INTUITIONISTIC FUZZY FILTERS IN BCH-ALGEBRAS

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ABSTRACT. We study the fundamental properties of intuitionistic fuzzy filters in BCH-algebras and give a characterization theorem of them.

1. Introduction

In 1966, Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([13, 14, 15]), and since then many researchers have investigated various properties of these algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [11] and [12], Q. P. Hu and X. Li introduced a wider class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. They have also studied some properties of these algebras. Following the introduction of fuzzy sets by L. A. Zadeh [31], the fuzzy set theory developed by L. A. Zadeh himself and others have found many applications in the domain of mathematics and elsewhere. N. Kuroki [22, 23] and A. Rosenfeld [28] introduced the notion of a fuzzy set in the realm of semigroup theory and group theory, respectively. Also, researchers in various disciplines of mathematics have been trying to extend their ideas to the broader framework of the fuzzy setting. In this thesis we study the fundamental properties of intuitionistic fuzzy filters in BCH-algebras and give a characterization theorem of them.

2. Preliminaries

Definition 2.1. *Let X be a set with a binary operation $*$ and a constant 0. Then $(X;*,0)$ is a* BCH-algebra *if it satisfies the following conditions;*

(H1) $x*x=0$,
(H2) $x*y=0$ *and* $y*x=0$ *imply* $x=y$,
(H3) $(x*y)*z=(x*z)*y$,
for all $x,y,z\in X$.

We can define a binary relation $\leq$ on X by letting $x\leq y$ if and only if $x*y=0$.
In a BCH-algebra X, the following hold

(p1) $x*0=x$,
(p2) $x*0=0$ imply $x=0$,
(p3) $0*(x*y)=(0*x)*(0*y)$,
for all $x,y\in X$.

A BCH-algebra X with additional equality $(x*y)*(z*u)=(x*z)*(y*u)$ for all $x,y,z,u\in X$ is called a *medial BCH-algebra.*

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We now review some fuzzy logic concepts. A fuzzy set in a set X is a function $\mu : X \to [0,1]$, and the *complement* of a fuzzy set μ in X, denoted by $\bar{\mu}$, is the fuzzy set in X given by $\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$. For a fuzzy set μ in X and $\alpha \in [0,1]$, define $U(\mu;\alpha)$ and $L(\mu;\alpha)$ to be the sets $U(\mu;\alpha) := \{x \in X \mid \mu(x) \geq \alpha\}$ and $L(\mu;\alpha) := \{x \in X \mid \mu(x) \leq \alpha\}$, which are called an *upper α-level cut* of μ and a *lower α-level cut* of μ, respectively.

An intuitionistic fuzzy set (briefly, IFS, see [1]) A in a non-empty set X is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$$

where the functions $\mu_A : X \to [0,1]$ and $\gamma_A : X \to [0,1]$ denote the degree of membership and the degree of nonmembership, respectively, and

$$0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \text{ for all } x \in X.$$

An intuitionistic fuzzy set $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ in X can be identified to an ordered pair (μ_A, γ_A) in $I^X \times I^X$. For the sake of simplicity, the IFS $A = \{(x, \mu_A(x), \gamma_A(x)) \mid x \in X\}$ will be denoted by $A = (\mu_A, \gamma_A)$.

Definition 2.2. *A non-empty subset B of a BCH-algebra X is called a* subalgebra *of X if $x, y \in B$ implies $x * y \in B$.*

Definition 2.3. *A fuzzy set μ in a BCH-algebra X is called a* fuzzy subalgebra *of X if $\mu(x * y) \geq \min\{\mu(x),\, \mu(y)\}$ for all $x, y \in X$.*

Definition 2.4. *A subset B of a BCH-algebra X is said to be* regular *if whenever $x * y \in B$ and $x \in B$ then $y \in B$ for all $x, y \in X$.*

Note that every regular subset contains the zero element 0.

Definition 2.5. *A fuzzy set μ in a BCH-algebra X is said to satisfy the* fuzzy regularity *if $\mu(y) \geq \min\{\mu(x * y), \mu(x)\}$ for all $x, y \in X$. A fuzzy subalgebra of X satisfying the fuzzy regularity is called a* fuzzy regular subalgebra *of X.*

3. Intuitionistic Fuzzy Filters

In what follows let X denote a BCH-algebra unless otherwise specified.

Definition 3.1. ([6]) *A* filter *of a BCH-algebra X is a nonempty subset F of X such that*

(u1) *If $x, y \in F$, then $x \wedge y \in F$ and $y \wedge x \in F$, where $x \wedge y = y * (y * x)$.*

(u2) *If $x \in F$ and $x \leq y$, then $y \in F$.*

Moreover, a filter F of X is said to be *closed* if it satisfies:

(u3) $0 * x \in F$ for all $x \in F$.

Definition 3.2. ([16]) *A nonempty subset F of X is called a* closed quasi filter *of X if it satisfies conditions (u2) and (u3).*

Definition 3.3. *An IFS $A = (\mu_A, \gamma_A)$ in X is called an* intuitionistic fuzzy subalgebra *of X if*

(F1) $\mu_A(x * y) \geq \min\{\mu_A(x),\, \mu_A(y)\}$

(F2) $\gamma_A(x * y) \leq \max\{\gamma_A(x),\, \gamma_A(y)\}$

for all $x, y \in X$.

Definition 3.4. *An IFS $A = (\mu_A, \gamma_A)$ in X is said to satisfy the* intuitionistic fuzzy regularity *if*

(F3) $\mu_A(y) \geq \min\{\mu_A(x * y),\, \mu_A(x)\}$

(F4) $\gamma_A(y) \leq \max\{\gamma_A(x * y),\ \gamma_A(x)\}$

for all $x, y \in X$. An intuitionistic fuzzy subalgebra of X satisfying the intuitionistic fuzzy regularity is called an intuitionistic fuzzy regular subalgebra *of X.*

Example 3.5. *(1) Consider a BCH-algebra $X = \{0, 1, 2, 3\}$ with Cayley table as follows:*

$*$	0	1	2	3
0	0	3	0	3
1	1	0	3	2
2	2	3	0	1
3	3	0	3	0

Define an IFS $A = (\mu_A, \gamma_A)$ in X by

$$\mu_A(x) = \begin{cases} \frac{2}{3} & \text{if } x = 0, 2, \\ 0 & \text{if } x = 1, 3, \end{cases} \qquad \gamma_A(x) = \begin{cases} 0 & \text{if } x = 0, 2, \\ \frac{1}{2} & \text{if } x = 1, 3. \end{cases}$$

Then $A = (\mu_A, \gamma_A)$ satisfies the intuitionistic fuzzy regularity, and moreover it is an intuitionistic fuzzy regular subalgebra of X.

(2) Consider a BCH-algebra $X = \{0, 1, 2, 3, 4\}$ with Cayley table as follows:

$*$	0	1	2	3	4
0	0	0	4	3	2
1	1	0	4	3	2
2	2	2	0	4	3
3	3	3	2	0	4
4	4	4	3	2	0

Define an IFS $A = (\mu_A, \gamma_A)$ in X by

$$\mu_A(x) = \begin{cases} t_1 & \text{if } x = 0, \\ t_2 & \text{if } x = 1, 3, \\ t_3 & \text{if } x = 2, 4, \end{cases} \qquad \gamma_A(x) = \begin{cases} s_1 & \text{if } x = 0, \\ s_2 & \text{if } x = 1, 3, \\ s_3 & \text{if } x = 2, 4, \end{cases}$$

where $t_1 > t_2 > t_3$ and $s_1 < s_2 < s_3$ in $[0, 1]$ such that $s_i + t_i \leq 1$ for $i = 1, 2, 3$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subalgebra of X which does not satisfy the intuitionistic fuzzy regularity.

Definition 3.6. (Jun and Dudek [17]) *An IFS $A = (\mu_A, \gamma_A)$ in X is called an* intuitionistic fuzzy closed ideal *of X if*

(F5) $\mu_A(0 * x) \geq \mu_A(x)$ *and* $\gamma_A(0 * x) \leq \gamma_A(x)$,

(F6) $\mu_A(y) \geq \min\{\mu_A(y * x),\ \mu_A(x)\}$ *and* $\gamma_A(y) \leq \max\{\gamma_A(y * x),\ \gamma_A(x)\}$,

for all $x, y \in X$

Definition 3.7. *An IFS $A = (\mu_A, \gamma_A)$ in X is called an* intuitionistic fuzzy filter *of X if*

(F7) $\min\{\mu_A(x \wedge y),\ \mu_A(y \wedge x)\} \geq \min\{\mu_A(x),\ \mu_A(y)\}$,

(F8) $\max\{\gamma_A(x \wedge y),\ \gamma_A(y \wedge x)\} \leq \max\{\gamma_A(x),\ \gamma_A(y)\}$,

(F9) $x * y = 0$ *implies* $\mu_A(x) \leq \mu_A(y)$ *and* $\gamma_A(x) \geq \gamma_A(y)$

for all $x, y \in X$. If, moreover, an intuitionistic fuzzy filter $A = (\mu_A, \gamma_A)$ satisfies (F5), we say that $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy closed filter *of X.*

Example 3.8. *Consider a BCH-algebra $X = \{0, 1, 2, 3\}$ with Cayley table as follows:*

$*$	0	1	2	3
0	0	0	3	2
1	1	0	3	2
2	2	2	0	3
3	3	3	2	0

Define an IFS $A = (\mu_A, \gamma_A)$ in X by

$$\mu_A(x) = \begin{cases} t_1 & \textit{if} \quad x = 0, 1, \\ t_2 & \textit{if} \quad x = 2, \\ 0 & \textit{if} \quad x = 3, \end{cases} \qquad \gamma_A(x) = \begin{cases} 0 & \textit{if} \quad x = 0, 1, \\ s_1 & \textit{if} \quad x = 2, \\ s_2 & \textit{if} \quad x = 3, \end{cases}$$

where $t_1 > t_2$ and $s_1 < s_2$ in $[0,1]$ such that $s_i + t_i \leq 1$ for $i = 1, 2$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of X.

Definition 3.9. *An IFS $A = (\mu_A, \gamma_A)$ in X is called an* intuitionistic fuzzy closed quasi filter *of X if it satisfies conditions (F5) and (F9).*

Proposition 3.10. *Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy closed quasi filter of X. Then*

(i) $\mu_A(y * x) = \mu_A(x * y)$ *and* $\gamma_A(y * x) = \gamma_A(x * y)$, $\forall x, y \in X$,
(ii) $\mu_A(x \wedge y) = \mu_A(x)$ *and* $\gamma_A(x \wedge y) = \gamma_A(x)$, $\forall x, y \in X$,
(iii) $x * y = 0$ *implies* $\mu_A(x) = \mu_A(y)$ *and* $\gamma_A(x) = \gamma_A(y)$, $\forall x, y \in X$.

Proof. (i) Let $x, y \in X$. Using (p3), (H3) and (H1), we can easily verify that $\big(0 * (x * y)\big) * (y * x) = 0$. It follows from (F5) and (F9) that

$$\mu_A(y * x) \geq \mu_A\big(0 * (x * y)\big) \geq \mu_A(x * y)$$

and

$$\gamma_A(y * x) \leq \gamma_A\big(0 * (x * y)\big) \leq \gamma_A(x * y).$$

Similarly we have $\mu_A(y * x) \leq \mu_A(x * y)$ and $\gamma_A(y * x) \geq \gamma_A(x * y)$. Hence (i) is true.

(ii) Using (i), (H1), (H3) and (p1), we have

$$\begin{aligned} \mu_A(x \wedge y) &= \mu_A\big(y * (y * x)\big) = \mu_A\big((y * x) * y\big) \\ &= \mu_A\big((y * y) * x\big) = \mu_A(0 * x) \\ &= \mu_A(x * 0) = \mu_A(x) \end{aligned}$$

and similarly $\gamma_A(x \wedge y) = \gamma_A(x)$.

(iii) Let $x, y \in X$ be such that $x * y = 0$. Using (p1) and (ii), we get

$$\mu_A(x) = \mu_A(x * 0) = \mu_A\big(x * (x * y)\big) = \mu_A(y \wedge x) = \mu_A(y)$$

and

$$\gamma_A(x) = \gamma_A(x * 0) = \gamma_A\big(x * (x * y)\big) = \gamma_A(y \wedge x) = \gamma_A(y).$$

This completes the proof. □

Theorem 3.11. *An intuitionistic fuzzy closed quasi filter $A = (\mu_A, \gamma_A)$ of X, which is also an intuitionistic fuzzy subalgebra, is an intuitionistic fuzzy closed ideal of X.*

Proof. It is sufficient to show that $A = (\mu_A, \gamma_A)$ satisfies (F6). We only show the case of μ_A. Let $x, y \in X$. Then

$$\begin{aligned} \mu_A(y) &= \mu_A(y * 0) && \text{[by (p1)]} \\ &= \mu_A(0 * y) && \text{[by Proposition 3.10(i)]} \\ &= \mu_A\big((x * x) * y\big) && \text{[by (H1)]} \\ &= \mu_A\big((x * y) * x\big) && \text{[by (H3)]} \\ &\geq \min\big\{\mu_A(x * y),\ \mu_A(x)\big\} && \text{[by (F1)]} \\ &= \min\big\{\mu_A(y * x),\ \mu_A(x)\big\} && \text{[by Proposition 3.10(i)]} \end{aligned}$$

. □

From Proposition 3.10, we know that every intuitionistic fuzzy closed quasi filter satisfies conditions (F7) and (F8), which means that every intuitionistic fuzzy closed quasi filter is an intuitionistic fuzzy closed filter. Since the converse is clear, we conclude that the notion of intuitionistic fuzzy closed quasi filter and intuitionistic fuzzy closed filter coincide.

Lemma 3.12. (Jun [17, Theorem 3.9]) *Every intuitionistic fuzzy closed ideal is an intuitionistic fuzzy subalgebra.*

Theorem 3.13. *Every intuitionistic fuzzy closed ideal $A = (\mu_A, \gamma_A)$ of X satisfying the inequalities*

$$\mu_A(y * x) \geq \mu_A(x * y) \text{ and } \gamma_A(y * x) \leq \gamma_A(x * y), \forall x, y \in X$$

is an intuitionistic fuzzy closed filter of X.

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy closed ideal of X satisfying the inequalities

$$\mu_A(y * x) \geq \mu_A(x * y) \text{ and } \gamma_A(y * x) \leq \gamma_A(x * y), \forall x, y \in X.$$

Since $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy subalgebra of X, we have

$$\begin{aligned} \mu_A(x \wedge y) &= \mu_A\big(y * (y * x)\big) \geq \min\big\{\mu_A(y),\, \mu_A(y * x)\big\} \\ &\geq \min\big\{\mu_A(y),\, \min\{\mu_A(y),\, \mu_A(x)\}\big\} \\ &= \min\big\{\mu_A(x),\, \mu_A(y)\big\} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(x \wedge y) &= \gamma_A\big(y * (y * x)\big) \leq \max\big\{\gamma_A(y),\, \gamma_A(y * x)\big\} \\ &\leq \max\big\{\gamma_A(y),\, \max\{\gamma_A(y),\, \gamma_A(x)\}\big\} \\ &= \max\big\{\gamma_A(x),\, \gamma_A(y)\big\}. \end{aligned}$$

Similarly we have

$$\mu_A(y \wedge x) \geq \min\big\{\mu_A(x),\, \mu_A(y)\big\} \text{ and } \gamma_A(y \wedge x) \leq \max\big\{\gamma_A(x),\, \gamma_A(y)\big\}.$$

Hence

$$\min\big\{\mu_A(x \wedge y),\, \mu_A(y \wedge x)\big\} \geq \min\big\{\mu_A(x),\, \mu_A(y)\big\}$$

and

$$\max\big\{\gamma_A(x \wedge y),\, \gamma_A(y \wedge x)\big\} \leq \max\big\{\gamma_A(x),\, \gamma_A(y)\big\}.$$

Let $x, y \in X$ be such that $x * y = 0$. Then $0 * x = 0 * y$. It follows from (p1) and the assumption that

$$\mu_A(y) = \mu_A(y * 0) \geq \mu_A(0 * y) = \mu_A(0 * x) \geq \mu_A(x * 0) = \mu_A(x)$$

and

$$\gamma_A(y) = \gamma_A(y * 0) \leq \gamma_A(0 * y) = \gamma_A(0 * x) \leq \gamma_A(x * 0) = \gamma_A(x).$$

This completes the proof. □

Lemma 3.14. *If an IFS $A = (\mu_A, \gamma_A)$ in X satisfies the intuitionistic fuzzy regularity, then $\mu_A(0) \geq \mu_A(x)$ and $\gamma_A(0) \leq \gamma_A(x)$ for all $x \in X$.*

Proof. Taking $y = 0$ in (F3) and (F4) and using (p1) induce the required results. □

Theorem 3.15. *If an IFS $A = (\mu_A, \gamma_A)$ in X satisfies intuitionistic fuzzy regularity and the condition (F6), then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy closed ideal of X.*

Proof. It is sufficient to show that $A=(\mu_A,\gamma_A)$ satisfies the condition (F5). We only show the case of μ_A. For any $x,y\in X$, we have

$$\begin{aligned}\mu_A(0*x) &\geq \min\{\mu_A\big(0*(0*x)\big),\,\mu_A(0)\}=\mu_A\big(0*(0*x)\big)\\ &\geq \min\{\mu_A\big((0*(0*x))*x\big),\,\mu_A(x)\}\\ &= \min\{\mu_A\big((0*x)*(0*x)\big),\,\mu_A(x)\}\\ &= \min\{\mu_A(0),\mu_A(x)\}=\mu_A(x)\end{aligned}$$

. □

The following example shows that the converse of Theorem 3.15 may not be true.

Example 3.16. *Consider a BCH-algebra $X=\{0,1,2,3,4\}$ with Cayley table as follows:*

$*$	0	1	2	3	4
0	0	0	4	3	2
1	1	0	4	3	2
2	2	2	0	4	3
3	3	3	2	0	4
4	4	4	3	2	0

Define an IFS $A=(\mu_A,\gamma_A)$ in X by

$$\mu_A(x)=\begin{cases} t_1 & \textit{if}\quad x=0,\\ t_2 & \textit{otherwise,}\end{cases}\qquad \gamma_A(x)=\begin{cases} s_1 & \textit{if}\quad x=0,\\ s_2 & \textit{otherwise,}\end{cases}$$

where $t_1>t_2$ and $s_1<s_2$ in $[0,1]$ such that $s_i+t_i\leq 1$ for $i=1,2$. Then $A=(\mu_A,\gamma_A)$ is an intuitionistic fuzzy closed ideal of X but not satisfy the intuitionistic fuzzy regularity.

Lemma 3.17. (Jun [16, Corollary 4.7]) *In a medial BCH-algebra X, we have $x*y=0*(y*x)$ for all $x,y\in X$.*

Theorem 3.18. *If X is medial, then every intuitionistic fuzzy closed ideal of X satisfies the intuitionistic fuzzy regularity.*

Proof. Let $A=(\mu_A,\gamma_A)$ be an intuitionistic fuzzy closed ideal of a medial BCH-algebra X and let $x,y\in X$. Then $\mu_A\big(0*(x*y)\big)\geq\mu_A(x*y)$ and $\gamma_A\big(0*(x*y)\big)\leq\gamma_A(x*y)$ by (F5). It follows from (F6) and Lemma 3.17 that

$$\begin{aligned}\mu_A(y) &\geq \min\{\mu_A(y*x),\,\mu_A(x)\}\\ &= \min\{\mu_A\big(0*(x*y)\big),\mu_A(x)\}\\ &\geq \min\{\mu_A(x*y),\,\mu_A(x)\}\end{aligned}$$

and similarly $\gamma_A(y)\leq\max\{\gamma_A(x*y),\,\gamma_A(x)\}$. This completes the proof. □

Theorem 3.19. *If an IFS $A=(\mu_A,\gamma_A)$ in X is an intuitionistic fuzzy filter of X, then the sets*

$$X_\mu:=\{x\in X\mid \mu_A(x)\geq\mu_A(0)\}\ \text{ and }\ X_\gamma:=\{x\in X\mid \gamma_A(x)\leq\gamma_A(0)\}$$

are filters of X.

Proof. Let $x,y\in X_\mu$. Then $\mu_A(x)\geq\mu_A(0)$ and $\mu_A(y)\geq\mu_A(0)$. It follows from (F7) that

$$\min\{\mu_A(x\wedge y),\,\mu_A(y\wedge x)\}\geq\min\{\mu_A(x),\,\mu_A(y)\}\geq\mu_A(0)$$

so that $\mu_A(x\wedge y)\geq\mu_A(0)$ and $\mu_A(y\wedge x)\geq\mu_A(0)$. This shows that $x\wedge y\in X_\mu$ and $y\wedge x\in X_\mu$. Let $x,y\in X$ be such that $x\in X_\mu$ and $x\leq y$. Since $x*y=0$, it follows from (F9) that $\mu_A(y)\geq\mu_A(x)\geq\mu_A(0)$ so that $y\in X_\mu$. Hence X_μ is a filter of X. The case of X_γ is similar. □

Using the notion of upper and lower level cuts, we state a characterization of intuitionistic fuzzy filters.

Theorem 3.20. *An IFS $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of X if and only if for all $s, t \in [0, 1]$, the nonempty sets $U(\mu_A; t)$ and $L(\gamma_A; s)$ are filters of X.*

Proof. Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy filter of X and $U(\mu_A; t)$ and $L(\gamma_A; s)$ are nonempty for any $s, t \in [0, 1]$. If $x, y \in U(\mu_A; t)$, then $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$. It follows from (F7) that

$$\min\{\mu_A(x \wedge y),\, \mu_A(y \wedge x)\} \geq \min\{\mu_A(x),\, \mu_A(y)\} \geq t$$

so that $\mu_A(x \wedge y) \geq t$ and $\mu_A(y \wedge x) \geq t$, that is, $x \wedge y \in U(\mu_A; t)$ and $y \wedge x \in U(\mu_A; t)$. Let $x, y \in X$ be such that $x \in U(\mu_A; t)$ and $x \leq y$. Then $\mu_A(x) \geq t$ and $x * y = 0$. Using (F9), we get $\mu_A(y) \geq \mu_A(x) \geq t$, and thus $y \in U(\mu_A; t)$. This proves that $U(\mu_A; t)$ is a filter of X. It can be proved similarly that $L(\gamma_A; s)$ is a filter of X.

Conversely, assume that $U(\mu_A; t)$ $(\neq \emptyset)$ and $L(\gamma_A; s)$ $(\neq \emptyset)$ are filters of X for every $s, t \in [0, 1]$. If (F7) is false, then

$$\min\{\mu_A(x_0 \wedge y_0),\, \mu_A(y_0 \wedge x_0)\} < \min\{\mu_A(x_0),\, \mu_A(y_0)\}$$

for some $x_0, y_0 \in X$. There exists t_0 such that

$$\min\{\mu_A(x_0 \wedge y_0),\, \mu_A(y_0 \wedge x_0)\} < t_0 < \min\{\mu_A(x_0),\, \mu_A(y_0)\}.$$

It follows that either

$$\mu_A(x_0 \wedge y_0) < t_0 < \min\{\mu_A(x_0),\, \mu_A(y_0)\}$$

or

$$\mu_A(y_0 \wedge x_0) < t_0 < \min\{\mu_A(x_0),\, \mu_A(y_0)\}$$

so that either

$$x_0, y_0 \in U(\mu_A; t_0) \text{ and } x_0 \wedge y_0 \notin U(\mu_A; t_0)$$

or

$$x_0, y_0 \in U(\mu_A; t_0) \text{ and } y_0 \wedge x_0 \notin U(\mu_A; t_0).$$

This is a contradiction; hence (F7) is valid. It is similar to see that (F8) is valid.

Finally, let $x_0, y_0 \in X$ be such that $x_0 * y_0 = 0$ and $\mu_A(x_0) > \mu_A(y_0)$. There is u_0 such that $\mu_A(x_0) > u_0 > \mu_A(y_0)$. It follows that $x_0 \in U(\mu_A; u_0)$ but $y_0 \notin U(\mu_A; u_0)$, which is a contradiction. If $a, b \in X$ satisfies $a * b = 0$ and $\gamma_A(a) < \gamma_A(b)$, then $\gamma_A(a) < v_0 < \gamma_A(b)$ for some v_0 and thus $a \in L(\gamma_A; v_0)$ and $b \notin L(\gamma_A; v_0)$. This is a contradiction. Thus (F9) is true, and the proof is complete. □

Theorem 3.21. *Let F be a filter of X and let $A = (\mu_A, \gamma_A)$ be an IFS in X defined by*

$$\mu_A(x) = \begin{cases} t_1 & \text{if } x \in F, \\ t_2 & \text{otherwise}, \end{cases} \qquad \gamma_A(x) = \begin{cases} s_1 & \text{if } x \in F, \\ s_2 & \text{otherwise}, \end{cases}$$

for all $x \in X$ and $s_i, t_i \in [0, 1]$ such that $t_1 > t_2$, $s_1 < s_2$ and $s_i + t_i < 1$ for $i = 1, 2$. Then $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of X and $U(\mu_A; t_1) = F = L(\gamma_A; s_1)$.

Proof. Let $x, y \in X$. If any one of x and y does not belong to F, then

$$\min\{\mu_A(x \wedge y),\, \mu_A(y \wedge x)\} \geq t_2 = \min\{\mu_A(x),\, \mu_A(y)\}$$

and

$$\max\{\gamma_A(x \wedge y),\, \gamma_A(y \wedge x)\} \leq s_2 = \max\{\gamma_A(x),\, \gamma_A(y)\}.$$

If $x, y \in F$, then $x \wedge y \in F$ and $y \wedge x \in F$ by (u1). Hence

$$\min\{\mu_A(x \wedge y),\, \mu_A(y \wedge x)\} = t_1 = \min\{\mu_A(x),\, \mu_A(y)\}$$

and

$$\max\{\gamma_A(x \wedge y),\ \gamma_A(y \wedge x)\} = s_1 = \max\{\gamma_A(x),\ \gamma_A(y)\}.$$

Let $x, y \in X$ be such that $x * y = 0$. If $x \notin F$, then $\mu_A(x) = t_2 \leq \mu_A(y)$ and $\gamma_A(x) = s_2 \geq \gamma_A(y)$. If $x \in F$, then $y \in F$ by (u2). Thus $\mu_A(x) = t_1 = \mu_A(y)$ and $\gamma_A(x) = s_1 = \gamma_A(y)$. Therefore $A = (\mu_A, \gamma_A)$ is an intuitionistic fuzzy filter of X. Obviously, $U(\mu_A; t_1) = F = L(\gamma_A; s_1)$. □

Theorem 3.22. *If an IFS $A = (\mu_A, \gamma_A)$ in X is an intuitionistic fuzzy filter of X, then*

$$\mu_A(x) := \sup\{t \in [0,1] \mid x \in U(\mu_A; t)\}$$

and

$$\gamma_A(x) := \inf\{t \in [0,1] \mid x \in L(\gamma_A; t)\}$$

for all $x \in X$.

Proof. Let $\delta := \sup\{t \in [0,1] \mid x \in U(\mu_A; t)\}$ and let $\varepsilon > 0$ be given. Then $\delta - \varepsilon < t$ for some $t \in [0,1]$ such that $x \in U(\mu_A; t)$. It follows that $\delta - \varepsilon < \mu_A(x)$ so that $\delta \leq \mu_A(x)$ since ε is arbitrary. On the other hand, since $\mu_A(x) \in \{t \in [0,1] \mid x \in U(\mu_A; t)\}$, we have $\mu_A(x) \leq \sup\{t \in [0,1] \mid x \in U(\mu_A; t)\} = \delta$. Therefore

$$\mu_A(x) = \sup\{t \in [0,1] \mid x \in U(\mu_A; t)\}.$$

Similarly we can prove

$$\gamma_A(x) = \inf\{t \in [0,1] \mid x \in L(\gamma_A; t)\}.$$

This completes the proof. □

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