PROPERTIES OF A RUDIN'S ORTHOGONAL FUNCTION WHICH IS A LINEAR COMBINATION OF TWO INNER FUNCTIONS

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ABSTRACT. ϕ is called a Rudin's (orthogonal) function if ϕ is a function in H^∞ and the different nonnegative powers of ϕ are orthogonal in H^2. When ϕ is a multiple of an inner function and $\phi(0) = 0$, ϕ is a Rudin's function. Sundberg and Bishop showed that a Rudin's function is not necessarily a multiple of an inner function. We study a Rudin's function which is a linear combination of two inner functions or a polynomial of an inner function.

§1. Introduction.

For $1 \le p \le \infty$, H^p denotes the usual Hardy space on the unit circle T and σ is a normalized Lebesgue measure on T. ϕ is called a Rudin's (orthogonal) function if ϕ is a function in H^∞ and the different nonnegative powers of ϕ are orthogonal in H^2. Two inner functions f and g are called (statistically) independent if

$$\int_T f^\ell \bar{g}^s d\sigma = 0$$

for all nonnegative integers $\ell, s, |\ell| + |s| > 0$. W. Rudin posed two problems (see [3]) : R1. Is a Rudin's (orthogonal) function necessarily a multiple of an inner function? R2. Do there exist two (statistically) independent inner functions? R1 has been negatively solved by C. Sundberg [6] and C. Bishop [1]. It is shown in [3] that if R1 is valid, then so is R2. We don't know whether the converse is true or not. Under some conditions, R1 has been positively solved. That is, when ϕ is a univalent function [2] and ϕ is in the disc algebra with boundary function in Lip α for some $\alpha > 1/2$ [3].

In §2, we give a few properties two (statistically) independent inner functions must satisfy. In §3, we study a Rudin's (independent) function which is a linear combination of two inner functions. In §4, we show that a Rudin's (orthogonal) function is a multiple of an inner function when it is a polynomial of an inner function.

§2. Statistically independent inner functions

In this section, we give a few properties two (statistically) independent inner functions must satisfy.

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Proposition 1. *If f and g are independent inner functions, then neither f nor g is a finite Blashke product.*

Proof. Let $H^2[g]$ be the closure of the set of all polynomials of g in H^2, then $f^j H^2[g]$ is orthogonal to $f^\ell H^2[g]$ if $j \neq \ell$ because f and g are independent inner functions. Set

$$M = \sum_{j=0}^{\infty} \oplus f^j H^2[g],$$

then M is a closed invariant subspace of H^2 under the multiplication by f. It is known (cf. [5, p11]) that the dimension of $M \ominus fM$ is less than or equal to that of $H^2 \ominus fH^2$. This implies that f can not be a finite Blaschke product.

Proposition 2. *If f and g are independent inner functions, then the set of all polynomials of f and g is not dense in H^2.*

Proof. Suppose the set of all polynomials of f and g is dense in H^2. Then

$$H^2 = \sum_{j=0}^{\infty} \oplus f^j H^2[g] = H^2[g] \oplus fH^2$$

because f and g are independent. Hence for any $n \geq 1$, g^n is orthogonal to fH^2 and so $f\bar{g}^n \in H_0^2 = L^2 \ominus \bar{H}^2$. Therefore $f = 0$ because $f \in \bigcap_{n=0}^{\infty} g^n H^2 = \{0\}$. This contradiction shows the proposition.

Proposition 3. *If f and g are independent inner functions, then there exists a positive integer n such that $z^n f$ and $z^n g$ are not independent.*

Proof. Suppose $z^n f$ and $z^n g$ are independent for any positive integer n. For any integer $\ell \geq 1$, f^ℓ is orthognal to $z^n g^{\ell+1}$ for all $n \geq 0$. Put $\phi = \bar{f} g$, then $\phi^\ell g$ belongs to H^2 for all integer $\ell \geq 0$. By [3, p177], ϕ belongs to H^2. Similarly we can show that $\bar{\phi}$ belongs to H^2. Therefore ϕ is constant and so $g = \alpha f$ for some constant α with absolute value 1. This contradicts that f and g are independent.

§3. A linear combination of two inner functions

Let f and g be inner functions with $f(0) = g(0) = 0$ and let a and b complex numbers. Put

$$\phi(a, b) = af + bg.$$

If f and g are independent, then for an arbitrary pair (a, b), $\phi(a, b)$ is a Rudin's function. Theorem 4 shows that the converse is not true formally.

Lemma 1. *For some pair (a, b), if $\phi(a, b)$ is a Rudin's function, then for any positive integer $\ell \geq 2$*

$$b^\ell \bar{a} \int_T \bar{f} g^\ell d\sigma + a^\ell \bar{b} \int_T f^\ell \bar{g} d\sigma = 0.$$

Proof. Since $(\phi^\ell, \phi) = 0$ for $\ell \geq 2$,

$$\sum_{j=0}^{\ell} \binom{\ell}{j} a^{\ell-j} \bar{a} b^j \int_T f^{\ell-j-1} g^j d\sigma + \sum_{j=0}^{\ell} \binom{\ell}{j} a^{\ell-j} b^j \bar{b} \int_T f^{\ell-j} g^{j-1} d\sigma = 0$$

because $\phi^\ell = \sum_{j=0}^{\ell} \binom{\ell}{j} a^{\ell-j} b^j f^{\ell-j} g^j$. If $1 \leq j \leq \ell-1$, then $f^{\ell-j-1} g^j \in H^\infty$ and $f^{\ell-j} g^{j-1} \in H^\infty$. This implies that $\int_T f^{\ell-j-1} g^j d\sigma = \int_T f^{\ell-j} g^{j-1} d\sigma = 0$ because $f(0) = g(0) = 0$. Therefore

$$\bar{a} b^\ell \int_T \bar{f} g^\ell d\sigma + a^\ell \bar{b} \int_T f^\ell \bar{g} d\sigma = 0.$$

Corollary 1. *For some pair (a, b), if $\phi(a, b) = az + bg$ is a Rudin's function where g is an inner function with $g(0) = 0$, then $a = 0$, $b = 0$ or $g = cz$ for some complex number c.*

Proof. By Lemma 1,

$$a^\ell \bar{b} \int_T z^\ell \bar{g} d\sigma = 0 \quad (\ell \geq 2)$$

and so if $a^\ell \bar{b} \neq 0$, then $g = cz$ for some complex number c.

Theorem 4. *For arbitrary pair (a, b), $\phi(a, b)$ is a Rudin's function if and only if*

$$\int_T \bar{f}^s g^t d\sigma = \int_T f^t \bar{g}^s d\sigma = 0$$

for all non-negative integers t, s, $t \geq s+1$.

Proof. When $\ell > k$,

$$\begin{aligned}
(\phi^\ell, \phi^k) &= \left(\sum_{j=0}^{\ell} \binom{\ell}{j} a^{\ell-j} b^j f^{\ell-j} g^j, \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i f^{k-i} g^i \right) \\
&= \sum_{j=0}^{t} \sum_{i=0}^{k} \binom{\ell}{j} \binom{k}{i} a^{\ell-j} \bar{a}^{k-i} b^j \bar{b}^i \int_T f^{\ell-k+i-j} g^{j-i} d\sigma \\
&= \sum_{-k \leq j-i < 0} \sum F(\ell, k, i, j) + \sum_{\ell-k < j-i \leq \ell} \sum F(\ell, k, i, j)
\end{aligned}$$

where $F(\ell, k, i, j) = \binom{\ell}{j} \binom{k}{i} a^{\ell-j} \bar{a}^{k-i} b^j \bar{b}^i \int_T f^{\ell-k+i-j} g^{j-i} d\sigma$. Because if $j - i = 0$ or $0 \leq j - i \leq \ell - k$, then $f^{\ell-k+i-j} g^{j-i} \in H^\infty$ and so $F(\ell, k, i, j) = 0$. In the last line, note the following. When $-k \leq j - i < 0$, $t = \ell - k + i - j \underset{\neq}{>} s$ if $s = -(j-i)$. When $\ell - k \underset{\neq}{<} j - i \leq \ell$, $t = j - i \underset{\neq}{>} s$ if $s = -(\ell - k + i - j)$. Hence

$$\int_T f^{\ell-k+i-j} g^{j-i} d\sigma = \int_T f^t \bar{g}^s d\sigma \quad \text{or} \quad \int_T \bar{f}^s g^t d\sigma$$

where $t \geq s+1$.

The 'if' part of the theorem is clear by the fact noted above. We will prove the 'only if' part by induction about s. If $s=1$, by Lemma 1 it holds because (a,b) is arbitrary. Suppose it holds for $2 \leq s \leq n-1$, that is, for $t \geq s+1$

$$\int_T \bar{f}^s g^t d\sigma = \int_T f^t \bar{g}^s d\sigma = 0.$$

Since ϕ is a Rudin's function, $(\phi^\ell, \phi^n) = 0$ if $\ell > n$. By the fact noted above,

$$(\phi^\ell, \phi^n) = \sum_{-n \leq j-i<0} \sum F(\ell,n,i,j) + \sum_{\ell-n<j-i\leq \ell} \sum F(\ell,n,i,j).$$

By the induction hypothesis,

$$\begin{aligned}(\phi^\ell, \phi^n) &= \sum_{j-i=-n} \sum F(\ell,n,i,j) + \sum_{j-i=\ell} \sum F(\ell,n,i,j) \\ &= F(\ell,n,n,0) + F(\ell,n,0,\ell).\end{aligned}$$

In fact, when $-(n-1) \leq j-i<0$, $t = \ell-n+i-j \gneq s = -(j-i)$ and so $F(\ell,n,i,j)=0$. When $\ell - n \lneq j-i \leq -1$, $t = j-i \gneq s = -(\ell-n+i-j) \leq n-1$ and so $F(\ell,n,i,j)=0$. Thus

$$\binom{\ell}{0}\binom{n}{n} a^\ell \bar{a}^0 b^0 \bar{b}^n \int_T f^\ell \bar{g}^n d\sigma + \binom{\ell}{\ell}\binom{n}{0} a^0 \bar{a}^n b^\ell \bar{b}^0 \int_T \bar{f}^n g^\ell d\sigma = 0.$$

Since (a,b) is arbitrary,

$$\int_T f^\ell \bar{g}^n d\sigma = \int_T \bar{f}^n g^\ell d\sigma = 0.$$

Question. *If $\phi(a,b) = af+bg$ is a Rudin's function, then are f and g necessarily independent inner functions?*

Theorem 5. *Let q and Q be inner functions with $q(0)=0$ and $Q(0) \neq 0$. If $f=q^s$ and $g=q^m Q$ where $m \geq s+1$, $s \geq 1$ and $\phi(a,b)=af+bg$ is a Rudin's function, then $a=0$ or $b=0$.*

Proof. We will prove the following claim :

$$a^\ell \bar{b}^{\ell-k} \int_T f^\ell \bar{g}^{\ell-k} d\sigma = 0 \quad (\ell \geq k).$$

Put $k = m-s$ and $\ell = m$, then

$$\int_T f^\ell \bar{g}^{\ell-k} d\sigma = \int_T q^{s\ell - m(\ell-k)} \bar{Q}^{\ell-k} d\sigma = \int_T \bar{Q}^s d\sigma \neq 0$$

because $s\ell - m(\ell-k) = 0$. This implies $a^\ell \bar{b}^{\ell-k} = 0$.

We will show the claim by induction about ℓ. When $\ell = k$, it is clear because $f(0)=0$. Suppose it holds for $k \leq \ell \leq n-1$. Since ϕ is a Rudin's function,

$$(\phi^n, \phi^{n-k}) = \sum_{j=0}^{n} \sum_{i=0}^{n-k} \binom{n}{j}\binom{n-k}{i} a^{n-j} \bar{a}^{n-k-i} b^j \bar{b}^i \int_T f^{n-(n-k)+i-j} \bar{g}^{i-j} d\sigma = 0.$$

When $i-j \le 0$,

$$\int_T f^{n-(n-k)+i-j}\bar{g}^{i-j}d\sigma = 0$$

because

$$f^{n-(n-k)+i-j}g^{j-i} = q^{s\{n-(n-k)+i-j\}}\bar{q}^{m(i-j)}Q^{j-i} = q^{sk+(m-s)(j-i)}Q^{j-i}.$$

When $0 \le i-j \le (n-1)-k$, by the induction hypothesis,

$$\int_T f^{n-(n-k)+i-j}\bar{g}^{i-j}d\sigma = 0.$$

If we put $i-j = s-k$ and $k \le s \le n-1$, then $n-(n-k)+i-j = s$. Thus

$$\begin{aligned} 0 &= (\phi^n, \phi^{n-k}) \\ &= \sum_{j=0}^{n}\sum_{i=0}^{n-k}\binom{n}{j}\binom{n-k}{i}a^{n-j}\bar{a}^{n-k-i}b^j\bar{b}^i\int_T f^{n-(n-k)+i-j}\bar{g}^{i-j}d\sigma \\ &= \binom{n}{0}\binom{n-k}{n-k}a^n\bar{a}^0b^0\bar{b}^{n-k}\int_T f^n\bar{g}^{n-k}d\sigma. \end{aligned}$$

§4. Rudin's orthogonal function.

In this section, we study a Rudin's function ϕ which is a polynomial of an inner function. Theorem 5 determines a Rudin's function when $\phi = aq^s + bq^m$, $m \ge s+1$ and q is an inner function with $q(0) = 0$. On the other hand, Theorem 6 solves affirmatively R1 when ϕ is a polynomial of an inner function. Proposition 7 gives another proof of Corollary 3 in [2] and another one of Proposition 3 in §2.

Theorem 6. *Let ϕ_0 be a Rudin's function and $\phi = \sum_{j=1}^{n} a_j\phi_0^j$ with $a_n \ne 0$. If ϕ is a Rudin's function, then $\phi = a_n\phi_0^n$.*

Proof. We may assume that $\phi = \sum_{j=1}^{n} a_j\phi_0^j$, $a_n = 1$ and $n > 1$. By induction, we will show that $a_\ell = 0$ for $1 \le \ell \le n-1$. Suppose $\ell = 1$. Since

$$\phi^n = \sum_{i=0}^{n}\binom{n}{i}(a_1\phi_0)^{n-i}\left(\sum_{k=2}^{n}a_k\phi_0^k\right)^i,$$

the smallest degree of ϕ^n is n because the smallest one of $\binom{n}{i}(a_1\phi_0)^{n-i}\left(\sum_{k=2}^{n}a_k\phi_0^k\right)^i$ is $n+i$. On the other hand, the degree of ϕ is n. Hence if ϕ is a Rudin's function,

$$(\phi^n, \phi) = a_1^n = 0$$

because ϕ_0 is a Rudin's function.

Suppose $a_1 = a_2 = \cdots = a_\ell = 0$ for $\ell < n-1$. Then since

$$\phi^n = \sum_{i=0}^{n} \binom{n}{i} (a_{\ell+1}\phi_0^{\ell+1})^{(n-i)} \left(\sum_{k=\ell+2}^{n} a_k \phi_0^k \right)^i ,$$

the smallest degree of ϕ^n is $n(\ell+1)$ because the smallest one of $\binom{n}{i}(a_{\ell+1}\phi_0^{\ell+1})^{(n-i)} \left(\sum_{k=\ell+2}^{n} a_k \phi_0^k \right)^i$ is $(\ell+1)(n-i) + (\ell+2)i = i + n(\ell+1)$. On the other hand, since

$$\phi^{\ell+1} = \sum_{j=0}^{\ell+1} \binom{\ell+1}{j} \phi_0^{n(\ell+1-j)} \left(\sum_{k=\ell+1}^{n-1} a_k \phi_0^k \right)^j ,$$

the largest degree of $\phi^{\ell+1}$ is $n(\ell+1)$ because the largest one of $\binom{\ell+1}{j}\phi_0^{n(\ell+1-j)} \left(\sum_{k=\ell+1}^{n-1} a_k \phi_0^k \right)^j$ is $(n-1)j + n(\ell+1-j) = n(\ell+1) - j$. Hence if ϕ is a Rudin's function,

$$(\phi^n, \phi^{\ell+1}) = (a_{\ell+1})^{n(\ell+1)} = 0$$

because ϕ_0 is a Rudin's function.

Proposition 7. *If $z^n\phi$ is a Rudin's function for all $n \geq 0$, then ϕ is a multiple of an inner function.*

Proof. Fix a positive integer k. For all $n \geq 0$, $z^{nk}\phi^k$ is orthogonal to $z^{n(k+1)}\phi^{k+1}$ because $z^n\phi$ is a Rudin's function. Thus ϕ^k is orthogonal to $\{z^n\phi^{k+1}\}_{n=0}^{\infty}$. Put $\phi = qh$ where q is inner and h is outer. Then by the Beurling's theorem [5, p11], ϕ^k is orthogonal to $q^{k+1}H^2$ and so $\bar{h}^{k+1}q \in H^2$ for all $k \geq 0$. By [4, p177], h is constant and so ϕ is inner.

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