h)$,

(2) $\widetilde{\Omega} = (\partial_a \psi) dx^a = \Psi_a dx^a$,

(3) $\nabla_j \Psi_i = 0$,

(4) $\nabla_j C_i{}^h = \Psi_j \delta^h_i - K_{aji}{}^h B^a$,

(5) $L_B \Gamma_{ji}{}^h = \nabla_j \nabla_i B^h + K_{aji}{}^h B^a = \Psi_j \delta^h_i + \Psi_i \delta^h_j$,

(6) $L_D \Gamma_{ji}{}^h = 0$,

where $(\widetilde{V}^h,\ \widetilde{V}^{\bar{h}}) := \widetilde{V}^a E_a + \widetilde{V}^{\bar{a}} E_{\bar{a}} = \widetilde{V}$.

Using these theorems, we at last come to the following

Theorem 5. *Let $(M,\ g)$ be a complete Riemannian manifold and TM its tangent bundle. Assume that TM admits a non-affine infinitesimal projective transformation with respect to one of the following lift connections:*
(1) *the horizontal lift connection.*
(2) *the complete lift connection.*
(3) *the diagonal lift connection.*
(4) *the lift connection* II+III.
Then M and TM are locally flat.

Proof. Here we prove only case (4) because other cases are proved with same technics as case (4). We put $X^h := A_a{}^h \Phi^a$. Then, using (4.19) and (4.34), we have

$$
\begin{aligned}
L_X g_{ji} &= \nabla_j X_i + \nabla_i X_j \\
&= (\nabla_j A_{ai}) \Phi^a + (\nabla_i A_{aj}) \Phi^a \\
&= (\Phi_a g_{ji} - \frac{1}{2} K_{baji} D^b) \Phi^a + (\Phi_a g_{ji} - \frac{1}{2} K_{baij} D^b) \Phi^a \\
&= 2(\Phi_a \Phi^a) g_{ji}.
\end{aligned} \tag{4.47}
$$

Similarly we put $Y^h := (\nabla_a B^h - C_a{}^h) \Psi^a$. Then, using (4.24), (4.31) and (4.45),

$$
\begin{aligned}
L_Y g_{ji} &= (\nabla_j \nabla_a B_i - \nabla_j C_{ai}) \Psi^a + (\nabla_i \nabla_a B_j - \nabla_i C_{aj}) \Psi^a \\
&= \{(-K_{bjai} B^b + \Psi_j g_{ai} + \Psi_a g_{ji}) - (\Psi_j g_{ai} - K_{bjai} B^b)\} \Psi^a \\
&\qquad + (\Psi_a g_{ji}) \Psi^a \\
&= 2(\Psi_a \Psi^a) g_{ji}
\end{aligned} \tag{4.48}
$$

Therefore X and Y are infinitesimal homothetic transformations.

To prove Theorem 5, we need the following well-known

Lemma 3 [K1]. *If a complete Riemannian manifold M admits a non-isometric infinitesimal homothetic transformation, then M is locally flat.*

Therefore M is locally flat by virtue of Lemma 3. In this case the lift connection coincides with the horizontal lift conection, and TM is also locally flat. In fact

$$\widetilde{K}(E_k,\ E_j)E_i = \widetilde{\nabla}_{E_k}\widetilde{\nabla}_{E_j}E_i - \widetilde{\nabla}_{E_j}\widetilde{\nabla}_{E_k}E_i - \widetilde{\nabla}_{[E_k,\ E_j]}E_i = K_{kji}{}^a E_a = 0,$$
$$\widetilde{K}(E_k,\ E_j)E_{\bar{i}} = \widetilde{\nabla}_{E_k}\widetilde{\nabla}_{E_j}E_{\bar{i}} - \widetilde{\nabla}_{E_j}\widetilde{\nabla}_{E_k}E_{\bar{i}} - \widetilde{\nabla}_{[E_k,\ E_j]}E_{\bar{i}} = K_{kji}{}^a E_{\bar{a}} = 0,$$
$$\widetilde{K}(E_k,\ E_{\bar{j}})E_i = \widetilde{K}(E_k,\ E_{\bar{j}})E_{\bar{i}} = \widetilde{K}(E_{\bar{k}},\ E_{\bar{j}})E_i = \widetilde{K}(E_{\bar{k}},\ E_{\bar{j}})E_{\bar{i}} = 0.$$

Q.E.D.

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