THE OPTIMAL MODEL WITH PASSENGER'S SATISFACTION OF RAILROAD

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Abstract. When railroad companies introduce new services such as express trains skipping stations, they generally did not have a logical methodology to determine the stations the express train should stop at. This paper proposes a mathematical model to determine the stations that the express train should stop, based on maximization of passenger's satisfaction level. Here, it is assumed that the degree of congestion in the trains represents the passenger's satisfaction. In this model, we discuss the condition of a specific station that some passengers can transfer to the express train. The numerical analysis is based on actual collected data and verifies the approximation of the model in real life situations.

1 Introduction

In general, the railroad company offers various services, such as rapid express and express services. Until quite recently, the rapid express service has not made an intermediate stop except neighborhood terminal station. But several railroad companies try to provide the new intermediate stop of rapid express train. Depending on the past experience and business result only, many railroad companies probably has made the decision of the number of stops. We will construct the models to determine whether this new service is good or bad. Passenger's satisfaction of railroad service is introduced into this model as the standard of optimization. The passenger's satisfaction here is strictly based upon the degree of congestion in a train. The seated customer is assumed to have high satisfaction with the service, and conversely, the unseated customer is assumed to have low satisfaction with the service. This mathematical model uses the function of passenger's satisfaction, subject to change by this degree of congestion, and applying it to all customers. This paper will introduce a function of passenger's satisfaction, subject to change by degree of congestion in a train, and attempts to make a contrast of the total satisfactions of all customers before and after the start of new rapid express service. Consequently, this result will determine whether this new service was good or bad, in terms of the passenger's satisfaction. By giving direct relationship between the number of customers making transfers to the new stop of the rapid express and the unit of distance away from the terminal station, it is possible to discuss the optimal stopping point for the new service.

2 Assumption of the railroad models

In general, railroad companies offer number of train services, e.g. rapid express, express, local services. This paper will focus on the case where rapid express service makes a brief stop on particular station. For simplification, this case only takes rapid express and express service into consideration. Also, the model only considers the service moving towards one particular terminal station. In urban city, there are several stations close to the terminal

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station, which will be abstracted with the terminal station into one group as the region Q, or terminal Q. The model I describes the state where the rapid express service train is not making an intermediate stop at station P. The station P is located in between the starting point and the terminal station. It is naturally considered that the station P is located in a satellite city of region Q. Their numbers of customers gotten on each express train are defined as follows:

α: the number of customers who has been on the rapid express train before station P
A: the number of customers who has been on express service before P and bound for Q
a: the number of customers who has been on express service before P,
and disembarked before Q
B: the number of customers who gets on express service at P and bound for Q
b: the number of customers who gets on express service that embarked on P,
and disembarked before Q

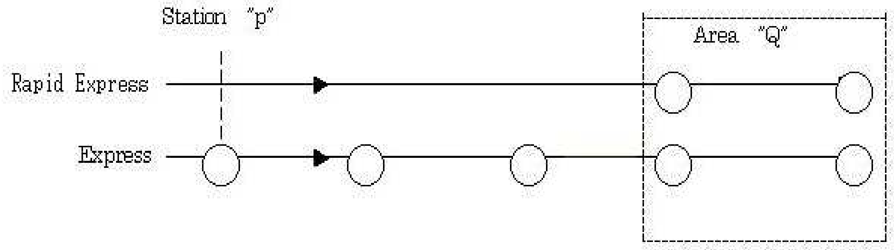


Figure 2-1: Model I

The assessment of the train service on the line may be represented by the accumulation of individual passenger's satisfaction. It is assumed that an individual satisfaction is determined by the degree of congestion and the riding time of a train. If x is the number of customers and t is the riding time on a train, satisfaction of a customer in the train is represented by $f_t(x)$. Figure that gives visual aid is as follows:

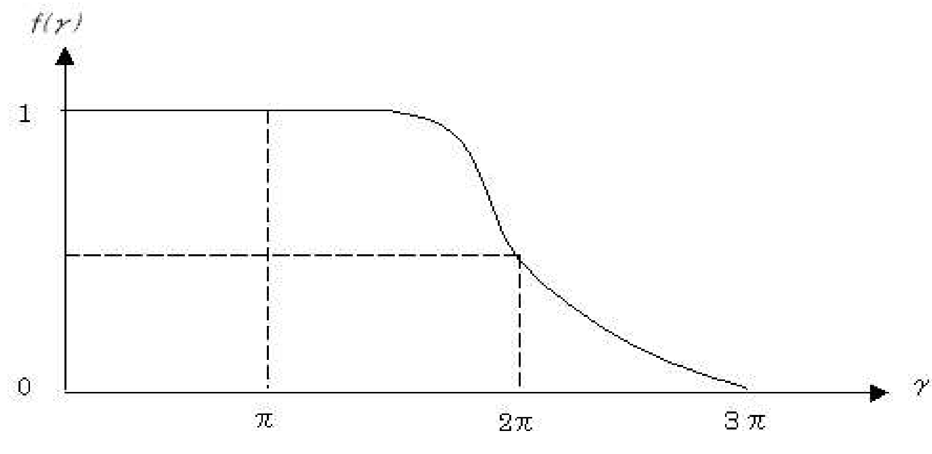


Figure 2-2: Satisfaction level

It is naturally considered that $f_t(x)$ is nonincreasing in x and t. If x is less than the number of seats, $f_t(x) = 1$, and if x is approximate to the maximum capacity of a train, $f_t(x) = 0$. In the rush hour, $f_t(x)$ may not be low if the riding time t is short. The sum of satisfactions of all passengers both a rapid express service and a express service in the model I, is put as ϕ. ϕ is calculated as follows:

$$\begin{aligned}\phi &= \min(\alpha, \pi) \times 1 + \max(\alpha - \pi, 0)f_t(\alpha)\\ &\quad + \min(a + b + A + B, \pi) \times 1 + \max(a + b + A + B - \pi, 0)f_t(a + b + A + B)\end{aligned}$$

In model I, it is introduced to the new service that a rapid express makes intermediate stop at station P.

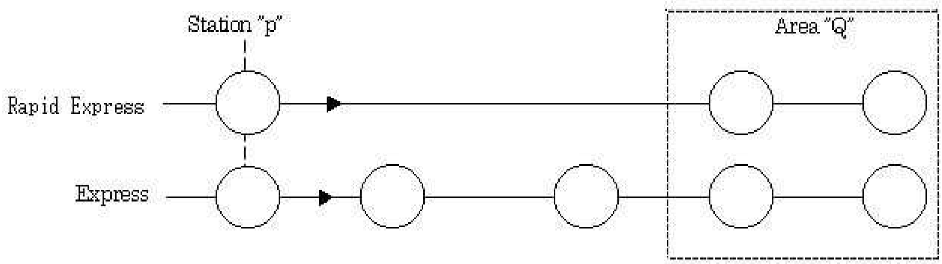


Figure 2-3: Model II

Model II is same under the assumptions of model I except for the new brief stop, therefore $f_t(x)$ can be directly used. However, the new stop at station P invites new situation where express service waits for the rapid express service to make transfer. In this case, it is assumed that the passenger on express services either makes transfer at station P, or balks the express service from station P to catch the later rapid express service. Under this premise, the sum of all passenger's satisfactions is put as ρ, which is calculated as follows.

$$\begin{aligned}\rho &= \min(\alpha + A + B, \pi) \times 1 + \max(\alpha + A + B - \pi, 0)f_t(\alpha + A + B)\\ &\quad + \min(a + b, \pi) \times 1 + \max(a + b - \pi, 0)f_t(a + b)\end{aligned}$$

For simplifying, let $a + b$ and $A + B$ be β and γ, respectively. β represents the number of passengers gotten on the express service from station P. γ represents the number of passengers that transfer from express service to the rapid express service at station P. Also, ϕ and ρ is considered as the function of γ. It is naturally considered that γ increases in proportion with the distance from the terminal Q. Hence, the formulae are transferred as follows:

$$\begin{aligned}\phi(\gamma) &= \min(\alpha, \pi) + \max(\alpha - \pi, 0)f_t(\alpha) + \min(\beta + \gamma, \pi) + \max(\beta + \gamma - \pi, 0)f_t(\beta + \gamma)\\ \rho(\gamma) &= \min(\alpha + \gamma, \pi) + \max(\alpha + \gamma - \pi, 0)f_t(\alpha + \gamma) + \min(\beta, \pi) + \max(\beta - \pi, 0)f_t(\beta)\end{aligned}$$

Now it is possible to compare the two models. By calculating the difference between the passenger's satisfaction level of $\phi(\gamma)$ and $\rho(\gamma)$, it is possible to determine whether the introduction of the new service of rapid express with intermediate stop at station P was a good or a bad decision. Hence, the difference can be calculated as follows:

$$\begin{aligned} M(\gamma) &= \rho(\gamma) - \phi(\gamma) \\ &= \max(\alpha+\gamma-\pi,0)f_t(\alpha+\gamma) - \max(\alpha-\pi,0)f_t(\alpha) + \min(\beta,\pi) - \min(\beta+\gamma,\pi) \\ &\quad + \max(\beta-\pi,0)f_t(\beta) - \max(\beta+\gamma-\pi,0)f_t(\beta+\gamma) + \min(\alpha+\gamma,\pi) - \min(\alpha,\pi) \end{aligned}$$

3 Analysis of two models

In actual situation, the degree of congestion is given by the following variables; maximum capacity, seating capacity, the actual passenger number, and the riding time on the train. If the number of passengers is less than the seating capacity π in one car, in which every passenger is able to take a seat, the passenger's satisfaction level is 1. In case where the number of passengers exceeds the passenger capacity in one car, for example in the rush hour, the limit of the passenger capacity is considered as 3π. Meanwhile, the passenger's satisfaction level simply decreases as the number of the passenger rises, i.e. the satisfaction level and the number of the passengers are inversely related. When the car is so crowded that passengers cannot more an inch, that is when the number of the passenger exceeds 3π, the customer satisfactory level is 0. Here, in order to simplify, we ignore t and use the linear function as follows;

$$f_t(x) = \begin{cases} 1 & if\, x \le \pi \\ -\frac{1}{2\pi}(x-3\pi) & if\, \pi \le x \le 3\pi \\ 0 & if\, 3\pi \le x \end{cases}$$

$M(\gamma)$is determined by α,β,γ,π. The analysis of the function $M(\gamma)$ evaluates whether the brief stop at station P of the rapid express service was good or bad. Furthermore, by finding the maximization of $M(\gamma)$, it is possible to determine the optimal intermediate stop for the rapid express. Here we will classify every situations depending the number of passengers and search the optimal γ which maximizes $M(\gamma)$:

(1) $\alpha \le \pi$, $\beta \le \pi$

$$\begin{aligned} M(\gamma) = \max\{\gamma-(\pi-\alpha),0\}f_t(\alpha+\gamma) - \max\{\gamma-(\pi-\beta),0\}f_t(\beta+\gamma) \\ + \beta - \min(\beta+\gamma,\pi) + \min(\alpha+\gamma,\pi) - \alpha \end{aligned}$$

(1.1) $\alpha \le \beta$

(i) $\gamma \le \pi - \beta$

$$M(\gamma) = \beta - (\beta+\gamma) + \alpha + \gamma - \alpha = 0$$

(ii) $\pi - \beta \le \gamma \le \pi - \alpha$

$$\begin{aligned} M(\gamma) &= -\{\gamma-(\pi-\beta)\}f_t(\beta+\gamma) + \beta - \pi + \gamma \\ &= \frac{1}{2\pi}\{\gamma-(\pi-\beta)\}^2 \end{aligned}$$

(iii) $\pi - \alpha \leq \gamma \leq 3\pi - \beta$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\}f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\}f_t(\beta + \gamma) + \beta - \pi + \pi - \alpha \\ &= -\tfrac{1}{2\pi}\{2(\alpha - \beta)\gamma + \alpha^2 - \beta^2 + (\alpha + \beta)\pi - (\alpha - \beta)3\pi\} + \beta - \alpha + \gamma \end{aligned}$$

(iv) $3\pi - \beta \leq \gamma \leq 3\pi - \alpha$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\}f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\}f_t(\beta + \gamma) + \beta - \pi + \pi - \alpha \\ &= -\tfrac{1}{2\pi}\{\gamma - (2\pi - \alpha)\}^2 - \alpha + \beta + \tfrac{1}{2}\pi \end{aligned}$$

(v) $3\pi - \alpha \leq \gamma$

$$M(\gamma) = \beta - \pi + \pi - \alpha = \beta - \alpha \geq 0$$

So, $M(\gamma)$ is presented graphically as Figure 3.1.

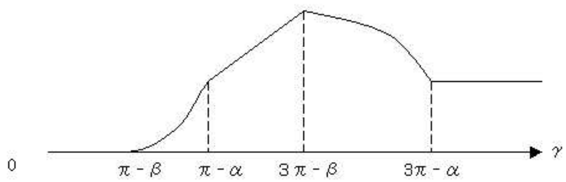


Figure 3.1

In this case, $\gamma = 3\pi - \beta$ is given to the value of maximization $M(\gamma) = -\frac{1}{2\pi}(\alpha^2 + \beta^2 + 4\pi\alpha - 4\pi\beta - 2\alpha\beta)$.

(1.2) $\beta \leq \alpha$

(i) $\gamma \leq \pi - \alpha$

$$M(\gamma) = \beta - (\beta + \gamma) + (\alpha + \gamma) - \alpha = 0$$

(ii) $\pi - \alpha \leq \gamma \leq \pi - \beta$

$$\begin{aligned} M(\gamma) &= -\{\gamma - (\pi - \alpha)\}f_t(\alpha + \gamma) + \beta - (\beta + \gamma) + \pi - \alpha \\ &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \alpha)\}\{\gamma - (3\pi - \alpha)\} - \alpha + \pi - \gamma \end{aligned}$$

(iii) $\pi - \beta \leq \gamma \leq 3\pi - \alpha$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\}f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\}f_t(\beta + \gamma) + \beta - \pi + \pi - \alpha \\ &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \alpha)\}\{\gamma - (3\pi - \alpha)\} - \alpha + \pi - \gamma \end{aligned}$$

(iv) $3\pi - \alpha \leq \gamma \leq 3\pi - \beta$

$$\begin{aligned} M(\gamma) &= -\{\gamma - (\pi - \beta)\}f_t(\beta + \gamma) + \beta - \pi + \pi - \alpha \\ &= \tfrac{1}{2\pi}\{\gamma - (2\pi - \beta)\}^2 - \tfrac{1}{2}\pi + \beta - \alpha \end{aligned}$$

(v) $3\pi - \beta \leq \gamma$

$$M(\gamma) = \beta - \pi + \pi - \alpha = \beta - \alpha \leq 0$$

So, $M(\gamma)$ is presented graphically as Figure 3.2.

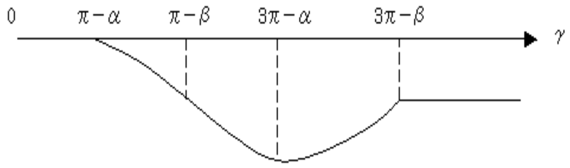


Figure 3.2

In this case, $0 \leq \gamma \leq \pi - \alpha$ is given to the value of maximization $M(\gamma) = 0$.

(2) $\alpha \leq \pi \leq \beta$

$$\begin{aligned} M(\gamma) = &\max\{\gamma - (\pi - \alpha), 0\} f_t(\alpha + \gamma) - \max\{\gamma - (\pi - \beta), 0\} f_t(\beta + \gamma) \\ &+ (\beta - \pi) f_t(\beta) + \min(\alpha + \gamma, \pi) - \alpha \end{aligned}$$

(2.1) $0 \leq \alpha \leq \pi \leq \beta \leq 2\pi + \alpha$

(i) $0 \leq \gamma \leq \pi - \alpha$

$$\begin{aligned} M(\gamma) &= -\{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) + (\beta - \pi) f_t(\beta) + \alpha + \gamma - \alpha \\ &= \tfrac{1}{2\pi}\{\gamma - (\pi - \beta)\}\{\gamma - (3\pi - \beta)\} - \tfrac{1}{2\pi}(\pi - \beta)(3\pi - \beta) + \gamma \end{aligned}$$

(ii) $\pi - \alpha \leq \gamma \leq 3\pi - \beta$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) + (\beta - \pi) f_t(\beta) + \pi - \alpha \\ &= \tfrac{1}{\pi}(\beta - \alpha)\gamma + \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) - \tfrac{1}{\pi}(\pi - \beta)(3\pi - \beta) + \pi - \alpha \\ &= \tfrac{1}{\pi}(\beta - \alpha)\gamma - \tfrac{1}{2\pi}(\pi - \alpha)^2 \end{aligned}$$

(iii) $3\pi - \beta \leq \gamma \leq 3\pi - \alpha$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) + (\beta - \pi) f_t(\beta) + \pi - \alpha \\ &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \alpha)\}\{\gamma - (3\pi - \alpha)\} - \tfrac{1}{2\pi}(\pi - \beta)(3\pi - \beta) + \pi - \alpha \\ &= -\tfrac{1}{2\pi}\{\gamma - (2\pi - \alpha)\}^2 - \alpha + 2\beta - \tfrac{1}{2\pi}\beta^2 \end{aligned}$$

(iv) $3\pi - \alpha \leq \gamma$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) + (\beta - \pi) f_t(\beta) + \pi - \alpha \\ &= -\tfrac{1}{2\pi}(\pi - \beta)(3\pi - \beta) + \pi - \alpha \geq 0 \end{aligned}$$

So, $M(\gamma)$ is presented graphically as Figure 3.3.

When $\pi \leq \beta - \alpha$

When $\pi \geq \beta - \alpha$

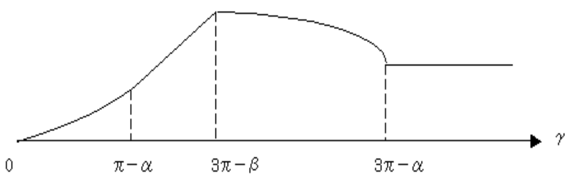


Figure 3.3

In this case, $\gamma = 2\pi - \alpha$ is given to the value of maximization $M(\gamma) = -\alpha + 2\beta - \frac{1}{2\pi}\beta^2$, if $\pi \leq \beta - \alpha$, $\gamma = 3\pi - \beta$ is given to the value of maximization $M(\gamma) = -\frac{1}{2\pi}(\pi^2 + \alpha^2 + 2\beta^2 + 4\pi\alpha - 6\pi\beta - 2\alpha\beta$, if $\pi \geq \beta - \alpha$.

(2.2) $0 \leq \alpha \leq \pi \leq 2\pi + \alpha \leq \beta \leq 3\pi$

(i) $0 \leq \gamma \leq 3\pi - \beta$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \beta)\}f_t(\beta + \gamma) + (\beta - \pi)f_t(\beta) + \alpha + \gamma - \alpha \\ &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \beta)\}\{\gamma - (3\pi - \beta)\} - \tfrac{1}{2\pi}(\pi - \beta)(3\pi - \beta) + \gamma \end{aligned}$$

(ii) $3\pi - \beta \leq \gamma \leq \pi - \alpha$

$$\begin{aligned} M(\gamma) &= -\{\gamma - (\pi - \beta)\}f_t(\beta + \gamma) + (\beta - \pi)f_t(\beta) + \alpha \\ &= -\tfrac{1}{2\pi}(\pi - \beta)(3\pi - \beta) + \gamma \end{aligned}$$

(iii) $\pi - \alpha \leq \gamma \leq 3\pi - \alpha$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\}f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\}f_t(\beta + \gamma) + (\beta - \pi)f_t(\beta) + \pi - \alpha \\ &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \alpha)\}\{\gamma - (3\pi - \alpha)\} - \tfrac{1}{2\pi}(\pi - \beta)(3\pi - \beta) + \pi - \alpha \\ &= -\tfrac{1}{2\pi}\{\gamma - (2\pi - \alpha)\}^2 - \alpha + 2\beta - \tfrac{1}{2\pi}\beta^2 \end{aligned}$$

(iv) $3\pi - \alpha \leq \gamma$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\}f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\}f_t(\beta + \gamma) + (\beta - \pi)f_t(\beta) + \pi - \alpha \\ &= -\tfrac{1}{2\pi}(\pi - \beta)(3\pi - \beta) + \pi - \alpha \geq 0 \end{aligned}$$

So, $M(\gamma)$ is presented graphically as Figure 3.4.

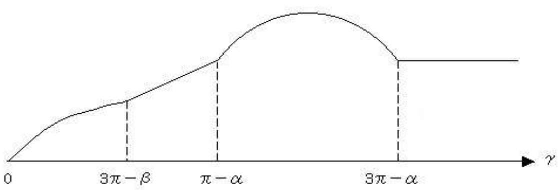


Figure 3.4

In this case, $\gamma = 2\pi - \alpha$ is given to the value of maximization $M(\gamma) = -\alpha + 2\beta - \frac{1}{2\pi}\beta^2$.

(2.3) $0 \le \alpha \le \pi \le 2\pi + \alpha \le 3\pi \le \beta$

(i) $0 \le \gamma \le \pi - \alpha$

$$M(\gamma) \ = \ \gamma \ge 0$$

(ii) $\pi - \alpha \le \gamma \le 3\pi - \alpha$

$$\begin{aligned} M(\gamma) &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \alpha)\}\{\gamma - (3\pi - \alpha)\} + \pi - \alpha \\ &= -\tfrac{1}{2\pi}\{\gamma - (2\pi - \alpha)\}^2 + \tfrac{3}{2}\pi - \alpha \end{aligned}$$

(iii) $3\pi - \alpha \le \gamma$

$$M(\gamma) \ = \ \pi - \alpha \ge 0$$

So, $M(\gamma)$ is presented graphically as Figure 3.5.

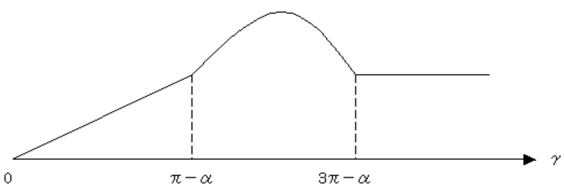


Figure 3.5

In this case, $\gamma = 2\pi - \alpha$ is given to the value of maximization $M(\gamma) = \frac{3}{2}\pi - \alpha$.

(3) $\beta \le \pi \le \alpha$

$$\begin{aligned} M(\gamma) = \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \max\{\gamma - (\pi - \beta), 0\} f_t(\beta + \gamma) \\ - (\alpha - \pi) f_t(\alpha) + \beta - \min(\beta + \gamma, \pi) \end{aligned}$$

(3.1) $\beta \leq \pi \leq \alpha \leq 2\pi + \beta$

(i) $0 \leq \gamma \leq \pi - \beta$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - (\alpha - \pi) f_t(\alpha) + \beta - (\beta + \gamma) \\ &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \alpha)\}\{\gamma - (3\pi - \alpha)\} - \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) - \gamma \end{aligned}$$

(ii) $\pi - \beta \leq \gamma \leq 3\pi - \alpha$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) - (\alpha - \pi) f_t(\alpha) + \beta - \pi \\ &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \alpha)\}\{\gamma - (3\pi - \alpha)\} + \tfrac{1}{2\pi}\{\gamma - (\pi - \beta)\}\{\gamma - (3\pi - \beta)\} \\ &\quad + \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) + \beta - \pi \end{aligned}$$

(iii) $3\pi - \alpha \leq \gamma \leq 3\pi - \beta$

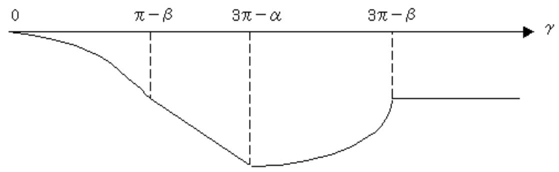
$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) - (\alpha - \pi) f_t(\alpha) + \beta - \pi \\ &= \tfrac{1}{2\pi}\{\gamma - (\pi - \beta)\}\{\gamma - (3\pi - \beta)\} + \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) + \beta - \pi \end{aligned}$$

(iv) $3\pi - \beta \leq \gamma$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) - (\alpha - \pi) f_t(\alpha) + \beta - \pi \\ &= \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) + \beta - \pi \leq 0 \end{aligned}$$

So, $M(\gamma)$ is presented graphically as Figure 3.6.

When $\pi \geq \alpha - \beta$



When $\pi \leq \alpha - \beta$

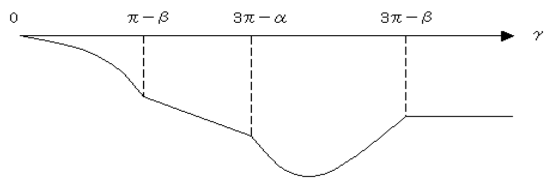


Figure 3.6

In this case, $\gamma = 0$ is given to the value of maximization $M(\gamma) = 0$.

(3.2) $\beta \leq \pi \leq 2\pi + \beta \leq \alpha \leq 3\pi$

(i) $0 \leq \gamma \leq 3\pi - \alpha$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) + (\pi - \alpha) f_t(\alpha) + \beta - (\beta + \gamma) \\ &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \alpha)\}\{\gamma - (3\pi - \alpha)\} + \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) - \gamma \end{aligned}$$

(ii) $3\pi - \alpha \leq \gamma \leq \pi - \beta$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - (\alpha - \pi) f_t(\alpha) + \beta - (\beta + \gamma) \\ &= \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) + \gamma \end{aligned}$$

(iii) $\pi - \beta \leq \gamma \leq 3\pi - \beta$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) - (\alpha - \pi) f_t(\alpha) + \beta - \pi \\ &= \tfrac{1}{2\pi}\{\gamma - (\pi - \beta)\}\{\gamma - (3\pi - \beta)\} - \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) + \beta - \pi \end{aligned}$$

(iv) $3\pi - \beta \leq \gamma$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) - (\alpha - \pi) f_t(\alpha) + \beta - \pi \\ &= -\tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) + \beta - \pi \leq 0 \end{aligned}$$

So, $M(\gamma)$ is presented graphically as Figure 3.7.

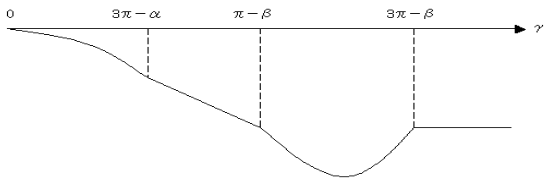


Figure 3.7

In this case, $\gamma = 0$ is given to the value of maximization $M(\gamma) = 0$.

(3.3) $\beta \leq \pi \leq 3\pi \leq \alpha$

(i) $0 \leq \gamma \leq \pi - \beta$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - (\alpha - \pi) f_t(\alpha) + \beta - (\beta + \gamma) \\ &= -\gamma \leq 0 \end{aligned}$$

(ii) $\pi - \beta \leq \gamma \leq 3\pi - \beta$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) - (\alpha - \pi) f_t(\alpha) + \beta - \pi \\ &= \tfrac{1}{2\pi}\{\gamma - (\pi - \beta)\}\{\gamma - (3\pi - \alpha)\} + \beta - \pi \end{aligned}$$

(iii) $3\pi - \alpha \leq \gamma$

$$\begin{aligned} M(\gamma) &= \{\gamma - (\pi - \alpha)\} f_t(\alpha + \gamma) - \{\gamma - (\pi - \beta)\} f_t(\beta + \gamma) - (\alpha - \pi) f_t(\alpha) + \beta - \pi \\ &= \beta - \pi \leq 0 \end{aligned}$$

So, $M(\gamma)$ is presented graphically as Figure 3.8.

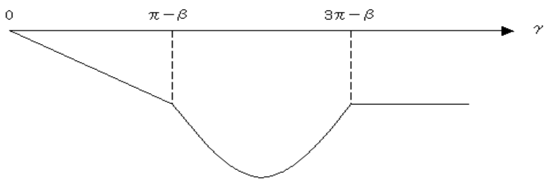


Figure 3.8

In this case, $\gamma = 0$ is given to the value of maximization $M(\gamma) = 0$.

(4) $\pi \leq \alpha$,$\pi \leq \beta$

$$M(\gamma) \;=\; \{\gamma-(\pi-\alpha)\}f_t(\alpha+\gamma)-\{\gamma-(\pi-\beta)\}f_t(\beta+\gamma)-(\pi-\beta)f_t(\beta)+(\pi-\alpha)f_t(\alpha)$$

(4.1) $\alpha \leq \beta \leq 3\pi$
(i) $0 \leq \gamma \leq 3\pi - \beta$

$$\begin{aligned} M(\gamma) &= -\tfrac{1}{2\pi}\{\gamma-(\pi-\alpha)\}\{\gamma-(3\pi-\alpha)\}+\tfrac{1}{2\pi}\{\gamma-(\pi-\beta)\}\{\gamma-(3\pi-\beta)\} \\ &= +\tfrac{1}{2\pi}(\pi-\alpha)(3\pi-\alpha)-\tfrac{1}{2}(\pi-\beta)(3\pi-\beta) \\ &= -\tfrac{1}{\pi}\{\beta-\alpha)\gamma \geq 0 \end{aligned}$$

(ii) $3\pi - \beta \leq \gamma \leq 3\pi - \alpha$

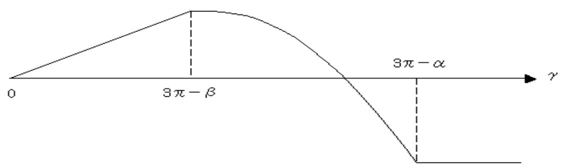
$$\begin{aligned} M(\gamma) &= -\tfrac{1}{2\pi}\{\gamma-(\pi-\alpha)\}\{\gamma-(3\pi-\alpha)\}+\tfrac{1}{2\pi}(\pi-\alpha)(3\pi-\alpha)-\tfrac{1}{2\pi}(\pi-\beta)(3\pi-\beta) \\ &= -\tfrac{1}{2\pi}\{\gamma-(2\pi-\alpha)\}^2+\tfrac{1}{2\pi}(\pi^2+\alpha^2-\beta^2-4\pi\alpha+4\pi\beta) \end{aligned}$$

(iii) $3\pi - \beta \leq \gamma$

$$M(\gamma) \;=\; -\frac{1}{2\pi}(\alpha-\beta)(4\pi-\alpha-\beta)$$

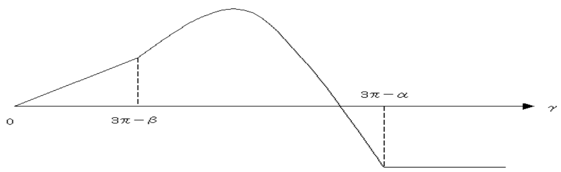
So, $M(\gamma)$ is presented graphically as Figure 3.9.

When $\beta - \alpha \leq \pi$ and $4\pi \leq \alpha + \beta$



When $\beta - \alpha \le \pi$ and $4\pi \ge \alpha + \beta$

When $\beta - \alpha \ge \pi$ and $4\pi \le \alpha + \beta$



When $\beta - \alpha \ge \pi$ and $4\pi \ge \alpha + \beta$

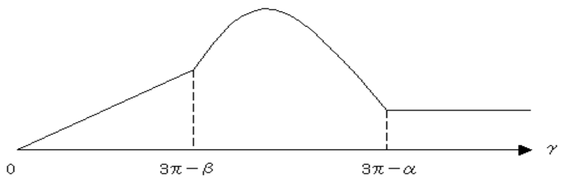


Figure 3.9

In this case, $\gamma = 3\pi - \beta$ is given to the value of maximization $M(\gamma) = \frac{1}{\pi}(-\beta^2 - 3\pi\alpha + 3\pi\beta + \alpha\beta)$ if $\beta - \alpha \le \pi$,$\gamma = 2\pi - \alpha$ is given to the value of maximization $M(\gamma) = \frac{1}{2\pi}(\pi^2 + \alpha^2 - \beta^2 - 4\pi\alpha + 4\pi\beta)$ if $\beta - \alpha \le \pi$.

(4.2) $\alpha \le 3\pi \le \beta$

(i) $0 \le \gamma \le 3\pi - \alpha$

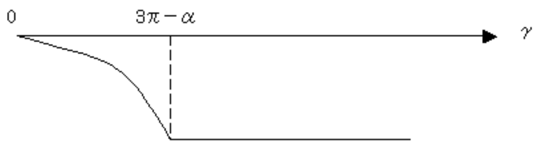
$$\begin{aligned} M(\gamma) &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \alpha)\}\{\gamma - (3\pi - \alpha)\} + \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) \\ &= -\tfrac{1}{2\pi}\{\gamma - (2\pi - \alpha)\}^2 + 2\pi - 2\alpha + \tfrac{1}{2\pi}\alpha^2 \end{aligned}$$

(ii) $3\pi - \alpha \le \gamma$

$$M(\gamma) = \frac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) \le 0$$

So, $M(\gamma)$ is presented graphically as Figure 3.10.

When $2\pi \leq \alpha$



When $\alpha \leq 2\pi$

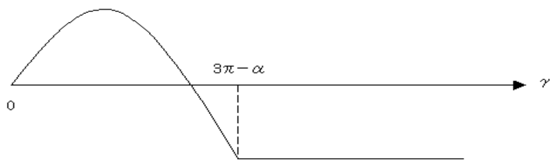


Figure 3.10

In this case, $\gamma = 2\pi - \alpha$ is given to the value of maximization $M(\gamma) = \frac{1}{2\pi}(2\pi - \alpha)^2$.

(4.3) $3\pi \leq \alpha \leq \beta$

$$M(\gamma) = 0$$

So, $M(\gamma)$ is presented graphically as Figure 3.11.

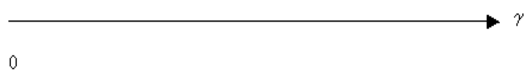


Figure 3.11

In this case, the value of maximization is $M(\gamma) = 0$D

(4.4) $\beta \leq \alpha \leq 3\pi$
(i) $0 \leq \gamma \leq 3\pi - \alpha$

$$\begin{aligned} M(\gamma) &= -\tfrac{1}{2\pi}\{\gamma - (\pi - \alpha)\}\{\gamma - (3\pi - \alpha)\} + \tfrac{1}{2\pi}\{\gamma - (\pi - \beta)\}\{\gamma - (3\pi - \beta)\} \\ &\quad + \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) - \tfrac{1}{2\pi}(\pi - \beta)(3\pi - \beta) \\ &= -\tfrac{1}{\pi}(\alpha - \beta)\gamma - \tfrac{1}{\pi}(\alpha - \beta)(\alpha + \beta - 4\pi) \end{aligned}$$

(ii) $3\pi - \alpha \leq \gamma \leq 3\pi - \beta$

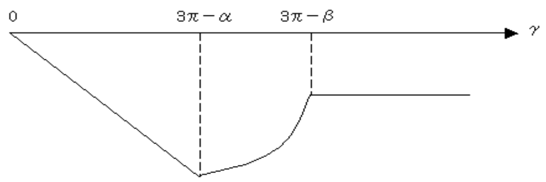
$$\begin{aligned} M(\gamma) &= \tfrac{1}{2\pi}\{\gamma - (\pi - \beta)\}\{\gamma - (3\pi - \beta)\} + \tfrac{1}{2\pi}(\pi - \alpha)(3\pi - \alpha) - \tfrac{1}{2\pi}(\pi - \beta)(3\pi - \beta) \\ &= -\tfrac{1}{2\pi}\{\gamma - (2\pi - \beta)\}^2 - \tfrac{1}{2\pi}(\pi^2 + \alpha^2 - \beta^2) \end{aligned}$$

(iii) $3\pi - \beta \leq \gamma$

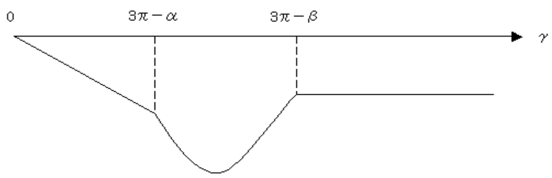
$$\begin{aligned} M(\gamma) &= \tfrac{1}{2\pi}(\pi-\alpha)(3\pi-\alpha) - \tfrac{1}{2\pi}(\pi-\beta)(3\pi-\beta) \\ &= -\tfrac{1}{2\pi}(\alpha-\beta)(4\pi-\alpha-\beta) \end{aligned}$$

So, $M(\gamma)$ is presented graphically as Figure 3.12.

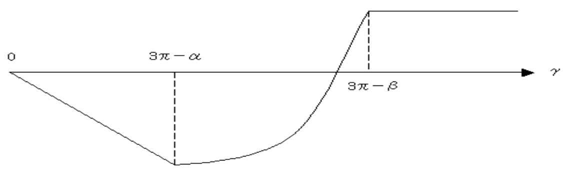
When $4\pi > \alpha + \beta$ and $\alpha - \beta \leq \pi$



When $4\pi > \alpha + \beta$ and $\alpha - \beta \geq \pi$



When $4\pi \leq \alpha + \beta$ and $\alpha - \beta \leq \pi$



When $4\pi \leq \alpha + \beta$ and $\alpha - \beta \geq \pi$

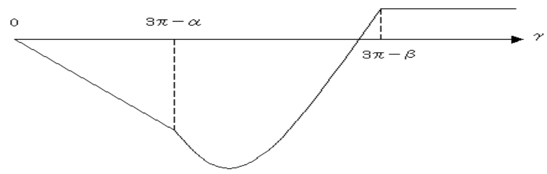


Figure 3.12

In this case, $\gamma = 2\pi - \beta$ is given to the value of maximization $M(\gamma) = -\frac{1}{2\pi}(\alpha-\beta)(4\pi-\alpha-\beta)$.

(4.5) $\beta \leq 3\pi \leq \alpha$

(i) $0 \leq \gamma \leq 3\pi - \beta$

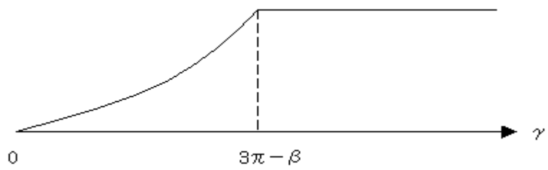
$$\begin{aligned} M(\gamma) &= \tfrac{1}{2\pi}\{\gamma-(\pi-\beta)\}\{\gamma-(3\pi-\beta)\} - \tfrac{1}{2\pi}(\pi-\beta)(3\pi-\beta) \\ &= \tfrac{1}{2\pi}\{\gamma-(2\pi-\beta)\}^2 - \tfrac{1}{2\pi}(2\pi^2-4\pi\beta+\beta^2) \end{aligned}$$

(ii) $3\pi - \beta \leq \gamma$

$$M(\gamma) = -\frac{1}{2\pi}(\pi-\beta)(3\pi-\beta) \geq 0$$

So, $M(\gamma)$ is presented graphically as Figure 3.13.

When $2\pi \leq \beta$



When $2\pi \geq \beta$

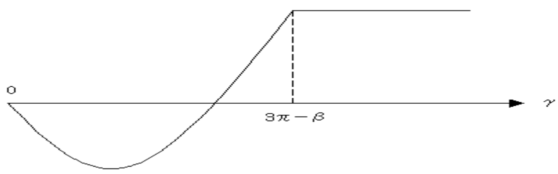


Figure 3.13

In this case, $\gamma = 3\pi - \beta$ is given to the value of maximization $M(\gamma) = -\frac{1}{2\pi}(\pi-\beta)(3\pi-\beta)$.

(4.6) $3\pi \leq \beta \leq \alpha$

$$M(\gamma) = 0$$

So, $M(\gamma)$ is presented graphically as Figure 3.14.

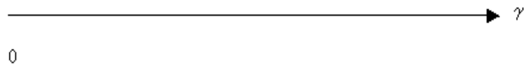


Figure 3.14

In this case, the value of maximization is $M(\gamma) = 0$.

This classification shows all possible situations. Analysis can be still made even in the second model where the rapid express service stops at station P where the passengers make transfers, by the following process. Even though the flow of the passengers in the former model is unknown, $M(\gamma)$ can by calculated by providing the value for α, β, γ. Furthermore, the classification table above gives numbers that determine the optimal decision. When the value of $M(\gamma)$ is positive, it signifies that introducing an intermediate stop at station P makes total passenger's satisfaction greater. Conversely, when the value of $M(\gamma)$ is negative, it signifies that the new service results in smaller total passenger's satisfaction. Therefore, by giving the fixed value for π and conducting a research for the value of α, β, γ, it is possible to find the dynamism of the passenger's satisfaction level.

Let's compare the maximum value of $M(\gamma)$ in ten situations where it's value exceeds zero. From the ten Figure s where $M(\gamma)$ exceeds zero, these $M(\gamma)$ are denoted as $M_A(\gamma)$‘$M_J(\gamma)$. Each of the minimum and maximum value is shown below.

(1.1) $0 \leq M_A(\gamma) \leq \frac{3}{2}\pi$
(2.1)-(i) $\frac{1}{2}\pi \leq M_B(\gamma) \leq 2\pi$
(2.1)-(ii) $0 \leq M_C(\gamma) \leq 2\pi$
(2.2) $\frac{1}{2}\pi \leq M_D(\gamma) \leq 2\pi$
(2.3) $\frac{1}{2}\pi \leq M_E(\gamma) \leq \frac{3}{2}\pi$
(4.1)-(i) $M_F(\gamma) = 0$
(4.1)-(ii) $0 \leq M_G(\gamma) \leq \frac{1}{2}\pi$
(4.2) $0 \leq M_H(\gamma) \leq 2\pi$
(4.4) $0 \leq M_I(\gamma) \leq \frac{1}{2}\pi$
(4.5) $-\frac{3}{2}\pi \leq M_J(\gamma) \leq \frac{1}{2}\pi$

Therefore, the value of the $M(\gamma)$ lies between$-\frac{3}{2}\pi \leq M(\gamma) \leq 2\pi$, and the maximum value of $M(\gamma)$ given that,

$0 \leqq \alpha \leqq \pi \leqq \beta \leqq 2\pi + \alpha$ and $\pi \leq \beta - \alpha$: $\alpha = 0$ and $\beta = 2\pi$
or $0 \leq \alpha \leq \pi \leq \beta \leq 2\pi + \alpha$ and $\beta - \alpha \leq \pi$: $\alpha = 0$ and $\beta = 2\pi$
or $0 \leq \alpha \leq \pi \leq 2\pi + \alpha \leq \beta \leq 3\pi$: $\alpha = 0$ and $\beta = 2\pi$
or $0 \leq \alpha \leq 3\pi \leq \beta$: $\alpha = 0$

is 2π in any cases. Therefore, it is now possible to calculate the condition in which the total passenger's satisfaction level is maximized. However, α and β is already known. It is not feasible to bring the passenger's satisfaction level to 2π in all cases, but it is possible to sustain the level in ($M(\gamma) = 0$) or to improve it.

Also, the number of transfers from the rapid express to the express at station P, and the distance from station P from terminal Q is in direct relationship. Therefore, by calculating the optimal value for γ, it is also possible to determine the optimal distance between station P and terminal Q. It is now apparent that in finding the best intermediate stop for the rapid express service, the experience and the instinct of the executive managers are no longer are dominant, but the calculation can provide a reasonable solution as well. Now this method must be verified using the real case.

4 Example

There is an actual case similar to which was discussed in the chapter 2 and 3. A large private railroad company in a large city has changed its railway schedule, and the rapid express service now makes a new stop. This new stop is considered station P. In the situation where the rapid express service is moving towards an urban city, this urban city is considered terminal Q. In this new schedule, the express service waits for the rapid express service that takes over at station P. That is to say, at station P the express service arrives at the station earlier, waits for the rapid express service to arrive so the passengers can make transfers, then the rapid express service departs from station P, and at last the express service too departs from the station. In this case, the research has been focused on a single car of the train, and actually collected numbers for the variables α,β, and γ. The research was done 12 times, and assumed that $\pi = 60$. The value of $M(\gamma)$ in each of the situation is calculated and summarized as the table below.

No.	α	β	γ	$M(\gamma)$
1	35	27	21	0
2	30	17	36	-0.3
3	47	36	30	-2.108
4	37	10	24	-0.008
5	40	11	19	0
6	47	11	19	-0.3
7	39	12	26	-0.208
8	46	9	20	-0.3
9	29	11	24	0
10	34	9	18	0
11	36	5	17	0
12	40	20	26	-0.3
Average	38.333	14.833	23.333	-0.023

Table 4.1

As a result, in every case the value of $M(\gamma)$ has either been equal or less than zero. That is by introducing a new stop at station P has resulted in less total passenger's satisfaction. The average of the 12 situations too has resulted in negative value. In overall, it may be argued that the brief stopping at station P was not the optimal decision. Next, this paper will attempt to calculate and determine a station that provides better result.

This calculation will focus on the worst result taken, where $\alpha = 47$, $\beta = 36$, and $\gamma = 30$ under the condition where $\pi = 60$. The relationship among the three valuable is $\alpha \leq \pi$C$\beta \leq \pi$Cand $\beta \leq \alpha$, therefore it is possible to classify this under (1-2) of the previous chapter. It is confirmed that the maximum value for $M(\gamma)$ under condition where

$0 \leq \gamma \leq \pi - \alpha$ is 0. Under this condition, $M(\gamma)$ will never be positive. This in result has made the passenger's satisfaction level into 0 or negative value, and the stop at station P has dissatisfied the passengers. In order to bring the optimal situation where $M(\gamma) = 0$, it is inevitable for the condition $0 \leq \gamma \leq \pi - \alpha$ to be met, meaning $0 \leq \gamma \leq 13$ must be fulfilled for an optimal result. However, in reality this variable number of transfer is $\gamma = 30$, high above the constraint. This situation may represent a case where too many passengers have been on the rapid express before station P, and the number of the passenger has increased at the new stop, causing heavy crowd in the train. On the other hand, the number of the passengers on the express service has reduced due to the transfer, and the satisfaction level has not changed in the express service. Given that the satisfaction level on the express service has stayed 0, the overall satisfaction level of both rapid express and express is negative. In order to bring up the satisfaction level, there must be some measure to reduce the number of transfers from the express. Consequently, the stop of the rapid express should be canceled, or otherwise make an intermediate stop somewhere closer to terminal Q. If these measures cannot be taken, it is inevitable for the railroad company to increase the seating capacity, by increasing the car number of the train or the service itself.

However, there are many other constraints that must be considered when the rapid express makes a new intermediate stop, and there is a risk of relying too much on the simple calculation. There are complex factors behind the actual decision making of the railroad company, e.g. environment around the station, infrastructure of the station, the congestion of the station itself, the population around the station, pressure from the local self-governing body, and the request from the passengers. This paper has proposed a mathematical method by creating a model as a means of decision making method. However, there are many other elements to be considered, and a managerial executive must always make the final decision.

5 Conclusion

This paper would like to suggest to the management of the railroad company that a decision making method of Operations Research applies to railroad service. It is an attempt to evaluate a stopping point of the train service under condition where various services exist, in terms of passenger's satisfaction level. Furthermore, this paper has also argued that the optimal stopping station can be found by calculation. However, there are many constraints crucial in such decision making that cannot be put into mathematical expression. It is also necessary to consider the fact that this calculation has ignored the capacity difference of the individual car. Furthermore, duration of the service, price of the service, and environmental factors that are influential to the passenger's satisfaction needs to be considered in the future, which requires more adjustments. It is of great interest, to construct a model that is closer to the actual model, and that this research will contribute to some degree for the development of the railroad system in the future.

References

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[2] Masaki Kobayashi and Koyu Uematsu, "Construction of a New Model for Rail Service and its Analysis II," *OIU Journal of International Studies*, Vol.14, Extra issue, pp. 49-58, (2001).

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