CLOSURE-PRESERVING SUM THEOREMS

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Abstract.

Let X be a space with a closure-preserving cover $\mathcal{F}$ consisting of countably compact closed subsets. In this paper we prove the following: (1) if X is normal and each $F \in \mathcal{F}$ is weakly infinite-dimensional, then X is weakly infinite-dimensional, (2) if X is collectionwise normal and each $F \in \mathcal{F}$ is a C-space, then X is a C-space.

1 Introduction

We assume that all spaces are normal. In this paper we study sum theorems for infinite-dimensional spaces. Usual and undefined terms can be found in [2].

A space X is *weakly infinite-dimensional* if for every countable collection $\{(A_i, B_i) : i \in \mathbb{N}\}$ of pairs of disjoint closed subsets of X there exists a partition L_i in X between A_i and B_i for each $i \in \mathbb{N}$ such that $\bigcap_{i=1}^{\infty} L_i = \emptyset$. If $\mathcal{A}$ and $\mathcal{B}$ are collections of subsets of a space X, then $\mathcal{A} < \mathcal{B}$ means that $\mathcal{A}$ is a refinement of $\mathcal{B}$, i.e. for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subset B$. Notice that $\mathcal{A}$ need not be a cover even if $\mathcal{B}$ is a cover. A space X is a *C-space* if for every countable collection $\{\mathcal{G}_i : i \in \mathbb{N}\}$ of open covers of X there exists a countable collection $\{\mathcal{H}_i : i \in \mathbb{N}\}$ of collections of pairwise disjoint open subsets of X such that $\mathcal{H}_i < \mathcal{G}_i$ and $\bigcup\bigcup_{i=1}^{\infty} \mathcal{H}_i = X$. A collection $\{A_s : s \in S\}$ of subsets of a space X is *closure-preserving* if $\mathrm{Cl}(\bigcup_{s \in S'} A_s) = \bigcup_{s \in S'} \mathrm{Cl} A_s$ for every $S' \subset S$. A collection $\{A_s : s \in S\}$ of subsets of a space X is *hereditarily closure-preserving* if every collection $\{B_s : s \in S\}$, where $B_s \subset A_s$ for every $s \in S$, is closure-preserving. Let us note that every locally finite collection is hereditarily closure-preserving and that every hereditarily closure-preserving collection is closure-preserving.

Let X be a space with a closed cover $\mathcal{F}$ consisting of weakly infinite-dimensional subspaces. Hadziivanov [3] proved that X is weakly infinite-dimensional provided that X is countably paracompact and $\mathcal{F}$ is locally finite. Polkowski [5] proved that X is weakly infinite-dimensional provided that X is hereditarily normal and $\mathcal{F}$ is hereditarily closure-preserving. By using the same method of Polkowski, we can easily show that Hadziivanov's result above holds under the assumption that $\mathcal{F}$ is hereditarily closure-preserving.

Addis and Gresham [1] proved that if a paracompact and hereditarily collectionwise normal space X can be represented as the union of a locally finite collection $\mathcal{F}$ of closed C-spaces, then X is a C-space. Komoda [4] proved this result under the assumption that X is either paracompact or hereditarily collectionwise normal and $\mathcal{F}$ is hereditarily closure-preserving.

On the other hand Telgársky and Yajima [6] proved that if a space X has a closure-preserving closed cover $\mathcal{F}$ such that each $F \in \mathcal{F}$ is countably compact and $\dim F \leq n$, then $\dim X \leq n$. The purpose of this paper is to give analogous results for weakly infinite-dimensional spaces and for C-spaces.

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2 The Main Theorems

The following lemma which was proved by Telgársky and Yajima [6] plays an important role in the proofs of our main theorems.

2.1. Lemma.(Telgársky and Yajima [6]). *Let X be a space with a closure-preserving cover $\mathcal{F}$ consisting of countably compact closed sets. Let $\{\mathcal{F}_i : i \in \mathbb{N}\}$ be a sequence of collections of subsets of X and $\{X_i : i \in \mathbb{N}\}$ a sequence of closed subsets of X satisfying the following conditions;*

$X_{i+1} \subset X_i$, $\mathcal{F}_i$ is a maximal disjoint subcollection of $\mathcal{F}|X_i = \{F \cap X_i : F \in \mathcal{F}\}$, and $X_{i+1} \cap \bigcup \mathcal{F}_i = \emptyset$.
Then we have $\bigcap_{i=1}^{\infty} X_i = \emptyset$.

Since every F_σ-subset of a normal space is also normal, one can easily prove the following lemma.

2.2. Lemma. *Let F be a closed subspace of a space X and Z a zero-set in X. Let A and B be disjoint closed subsets of X such that $Z \cap A = \emptyset = Z \cap B$. If $Z \cap F$ is a partition in F between $A \cap F$ and $B \cap F$, then there exists a partition L in X between A and B such that $L \cap F \subset Z$.*

2.3. Theorem. *If a space X has a closure-preserving closed cover $\mathcal{F}$ such that each $F \in \mathcal{F}$ is countably compact and weakly infinite-dimensional, then X is weakly infinite-dimensional.*

Proof. Let $\{(A_{ij}, B_{ij}) : i, j \in \mathbb{N}\}$ be a countable collection of pairs of disjoint closed subsets of X. For every $i \in \mathbb{N}$, inductively, we shall construct a closed subset X_i, a collection $\mathcal{F}_i$ and a closed subset L_{ij} for each $j \in \mathbb{N}$ satisfying the following conditions;

$\mathcal{F}_i$ is a maximal disjoint subcollection of $\mathcal{F}|X_i$, and

L_{ij} is a partition in X between A_{ij} and B_{ij} for each $j \in \mathbb{N}$ such that $\bigcup \mathcal{F}_i \cap \bigcap_{j=1}^{\infty} L_{ij} = \emptyset$.

First we set $X_1 = X$. Assume that a closed subset X_i has been constructed. By Zorn's Lemma, take a maximal disjoint subcollection $\mathcal{F}_i$ of $\mathcal{F}|X_i$. For every $F \cap X_i \in \mathcal{F}_i$, where $F \in \mathcal{F}$, there exist $\ell(F) \in \mathbb{N}$ and a partition $T_{ij}(F)$ in $F \cap X_i$ between $A_{ij} \cap F \cap X_i$ and $B_{ij} \cap F \cap X_i$ for each $j \in \mathbb{N}$ such that $\bigcap_{j=1}^{\ell(F)} T_{ij}(F) = \emptyset$, because $F \cap X_i$ is countably compact and weakly infinite-dimensional. For every $\ell \in \mathbb{N}$ let us set

$$\mathcal{F}_{i\ell} = \{F \cap X_i \in \mathcal{F}_i : \ell(F) = \ell\}, K_{i\ell} = \bigcup \mathcal{F}_{i\ell}, \text{ and}$$

$$T_{ij\ell} = \bigcup \{T_{ij}(F) : F \cap X_i \in \mathcal{F}_{i\ell}\} \text{ for each } j \in \mathbb{N}.$$

Since $\mathcal{F}_{i\ell}$ is closure-preserving and pairwise disjoint, $\mathcal{F}_{i\ell}$ is discrete in X. Thus the set $T_{ij\ell}$ is a partition in $K_{i\ell}$ between $A_{ij} \cap K_{i\ell}$ and $B_{ij} \cap K_{i\ell}$. Since $\{K_{i\ell} : \ell \in \mathbb{N}\}$ is discrete in X, there exists a discrete collection $\{U_{i\ell} : \ell \in \mathbb{N}\}$ of open subsets of X such that $K_{i\ell} \subset U_{i\ell}$. Since $\bigcap_{j=1}^{\ell} T_{ij\ell} = \emptyset$, take a zero-set $Z_{ij\ell}$ in X such that $T_{ij\ell} \subset Z_{ij\ell} \subset U_{i\ell}, \bigcap_{j=1}^{\ell} Z_{ij\ell} = \emptyset$ and $Z_{ij\ell} \cap A_{ij} = \emptyset = Z_{ij\ell} \cap B_{ij}$. Let us set $Z_{ij} = \bigcup \{Z_{ij\ell} : \ell \in \mathbb{N}\}$ and $K_i = \bigcup \{K_{i\ell} : \ell \in \mathbb{N}\}$. Since $\{U_{i\ell} : \ell \in \mathbb{N}\}$ is discrete in X, Z_{ij} is a zero-set in X and $Z_{ij} \cap K_i$ is a partition in K_i between $A_{ij} \cap K_i$ and $B_{ij} \cap K_i$. By Lemma 2.2, take a partition L_{ij} in X between A_{ij} and B_{ij} such that $L_{ij} \cap K_i \subset Z_{ij}$. Then we have $\bigcup \mathcal{F}_i \cap \bigcap_{j=1}^{\infty} L_{ij} = K_i \cap \bigcap_{j=1}^{\infty} L_{ij} \subset K_i \cap \bigcap_{j=1}^{\infty} Z_{ij} = \emptyset$. Finally, we put $X_{i+1} = \bigcap_{j=1}^{\infty} L_{ij} \cap X_i$.

By Lemma 2.1, $\bigcap_{i=1}^{\infty} X_i = \emptyset$, therefore we have $\bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} L_{ij} = \bigcap_{i=1}^{\infty} X_i = \emptyset$. This implies that the space X is weakly infinite-dimensional. Theorem 2.3 has been proved.

Next we consider a sum theorem for C-spaces.

2.4. Lemma. *Let F be a closed subset of a collectionwise normal space X and $\{\mathcal{G}_i : i \in \mathbb{N}\}$ be a collection of open covers of X. If F is a countably paracompact C-space, then there exists a discrete collection $\mathcal{H}_i$ of open subsets of X for each $i \in \mathbb{N}$ such that $\mathcal{H}_i < \mathcal{G}_i$ and $F \subset \bigcup\bigcup_{i=1}^{\infty} \mathcal{H}_i$.*

Proof. Take a collection $\mathcal{U}_i$ of pairwise disjoint open subsets of F for each $i \in \mathbb{N}$ such that $\mathcal{U}_i < \mathcal{G}_i|F$ and $\bigcup\bigcup_{i=1}^{\infty} \mathcal{U}_i = F$. For ever $U \in \mathcal{U}_i$ take $G(U) \in \mathcal{G}_i$ with $U \subset G(U)$. We set $U_i = \bigcup \mathcal{U}_i$ for every $i \in \mathbb{N}$. Obviously, $\{U_i : i \in \mathbb{N}\}$ is a countable open cover of F. Thus there exists an open cover $\{V_i : i \in \mathbb{N}\}$ of F with $\mathrm{Cl} V_i \subset U_i$. Let us set $U' = U \cap V_i$ for every $U \in \mathcal{U}_i$. The collection $\{\mathrm{Cl}_X U' : U \in \mathcal{U}_i\}$ is discrete in X. Hence we can take a discrete collection $\{H(U) : U \in \mathcal{U}_i\}$ of open subsets of X such that $\mathrm{Cl}_X U' \subset H(U) \subset G(U)$. Let us set $\mathcal{H}_i = \{H(U) : U \in \mathcal{U}_i\}$. Obviously, we have $\mathcal{H}_i < \mathcal{G}_i$ and $F \subset \bigcup\bigcup_{i=1}^{\infty} \mathcal{H}_i$.

2.5. Theorem. *If a collectionwise normal space X has a closure-preserving closed cover $\mathcal{F}$ such that each $F \in \mathcal{F}$ is a countably compact C-space, then X is a C-space.*

Proof. Let $\{\mathcal{G}_{ij} : i, j \in \mathbb{N}\}$ be a countable collection of open covers of X. For every $i \in \mathbb{N}$, inductively, we shall construct a closed subset X_i, a collection $\mathcal{F}_i$ and a collection $\mathcal{H}_{ij}$ for each $j \in \mathbb{N}$ satisfying the following conditions;

$\mathcal{F}_i$ is a maximal disjoint subcollection of $\mathcal{F}|X_i$, and

$\mathcal{H}_{ij}$ is a collection of pairwise disjoint open subsets of X for each $j \in \mathbb{N}$ such that $\mathcal{H}_{ij} < \mathcal{G}_{ij}$ and $\bigcup \mathcal{F}_i \subset \bigcup\bigcup_{j=1}^{\infty} \mathcal{H}_{ij}$.

First we set $X_1 = X$. Assume that a closed subset X_i has been constructed. By Zorn's Lemma, take a maximal disjoint subcollection $\mathcal{F}_i$ of $\mathcal{F}|X_i$. Since $\mathcal{F}_i$ is closure-preserving and pairwise disjoint, $\mathcal{F}_i$ is discrete in X. Since $F \cap X_i \in \mathcal{F}_i$ is a countably compact C-space, $\bigcup \mathcal{F}_i$ is a countably paracompact C-space. By Lemma 2.4, there exists a collection $\mathcal{H}_{ij}$ of pairwise disjoint open subsets of X such that $\mathcal{H}_{ij} < \mathcal{G}_{ij}$ and $\bigcup \mathcal{F}_i \subset \bigcup\bigcup_{j=1}^{\infty} \mathcal{H}_{ij}$. Finally, we put $X_{i+1} = X_i - \bigcup\bigcup_{j=1}^{\infty} \mathcal{H}_{ij}$. By Lemma 2.1, $\bigcap_{i=1}^{\infty} X_i = \emptyset$, therefore we have $\bigcup\bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \mathcal{H}_{ij} = X$. This implies that the space X is a C-space.

2.6. Problem. Does Theorem 2.5 hold under the assumption that X is normal?

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