SELF-ADJOINT INTERPOLATION PROBLEMS IN CSL-ALGEBRA ALG$\mathcal{L}$

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Abstract. Given vectors x and y in a Hilbert space, an interpolating operator is a bounded operator T such that $Tx = y$. An interpolating operator for $\mathbf{N}$ vectors satisfies the equation $Tx_i = y_i$, for $i = 1, 2, \cdots, n$. In this article, we investigate self-adjoint interpolation problems in CSL-Algebra Alg$\mathcal{L}$.

1. Introduction

Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors on $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which x and y is there a bounded operator $T \in \mathcal{C}$ such that $Tx = y$. A variation, the '$\mathbf{N}$-vector interpolation problem', asks for an operator T such that $Tx_i = y_i$ for fixed finite collections $\{x_1, x_2, \cdots, x_n\}$ and $\{y_1, y_2, \cdots, y_n\}$. The $\mathbf{N}$-vector interpolation problem was considered for a C^*-algebra $\mathcal{U}$ by Kadison[9]. In case $\mathcal{U}$ is a nest algebra, the (one-vector) interpolation problem was solved by Lance[10]: his result was extended by Hopenwasser[4] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[11] obtained conditions for interpolation in case T is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[5] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition (attributed to S. Power) for interpolation $\mathbf{N}$-vectors, although necessity was not proved in that paper.

In this article, we investigate the self-adjoint interpolation problems in CSL-Algebra Alg$\mathcal{L}$: Given vectors x and y in a Hilbert space and a commutative subspace lattice $\mathcal{L}$ on $\mathcal{H}$, when is there a self-adjoint operator A in Alg$\mathcal{L}$ such that $Ax = y$?

First, we establish some notations and conventions. A commutative subspace lattice $\mathcal{L}$, or CSL $\mathcal{L}$ is a strongly closed lattice of pairwise-commuting projections acting on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. If $\mathcal{L}$ is CSL, Alg$\mathcal{L}$ is called a CSL-algebra. The symbol Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let x and y be

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vectors in a Hilbert space. Then $< x, y >$ means the inner product of vectors x and y. In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

2. Results

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a commutative subspace lattice of orthogonal projections acting on $\mathcal{H}$ containing 0 and I. Then Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. Let M be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of M. Let $\mathbf{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers.

Definition. Let $\mathcal{H}$ be a Hilbert space and let A be an operator acting on $\mathcal{H}$. Then A is called *a self-adjoint operator* if $A^* = A$.

Theorem 1. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $Ax = y$, A is self-adjoint and every E in $\mathcal{L}$ reduces A, then*

$$\sup\left\{\frac{\|\sum_{i=1}^{n}\alpha_i E_i y\|}{\|\sum_{i=1}^{n}\alpha_i E_i x\|} : n\in N, \alpha_i\in\mathbb{C} \text{ and } E_i\in\mathcal{L}\right\} < \infty \text{ and } < Ex, y > = < Ey, x >$$

for every E in $\mathcal{L}$.

Proof. We can get the first result by Theorem 1 [8] under the given hypothesis. So we need to show that $< Ex, y > = < Ey, x >$ for every E in $\mathcal{L}$ whenever $A^* = A$. Since $AE = EA$, $A^*E = EA^*$ for every E in $\mathcal{L}$. Since $Ax = A^*x = y$, $A^*Ex = AEx = Ey$ for every E in $\mathcal{L}$. Hence $< Ey, x > = < A^*Ex, x > = < Ex, Ax > = < Ex, y >$ for every E in $\mathcal{L}$.

Let x and y be vectors of a Hilbert space $\mathcal{H}$. Let

$$\mathcal{M} = \left\{\sum_{i=1}^{n}\alpha_i E_i x : n \in N, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\} \text{ and}$$
$$\mathcal{M}_1 = \left\{\sum_{i=1}^{n}\alpha_i E_i y : n \in N, \alpha_i \in \mathbb{C} \text{ and } E_i \in \mathcal{L}\right\}.$$

Theorem 2. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. Assume that $\mathcal{M}_1 \subset \overline{\mathcal{M}}$. If*

$$\sup\left\{\frac{\|\sum_{i=1}^{n}\alpha_i E_i y\|}{\|\sum_{i=1}^{n}\alpha_i E_i x\|} : n\in N, \alpha_i\in\mathbb{C} \text{ and } E_i\in\mathcal{L}\right\} < \infty \text{ and } <Ex, y> = <Ey, x>$$

for every E in $\mathcal{L}$, then there is an operator A in Alg$\mathcal{L}$ such that $y = Ax$, $A^* = A$ and every E in $\mathcal{L}$ reduces A.

Proof. We can get results except that $A^* = A$ by Theorem 1 [8] under the given hypothesis. So we need to prove that if $< Ex, y > = < Ey, x >$ for every E in $\mathcal{L}$, then $A^* = A$. Since

$< Ex, y >=< Ey, x >$ for every E in $\mathcal{L}$,

$$\begin{aligned} < A(\sum_{i=1}^{n} \alpha_i E_i x), x > &=< \sum_{i=1}^{n} \alpha_i E_i A x, x > \\ &=< \sum_{i=1}^{n} \alpha_i E_i y, x > \\ &=< \sum_{i=1}^{n} \alpha_i E_i x, y > . \end{aligned}$$

Since $y \in \overline{\mathcal{M}}$, $A^* x = y$. Since $EA = AE$, $EA^* = A^* E$ for every E in $\mathcal{L}$. So

$$\begin{aligned} A^*(\sum_{i=1}^{n} \alpha_i E_i x) &= \sum_{i=1}^{n} \alpha_i A^* E_i x \\ &= \sum_{i=1}^{n} \alpha_i E_i A^* x \\ &= \sum_{i=1}^{n} \alpha_i E_i y. \end{aligned}$$

Since $\mathcal{M}_1 \subset \overline{\mathcal{M}}$, $A^* f = 0$ for every f in $\overline{\mathcal{M}}^{\perp}$. Hence $A^* = A$.

Corollary 3. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. Assume that $\mathcal{M}$ is dense in $\mathcal{H}$. If*

$$\sup\left\{ \frac{\|\sum_{i=1}^{n} \alpha_i E_i y\|}{\|\sum_{i=1}^{n} \alpha_i E_i x\|} : n \in N, \alpha_i \in \mathbb{C} \textit{ and } E_i \in \mathcal{L} \right\} < \infty \textit{ and } <Ex, y>=<Ey, x>$$

for every E in $\mathcal{L}$, then there is an operator A in Alg$\mathcal{L}$ such that $y = Ax$, $A^* = A$ and every E in $\mathcal{L}$ reduces A.

If we summarize Theorems 1, 2 and Corollary 3, we can get the following theorem.

Theorem 4. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. Assume that $\mathcal{M}_1 \subset \overline{\mathcal{M}}$ or $\mathcal{M}$ is dense in $\mathcal{H}$. Then the following statements are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$, $A^* = A$ and every E in $\mathcal{L}$ reduces A.

(2) $$\sup\left\{ \frac{\|\sum_{i=1}^{n} \alpha_i E_i y\|}{\|\sum_{i=1}^{n} \alpha_i E_i x\|} : n \in N, \alpha_i \in \mathbb{C} \textit{ and } E_i \in \mathcal{L} \right\} < \infty \textit{ and}$$

$< Ex, y >=< Ey, x >$ for every E in $\mathcal{L}$.

Theorem 5. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be vectors in $\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ for all $p = 1, 2, \cdots, n$, $A^* = A$ and every E in $\mathcal{L}$ reduces A,* *then* $\sup\left\{\dfrac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|}: m_i \in N, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\} < \infty$ *and*

$< Ex_p, y_j >=< Ey_p, x_j >$ *for every* E *in* $\mathcal{L}$ *and all* $p, j = 1, 2, \cdots, n$.

Proof. By Theorem 2 [8], we know that

$\sup\left\{\dfrac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|}: m_i \in N, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\} < \infty$. Since $A^* = A$, $< Ey_p, x_j >=< EAx_p, x_j >=< AEx_p, x_j >=< Ex_p, A^*x_j >=< Ex_p, y_j >$ for every E in $\mathcal{L}$ and all $p, j = 1, 2, \cdots, n$.

Theorem 6. *Let* $\mathcal{H}$ *be a Hilbert space and* $\mathcal{L}$ *be a commutative subspace lattice on* $\mathcal{H}$. *Let* $x_1, x_2, \cdots, x_n$ *and* $y_1, y_2, \cdots, y_n$ *be vectors in* $\mathcal{H}$. *Let*

$$\mathcal{K} = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i : m_i \in N, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\} \text{ and}$$
$$\mathcal{K}_1 = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i : m_i \in N, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\}.$$

Assume that $\mathcal{K}_1 \subset \overline{\mathcal{K}}$.

If $\sup\left\{\dfrac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|}: m_i \in N, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\} < \infty$ *and*

$< Ex_p, y_j >=< Ey_p, x_j >$ *for every* E *in* $\mathcal{L}$ *and all* $p, j = 1, 2, \cdots, n$, *then there exists an operator* A *in Alg*$\mathcal{L}$ *such that* $Ax_p = y_p$ *for all* $p = 1, 2, \cdots, n$, $A^* = A$ *and every* E *in* $\mathcal{L}$ *reduces* A.

Proof. By Theorem 2 [8], there exists an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ for all $p = 1, 2, \cdots, n$ and every E in $\mathcal{L}$ reduces A. We want to show that $A^* = A$ if $< Ex_p, y_j >=< Ey_p, x_j >$ for every E in $\mathcal{L}$ and all $p, j = 1, 2, \cdots, n$. First, we will show that $A^*x_p = y_p$ for all $p = 1, 2, \cdots, n$. Since $< Ex_p, y_j >=< Ey_p, x_j >$ for all E in $\mathcal{L}$ and all $p, j = 1, 2, \cdots, n$,

$$\begin{aligned}< A(\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i), x_j > &=< \sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}Ax_i, x_j > \\ &=< \sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i, x_j > \\ &=< \sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i, y_j > .\end{aligned}$$

Since $\{y_1, y_2, \cdots, y_n\} \subset \overline{\mathcal{K}}$, $y_j = A^*x_j$ for all $j = 1, 2, \cdots, n$. Since $\mathcal{K}_1 \subset \overline{\mathcal{K}}$, $A^*f = 0$ for every f in $\overline{\mathcal{K}}^{\perp}$. Hence $A^* = A$.

Corollary 7. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be vectors in $\mathcal{H}$. Assume that*

$\mathcal{K} = \left\{ \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i : m_i \in N, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C} \right\}$ *is dense in $\mathcal{H}$. If*

$\sup \left\{ \frac{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i \|}{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i \|} : m_i \in N, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C} \right\} < \infty$ *and*

$< Ex_q, y_j > = < Ey_q, x_j >$ for every E in $\mathcal{L}$ and all $q, j = 1, 2, \cdots, n$, then there exists an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ for all $p = 1, 2, \cdots, n$, $A^* = A$ and every E in $\mathcal{L}$ reduces A.

If we summarize Theorems 5, 6 and Corollary 7, we can get the following theorem.

Theorem 8. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ be vectors in $\mathcal{H}$. Assume that $\mathcal{K}_1 \subset \overline{\mathcal{K}}$ or $\mathcal{K}$ is dense in $\mathcal{H}$. Then the following statements are equivalent.*

(1) There exists an operator A in Alg$\mathcal{L}$ such that $Ax_p = y_p$ for all $p = 1, \cdots, n$, $A^* = A$ and every E in $\mathcal{L}$ reduces A.

(2) $\sup \left\{ \frac{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i \|}{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i \|} : m_i \in N, l \leq n, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C} \right\} < \infty$

and $< Ex_p, y_j > = < Ey_p, x_j >$ for every E in $\mathcal{L}$ and all $p, j = 1, 2, \cdots, n$.

If we modify proofs of Theorems 5, 6, 7 and 8 a little bit , we can prove the following theorems. So we will omit their proofs.

Theorem 9. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$. If there is an operator A in Alg$\mathcal{L}$ such that $Ax_n = y_n$ for all $n = 1, 2, \cdots$, $A^* = A$ and every E in $\mathcal{L}$*

reduces A, then $\sup \left\{ \frac{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i \|}{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i \|} : m_i, l \in N, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C} \right\} <$

∞ and $< Ex_p, y_j > = < Ey_p, x_j >$ for every E in $\mathcal{L}$ and all $p, j = 1, 2, \cdots$.

Theorem 10. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$. Let*

$$\mathcal{U} = \left\{ \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} x_i : m_i, l \in N, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C} \right\} \text{ and}$$
$$\mathcal{U}_1 = \left\{ \sum_{k=1}^{m_i} \sum_{i=1}^{l} \alpha_{k,i} E_{k,i} y_i : m_i, l \in N, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C} \right\}.$$

Assume that $\mathcal{U}_1 \subset \overline{\mathcal{U}}$.

If $\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i, l \in N, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\} < \infty$

and $< Ex_p, y_j >=< Ey_p, x_j >$ *for every* E *in* $\mathcal{L}$ *and all* p, j, *then there is an operator* A *in* $Alg\mathcal{L}$ *such that* $Ax_n = y_n$ *for all* $n = 1, 2, \cdots$, $A^* = A$ *and every* E *in* $\mathcal{L}$ *reduces* A.

Corollary 11. *Let* $\mathcal{H}$ *be a Hilbert space and let* $\mathcal{L}$ *be a commutative subspace lattice on* $\mathcal{H}$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two infinite sequences of vectors in* $\mathcal{H}$. *Let*

$\mathcal{U}=\left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i : m_i, l \in N, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\}$. *Assume that* $\mathcal{U}$ *is dense*

in $\mathcal{H}$. *If* $\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i, l \in N, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\} < \infty$

and $<Ex_p, y_j>=<Ey_p, x_j>$ *for every* E *in* $\mathcal{L}$ *and all* p, j, *then there is an operator* A *in* $Alg\mathcal{L}$ *such that* $Ax_n = y_n$ *for all* $n=1,2,\cdots$, $A^*=A$ *and every* E *in* $\mathcal{L}$ *reduces* A.

Theorem 12. *Let* $\mathcal{H}$ *be a Hilbert space and let* $\mathcal{L}$ *be a commutative subspace lattice on* $\mathcal{H}$. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be two infinite sequences of vectors in* $\mathcal{H}$.

Let $\mathcal{U} = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i : m_i, l \in N, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\}$ *and let*

$\mathcal{U}_1 = \left\{\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i : m_i, l \in N, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\}$. *Assume that* $\mathcal{U}_1 \subset \overline{\mathcal{U}}$

or $\mathcal{U}$ *is dense in* $\mathcal{H}$. *Then the following statements are equivalent.*

(1) There is an operator A *in* $Alg\mathcal{L}$ *such that* $Ax_j = y_j$ *for all* $j = 1, 2, \cdots$, $A^* = A$ *and every* E *in* $\mathcal{L}$ *reduces* A.

(2) $\sup\left\{\frac{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}y_i\|}{\|\sum_{k=1}^{m_i}\sum_{i=1}^{l}\alpha_{k,i}E_{k,i}x_i\|} : m_i, l \in N, E_{k,i} \in \mathcal{L} \text{ and } \alpha_{k,i} \in \mathbb{C}\right\} < \infty$ *and*

$< Ex_p, y_j >=< Ey_p, x_j >$ *for every* E *in* $\mathcal{L}$ *and all* p, j.

From Theorem 2, we can get the following theorem.

Theorem 13. *Let* $\mathcal{H}$ *be a Hilbert space and let* $\mathcal{L}$ *be a commutative subspace lattice on* $\mathcal{H}$. *Let* $x_1, \cdots, x_n$ *and* y *be vectors in* $\mathcal{H}$.

If $\sup\left\{\frac{\|\sum_{i=1}^{m}\alpha_i E_i y\|}{\sum_{k=1}^{n}\|\sum_{i=1}^{m}\alpha_i E_i x_k\|} : m \in N, E_i \in \mathcal{L} \text{ and } \alpha_i \in \mathbb{C}\right\} < \infty$ *and*

$< Ex_p, y >=< Ey, x_p >$ *for every* E *in* $\mathcal{L}$ *and all* $p = 1, 2, \cdots, n$, *then there are operators* $A_1, \cdots, A_n$ *in* $Alg\mathcal{L}$ *such that* $y = \sum_{k=1}^{n} A_k x_k$, $A_l^* = A_l$ *and every* E *in* $\mathcal{L}$ *reduces* A_l *for all* $l = 1, 2, \cdots, n$.

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