KUBO-ANDO THEORY FOR CONVEX FUNCTIONAL MEANS

JUN ICHI FUJII

Received May 17, 2002

ABSTRACT. Inspired by the geometric mean due to Atteia and Raïssouli, we discuss a general theory of convex functional means on a Hilbert space like the Kubo-Ando theory of operator means. Though our construction is based on the integral representation in Kubo-Ando theory, it is an exact extension not only for operator means but also for Atteia-Raïssouli's ones. We give an example where our geometric mean can be defined even if their geometric one cannot. We show that our convex functional means satisfy monotonicity, semi-continuity, homogeneity, subadditivity, joint concavity, transformer inequality and normalization. One of outstanding properties of these means appeares in those for constant functions, which suggests us to weights for operator means.

1. INTRODUCTION.

Since we met Ando's lecture note [2], we have been studying operator means, which is now known as the Kubo-Ando theory. For positive operators on a (complex) Hilbert space H, the theory of operator means is established axiomatically by Kubo and Ando [11]: An (*operator*) *connection* m is a binary operation on positive operators satisfying the following axioms:

monotonicity: $A_1 \le A_2$ and $B_1 \le B_2$ imply $A_1 \ m \ B_1 \le A_2 \ m \ B_2$.
semi-continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \ m \ B_n \downarrow A \ m \ B$.
transformer inequality: $T^*(A \ m \ B)T \le (T^*AT)m(T^*BT)$.

An *operator mean* is a connection m satisfying

normalization: $A \ m \ A = A$.

It is easy to show the transformer equality if T is invertible. In particular, we have:

homogeneity: $\alpha(A \ m \ B) = (\alpha A)m(\alpha B)$ for every positive number α.

For an operator mean m, the corresponding numerical function $f_m(x) = 1 \ m \ x$ is *operator monotone*:

$$0 \le A \le B \qquad \text{implies} \qquad f_m(A) \le f_m(B).$$

This correspondence $m \mapsto f_m$ is bijective. In fact, if f is a continuous nonnegative opeartor monotone functional on $[0, \infty)$ with $f(1) = 1$, then a binary operation m defined by

$$A \ m \ B = A^{1/2} f\left(A^{-1/2} B A^{-1/2}\right) A^{1/2}$$

for positive invertible operators A and B induces an operator mean $A \ m \ B$. As in [7], the operator meams bijectively correspond to the operator concave functionals on $[0, 1]$ with $F(1/2) = 1/2$ by

$$F(x) = (1 - x) \ m \ x,$$

2000 *Mathematics Subject Classification.* 32A70, 47A65, 39A62.
Key words and phrases. convex functional, parallel addition, operator mean, functional mean.

which shows another construction of operator means by Izumino's method [8]. This fact is also a bridge between means for operators and ones for positive forms by Pusz and Woronowicz [14, 15], and two theories are essentially equivalent.

Recently, Atteia and Raïssouli introduced the geometric convex functional mean in [3] which is characterized as the arithmetico-harmonic mean like the iteration method in [6]. As in [10] for example, the notion 'Legendre-Fenchel conjugate' f^* of a convex functional f on a real vector space

$$f^*(y^*) = \sup_x \langle x, y \rangle - f(x)$$

is considered as that of 'inverse' in some sense. So they used essentially the harmonic mean τ_h defined by

$$f \tau_h g = \left(\frac{f^* + g^*}{2} \right)^* .$$

(In [16], they redefine these means on a complex Hilbert space.) Inspired by their convex functional means, we introduce a class of convex functional means σ_m corresponding to the operator means m and show similar properties of σ_m. Conversely we may define a general class Σ_0 of convex functional means axiomatically satisfying these common properties for σ_m like the Kubo-Ando theory [11]. Finally we see what subclass Σ corresponds to that of operator means.

2. The class Γ.

For our discussion, we summerize properties of a class of convex functionals. Let f be a lower-bounded convex functional on a (complex) Hilbert space H and

$$\text{dom } f = \{ \ x \in H \mid f(x) < \infty \ \}$$

the *domain* of f. If dom $f = H$, then f is called *finite*. Throughout this note, we assume that f is *proper*, i.e., dom f is not empty. and f is *lower semi-continuous*, i.e., the *epigraph*

$$\text{epi } f = \{ \ (x, \alpha) \in H \times \mathbb{R} \mid f(x) \leq \alpha \ \}$$

of f is closed. Then let $\Gamma = \Gamma(H)$ be the proper lower-bounded lower semi-continuous convex functionals on H. An *indicator* $\mathbf{1}_C$ for a closed convex subset C of H is defined by $\mathbf{1}_C(x) = 0$ if $x \in C$ and $\mathbf{1}_C(x) = \infty$ otherwise and it is a simple example in Γ. A typical and important example in Γ is f_A for a bounded linear positive operator A:

$$f_A(x) = \frac{1}{2} \langle Ax, x \rangle .$$

The functional f_A is called *quadratic* in the sense that $f(\gamma x) = |\gamma|^2 f(x)$ for all complex number γ. For a subspace C, the indicator $\mathbf{1}_C$ is also quadratic.

As the researchers on convex functionals have been discussing, the class Γ is stable in certain algebraic and topological senses. First it is closed for an key operation called *Legendre-Fenchel conjugate* f^*. Here, to condider complex spaces, it is defined as

$$f^*(y^*) = \sup_{x \in H} \text{Re } \langle x, y \rangle - f(x),$$

where we write y^* if y is considered as a functional on H. Then f^* is also a lower semi-continuous lower-bounded convex functional and $f^{**} = (f^*)^* = f$. Note that f^* is lower semi-continuous even if f is not. f^{**} is often called the *closure* of f, denote by cl$f$, since epi (f^{**}) coincides with the closure of epi (f).

Moreover, note that the conjugate operation preserves the quadraticity and if it is induced by a positive invertible operator, then the conjugate is by its inverse, which is confirmed in [16]:

Lemma 2.1. *If f is quadratic, then so is f^* and $f_A^* = f_{A^{-1}}$.*

Proof. Here we show the former statement. Suppose $f(\gamma x) = |\gamma|^2 f(x)$. Then

$$f^*(\gamma y^*) = \sup_x \mathrm{Re}\ \langle x, \bar{\gamma} y\rangle - f(x) = |\gamma|^2 \sup_x \mathrm{Re}\ \left\langle \frac{1}{\bar{\gamma}} x, y \right\rangle - f\left(\frac{1}{\bar{\gamma}} x\right) = |\gamma|^2 f^*(y^*). \qquad \square$$

Thus this conjugate operation * is considered as the inverse in the sense that $f_A^* = f_{A^{-1}}$. The following properties are easily obtained: For $f, g \in \Gamma$ and every positive number $\alpha > 0$, we have

(1*) $f \le g$ implies $f^* \ge g^*$.
(2*) $(f \pm \alpha)^* = f^* \mp \alpha$.
(3*) $(\alpha f)^*(\alpha y^*) = \alpha f^*(y^*)$.

For a finite dimensional case, the *epi-convergence*, which is characterized by

$$\mathrm{epi}\ \left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty} (\mathrm{epi}\ f_n)$$

for a product topology in $H \times \mathbb{R}$, has been often discussed in this class Γ as in a standard text like [17] or [10]. Moreover the Legendre-Fenchel conjugate preserves this convergence. To extend this property to an infinite dimensional case, Mosco [13] introduced the convergence $\text{M-lim}_{n\to\infty} f_n$, which is now called *Mosco convergence*, if the following conditions are satisfied:

(i) for each $x \in H$, there exists $x_n \in H$ with $\underset{n\to\infty}{\text{s-lim}}\, x_n = x$ and $f(x) = \lim_{n\to\infty} f_n(x_n)$,
(ii) $f(x) \le \liminf_{n\to\infty} f_n(x_n)$ for $\underset{n\to\infty}{\text{w-lim}}\, x_n = x$.

Then it is also shown that

$$\underset{n\to\infty}{\text{M-lim}}\, f_n = f \quad \text{if and only if} \quad \underset{n\to\infty}{\text{M-lim}}\, f_n^* = f^*.$$

Considering such stability for Γ, Atteia and Raïssouli [3] introduced convex functional means as the geometric mean is expressed by the limit in monotone convergence as the arithmetico-harmonic one.

3. Parallel addition.

The parallel addition for operators is introduced by Anderson-Duffin [1] and Fillmore-Williams [5]. For invertible operators, it is represented by

$$A : B = \left(A^{-1} + B^{-1}\right)^{-1},$$

and in general it is characterized by the formula:

$$(*) \qquad \langle A : B\ x, x\rangle = \inf_{y+z=x} \langle Ay, y\rangle + \langle Bz, z\rangle\,.$$

Typical extremal operator means are the *arithmetic* one $A \nabla B$ and the *harmonic* one $A\ !\ B = 2(A : B)$. By Löwner's theory [12], Kubo and Ando [11] showed that the operator means correspond also bijectively onto the Radon probability measures on $[0, \infty]$ by the following integral representation: For an operator mean m, there exists the Radon probability measure μ_m on $[0, \infty]$ with

$$A\ m\ B = aA + bB + \int_{(0,\infty)} (tA) : B \frac{1+t}{t} d\mu_m(t)$$

where $a = \mu_m(\{0\})$ and $b = \mu_m(\{\infty\})$. Thus, roughly speaking, an operator mean is considered as a convex combination of the arithmetic mean and the harmonic one, or equivalently the addition and the parallel addition.

The above formula $(*)$ suggests us that we can make the similar discussion in convex functionals since the notion 'inf-convolution' $f * g$ is defined as

$$f * g(x) = \inf_{y+z=x} f(y) + g(z).$$

Unfortunately Γ is not closed under this operation. In fact,

$$\mathrm{dom}\,(f * g) = \mathrm{dom}\,(f) + \mathrm{dom}\,(g),$$

and consequently

$$\mathbf{1}_{C_1} * \mathbf{1}_{C_2} = \mathbf{1}_{C_1+C_2}$$

but it is known that, in an infinite space, $C_1 + C_2$ is not always closed even if both C_1 and C_2 is closed. For example, for a bounded linear operator A on a Hilbert space H whose range is not closed, put $C_1 = H \oplus \{0\}$ and $C_2 = \{(x, Ax) | x \in H\}$ the graph of A. Then both C_1 and C_2 are closed in $H \oplus H$ but $C_1 + C_2$ is not, see [9]. Thus $f * g$ is not always lower semi-continuous. Moreover, even if $f * g$ is lower semi-continuous, we find it unsuitable for a parallel addition for convex functionals in spite of the formula $(*)$ since $2(f * f) \neq f$ and $2f * g \neq f \tau_h g$ if convex functions are not quadratic. However the following result is known for a real case. Moreover we pay attention to the following useful method to prove inequality or equality for parallel additions. So we give a proof of it:

Lemma 3.1. *If $f, g \in \Gamma$, then* $\mathrm{cl} f * g \equiv (f * g)^{**} = (f^* + g^*)^*$.

Proof. Since

$$\begin{aligned}(f^* + g^*)(y^*) &= \sup_{v,w} \mathrm{Re}\ \langle v + w, y\rangle - (f(v) + g(w)) \\ &\leq \sup_{v,w} \mathrm{Re}\ \langle v + w, y\rangle - (f * g)(v + w) = (f * g)^*(y^*),\end{aligned}$$

we have $(f^* + g^*)^* \geq f * g^{**}$ by (1^*). Conversely

$$\begin{aligned} f * g(x) &= \inf_{y+z=x} f^{**}(y) + g^{**}(z) \\ &= \inf_{y+z=x} \left(\sup_v \mathrm{Re}\ \langle y, v\rangle - f^*(v^*)\right) + \left(\sup_w \mathrm{Re}\ \langle z, w\rangle - g^*(w^*)\right) \\ &\geq \inf_{y+z=x} \left(\sup_v \mathrm{Re}\ \langle y + z, v\rangle - (f^* + g^*)(v^*)\right) \\ &= \sup_v \mathrm{Re}\ \langle x, v\rangle - (f^* + g^*)(v^*) = (f^* + g^*)^*(x). \end{aligned}$$

Taking the conjugation twice, we have $(f * g)^{**} \geq (f^* + g^*)^{***} = (f^* + g^*)^*$. Thus $(f * g)^{**} = (f^* + g^*)^*$. $\square$

Considering these and (3^*), we define the *parallel addition* for f and g by

$$(f : g)(x) = \frac{1}{4}\,{}_2\left((f^* + g^*)^*\right)(x) \equiv \frac{1}{4}\,(f^* + g^*)^*\,(2x).$$

Note that a functional ${}_T f$ defined by ${}_T f(x) \equiv f(Tx)$ for a bounded linear operator T also belongs to Γ for $f \in \Gamma$. By (3^*), this implies the following formulae:

$$(f : g)(x) = \frac{1}{4}(f * g)^{**}(2x) = \frac{1}{2}\left(\frac{f^* + g^*}{2}\right)^*(x) = \frac{1}{2}(f \tau_h g)(x).$$

Contrary to such differences, all of them are extensions for the parallel addition for operators:

Theorem 3.2. *If A and B are positive operators, then*

$$f_A : f_B = f_A * f_B = (f_A^* + f_B^*)^* = f_{A:B}.$$

Proof. By the formula $(*)$, we have

$$f_A * f_B(x) = \inf_{y+z=x} f_A(y) + f_B(z) = \frac{1}{2} \inf_{y+z=x} (\langle Ay, y\rangle + \langle Bz, z\rangle)$$
$$= \frac{1}{2} \langle (A:B)x, x\rangle = f_{A:B}(x),$$

so that $f_A * f_B$ is lower semi-continuous and hence $(f_A^* + f_B^*)^* = f_A * f_B = f_{A:B}$ by Lemma 3.1. In addition,

$$f_A : f_B(x) = \frac{1}{4}(f_A^* + f_B^*)^*(2x) = \frac{1}{4} \langle (A:B)2x, 2x\rangle = \langle (A:B)x, x\rangle = f_{A:B}(x). \qquad \square$$

Since the harmonic mean should be twice the parallel addition, we claim $2(f:f) = f$. In general, $2(f * f) = f$ does not always holds, but $2(f:f) = f$ does, which is the reason we need the above definition:

Lemma 3.3. $f : f = \dfrac{1}{2} f$ *for all* $f \in \Gamma$.

Proof. By (3*), we have $2(f:f)(x) = \frac{1}{2}(2f^*)^*(2x) = \frac{1}{2} \times 2f^{**}(x) = f(x)$. $\square$

The following estimation, which shows that the parallel addition is an operation in Γ, is obtained immediately by $4(f:g) = {}_2(f*g)^{**} \leq {}_2(f*g)$:

Theorem 3.4. *If* $t \in \operatorname{dom} f$ *and* $s \in \operatorname{dom} g$ *for* $f, g \in \Gamma$, *then*

$$4(f:g)(x) \leq f(2x-s) + g(s) \quad \textit{and} \quad 4(f:g)(x) \leq f(t) + g(2x-t).$$

In particular, f *(resp,* g*) are normalized in the sense* $f(0) = 0$*, then*

$$f : g(x) \leq \frac{1}{4} g(2x) \quad (\text{resp.,} \quad f : g(x) \leq \frac{1}{4} f(2x).)$$

The last inequality is represented into a simple one if g (resp., f) is *quadratic* in the sense that $f(\gamma x) = |\gamma|^2 f(x)$ for every complex number γ:

$$f : g \leq g \qquad (\text{resp., } f : g \leq f).$$

We are very interseted in quadratic functionals, so we discuss them in the next section.

4. Convex functional means via operator ones.

To consider general means, we show that the parallel addition has properties like operator means:

Lemma 4.1. *For* $f, g, h, k, f_k, g_k \in \Gamma$ *and* $\beta \in (0,1)$*, the parallel addition satisfies*

- **monotonicity:** $f \leq h$ *and* $g \leq k$ *imply* $f : g \leq h : k$.
- **semi-continuity:** $f_n \downarrow f$ *and* $g_n \downarrow g$ *imply* $f_n : g_n \downarrow f : g$.
- **subadditivity:** $(f+h) : (g+k) \geq f : g + h : k$.
- **joint concavity:** $(\beta f + (1-\beta)h) : (\beta g + (1-\beta)k) \geq \beta(f:g) + (1-\beta)(h:k)$.

Proof. If $f \leq h$ and $g \leq k$, then (1*) shows

$$4(f:g)(x) = (f^* + g^*)^*(2x) \leq (h^* + k^*)^*(2x) = 4(h:k)(x).$$

The semicontinuity is obtained by the fact that monotone convergence implies Mosco convergence, see [4]. The subadditivity of inf-convolution follows from

$$\begin{aligned} {}_2\big((f+h)*(g+k)\big)(x) &= \inf_{y+z=2x} (f+h)(y) + (g+k)(z) \\ &\geq \inf_{y+z=2x} f(y) + g(z) + \inf_{y+z=2x} h(y) + k(z) \\ &= {}_2(f*g)(x) + {}_2(h*k)(x). \end{aligned}$$

Taking double conjugate, we have

$$4(f+h):(g+k) \geq 4(f:g)+4(h:k).$$

Combining the homogeneity and the subadditivity, we have the joint concavity. $\square$

Since the product operation is lacking in the convex functionals, we use instead the operation

$$_T f(x) = f(Tx)$$

for $f \in \Gamma$ and a (bounded linear) operator T on H. Then we show another inverse property of conjugate:

Lemma 4.2. *$(_T f)^* = _{(T^{-1})^*} (f^*)$ for invertible T.*

Proof. The required formula follows from

$$(_T f)^*(y^*) = \sup_x \mathrm{Re}\ \langle x, y\rangle - f(Tx) = \sup_x \mathrm{Re}\ \left\langle Tx, (T^{-1})^* y\right\rangle - f(Tx) = f((T^{-1})^* y^*). \quad \square$$

Now, considering $_T f_A = f_{T^*AT}$, we can discuss the transformer inequality:

Lemma 4.3. *For $f, g \in \Gamma$, the parallel addition satisfies*

transformer inequality: $_T(f:g) \leq (_T f):(_T g)$.
transformer equality: $_T(f:g) = (_T f):(_T g)$ *if T is invertible.*
homogeneity: $(\alpha f : \alpha g) = \alpha(f:g)$ *for $\alpha > 0$.*
quadratic preserving: *If f and g is quadratic, then so is $f:g$.*

Proof. Since the range of T is a subspace of H, we have

$$\begin{aligned} 4_T(f:g)(x) = 4(f:g)(Tx) \leq _2 (f * g)(Tx) &= \inf_{v+w=2Tx} f(v) + g(w) \\ &\leq \inf_{y+z=2x} f(Ty) + g(Tz) = _2 \left((_T f) * (_T g) \right)(x), \end{aligned}$$

which implies the required inequality by taking the conjugation twice. Since $_X(f+g) = _X f + _X g$, Lemma 4.2 implies

$$\begin{aligned} 4_T(f:g)(x) &= _T \left((f^* + g^*)^* \right)(2x) = \left(_{(T^{-1})^*}(f^* + g^*) \right)^* (2x) \\ &= ((_T f)^* + (_T g)^*)^*(2x) = 4(_T f):(_T g)(x). \end{aligned}$$

The homogeneity follows from

$$\begin{aligned} 4(\alpha f : \alpha g)(x) &= ((\alpha f)^* + (\alpha g)^*)^*(2x) = \left(\alpha_{1/\alpha} f^* + \alpha_{1/\alpha} g^* \right)^* (2x) \\ &= \alpha \left(_{1/\alpha} f^* + _{1/\alpha}\, g^* \right)^* (2x/\alpha) = \alpha_\alpha \left(f^* + g^* \right)^* (2x/\alpha) \\ &= \alpha \left(f^* + g^* \right)^* (2x) = 4\alpha(f:g)(x). \end{aligned}$$

and the quadraticity is preserved by

$$|\gamma|^2(f:g) = (|\gamma|^2 f):(|\gamma|^2 g) = _\gamma f : _\gamma g = _\gamma(f:g). \quad \square$$

Now we can define a class of convex functional means σ_m corresponding to that of operator means m: First note that $f, g \in \Gamma$ does not always imply $f + g \in \Gamma$. In fact, if $\mathrm{dom}\ f \cap \mathrm{dom}\ g = \emptyset$, then $f + g \equiv \infty$. So, we assume here

$$\mathrm{dom}\ f \cap \mathrm{dom}\ g \neq \emptyset$$

when we discuss means of functionals in Γ. Such a pair (f, g) is denoted by $(f, g) \in \Gamma^2$. Let m be an operator mean in the sense of Kubo and Ando. Considering its integral representation, we define the *convex functional mean σ_m associated with m* by

$$(f\ \sigma_m g)(x) = af(x) + bg(x) + \int_{(0,\infty)} \left(_{\frac{1+t}{2}} f : t_{\frac{1+t}{2t}} g \right)(x) \frac{4}{1+t} d\mu_m(t).$$

If f and g are quadratic, then we easily have

$$f\ \sigma_m g(x) = af(x) + bg(x) + \int_{(0,\infty)} \big((tf) : g\big)(x)\frac{1+t}{t}d\mu_m(t).$$

In fact, since ${}_sf(x) = f(sx) = s^2 f(x)$ for quadratic f and $s > 0$, we have

$$\left({}_{\frac{1+t}{2}}f : t\,{}_{\frac{1+t}{2t}}g\right)(x)\frac{4}{1+t} = \left(\frac{(1+t)^2}{4}f : \frac{(1+t)^2}{4t}g\right)(x)\frac{4}{1+t} = (tf : g)\frac{1+t}{t}.$$

The above modification is needed for the normalization of the mean as we show in the below.

For the arithmetic mean σ_∇, the measure is decided by $a = b = 1/2$ and $\mu_\nabla((0,\infty)) = 0$. For the harmonic mean $\sigma_!$, we have $\mu_! = \delta_{\{1\}}$, the Dirac measure. For the geometric mean $\sigma_\sharp$, the corresponding measure $\mu_\sharp$ is decided by

$$d\mu_\sharp(t) = \frac{\sin\frac{\pi}{2}}{\pi\sqrt{t}(1+t)}dt.$$

Here we denote the geometric operator mean by $\sharp$.

By the above lemmas, we have fundamental properties similarly to operator means, which forms basic part of an extension of the Kubo-Ando theory:

Theorem 4.4. *For $(f,g),(h,k),(f_k,g_k) \in \Gamma^2$, $\alpha > 0$ and $\beta \in (0,1)$, a convex functional mean σ_m for an operator mean m is LBPL preserving and has the following properties:*

- **monotonicity:** $f \le h$ *and* $g \le k$ *imply* $f\sigma_m g \le h\sigma_m k$.
- **semi-continuity:** $f_n \downarrow f$ *and* $g_n \downarrow g$ *imply* $f_n\sigma_m g_n \downarrow f\sigma_m g$.
- **homogeneity:** $(\alpha f)\sigma_m(\alpha g) = \alpha(f\sigma_m g)$.
- **subadditivity:** $(f+h)\sigma_m(g+k) \ge f\sigma_m g + h\sigma_m k$.
- **joint concavity:** $(\beta f + (1-\beta)h)\sigma_m(\beta g + (1-\beta)k) \ge \beta(f\sigma_m g) + (1-\beta)(h\sigma_m k)$.
- **transformer inequality:** ${}_T(f\sigma_m g) \le ({}_Tf)\sigma_m({}_Tg)$.
- **normalization:** $f\sigma_m f = f$.
- **quadratic preserving:** *If f and g is quadratic, then so is $f\sigma g$.*

Proof. It suffices to show the normalization. By Lemma 4.3, we have

$$\begin{aligned}
\frac{4}{1+t}\left({}_{\frac{1+t}{2}}f : t\,{}_{\frac{1+t}{2t}}f\right)(x) &= \frac{1}{1+t}\left(\left({}_{\frac{1+t}{2}}f\right)^* + \left(t\,{}_{\frac{1+t}{2t}}f\right)^*\right)^*(2x) \\
&= \frac{1}{1+t}\left(\left({}_{\frac{1+t}{2}}f\right)^* + t\left({}_{\frac{1+t}{2}}f\right)^*\right)^*(2x) = \frac{1}{1+t}\left((1+t)\left({}_{\frac{1+t}{2}}f\right)^*\right)^*(2x) \\
&= \left(\left({}_{\frac{1+t}{2}}f\right)^*\right)^*\left(\frac{2}{1+t}x\right) = \left({}_{\frac{2}{1+t}}f^*\right)^*\left(\frac{2}{1+t}x\right) \\
&= {}_{\frac{1+t}{2}}(f^{**})\left(\frac{2}{1+t}x\right) = f^{**}(x) = f(x).
\end{aligned}$$

Since μ_m is a probability measure, we have $f\sigma_m f = f$. □

Considering constant functions $m\mathbf{I}$ where $\mathbf{I}(x) \equiv 1$, we have:

Corollary 4.5. *If $(f,g) \in \Gamma^2$, then $f\sigma_m g$ is also lower bounded;*

$$f, g \ge m \quad \text{implies} \quad f\sigma_m g \ge m,$$

and belongs to Γ. In particular, $f, g \ge 0$ implies $f\sigma_m g \ge 0$.

Recalling that

$$A\ m\ B = A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right)A^{1/2}$$

for operator mean m for invertible operators, we reconstruct convex functional ones:

Corollary 4.6. *If A is invertible, then*

$$f_{AmB} = {}_{A^{1/2}}f_{Im(A^{-1/2}BA^{-1/2})} = \left({}_{A^{1/2}}f_I\right)\sigma_m\left({}_{A^{1/2}}f_{A^{-1/2}BA^{-1/2}}\right).$$

Here we discuss constant functions $\mathbf{I}$ including an extension of Corollary 4.5. Before considering means for constant functions generally, we give a case of the parallel addition:

Lemma 4.7. $(f+t\mathbf{I}):(g+t\mathbf{I}) = f:g+t\mathbf{I}, \quad t\mathbf{I}:s\mathbf{I} = \frac{t+s}{4}\mathbf{I}$ *for $t,s \in \mathbb{R}$ and $f \in \Gamma$.*

Proof. Observing the proof of the subadditivity, we easily have the former. By $(t\mathbf{I})^* = \mathbf{1}_{\{0\}} - t\mathbf{I}$, we have

$$4(t\mathbf{I}:s\mathbf{I}) = \left((t\mathbf{I})^* + (s\mathbf{I})^*\right)^* = \left(\mathbf{1}_{\{0\}} - (t+s)\mathbf{I}\right)^* = (t+s)\mathbf{I}. \qquad \square$$

Thereby we have the additivity, or translation invariance of constants:

Theorem 4.8. *If c is a real number, m is an operator mean and $(f,g) \in \Gamma^2$, then*

$$(f+c\mathbf{I})\sigma_m(g+c\mathbf{I}) = f\sigma_m g + c\mathbf{I}.$$

Proof. We obtain the result by

$$\begin{aligned}
&(f+c\mathbf{I})\sigma_m(g+c\mathbf{I})(x)\\
&= a(f+c\mathbf{I})(x) + b(g+c\mathbf{I})(x) + \int_{(0,\infty)} \left({}_{\frac{1+t}{2}}(f+c\mathbf{I}) : t\,{}_{\frac{1+t}{2t}}(g+c\mathbf{I})\right)(x)\frac{4}{1+t}d\mu_m(t)\\
&= af(x) + bg(x) + (a+b)c\int_{(0,\infty)} \left(\left({}_{\frac{1+t}{2}}f + c\mathbf{I}\right) : \left(t\,{}_{\frac{1+t}{2t}}g + tc\mathbf{I}\right)\right)(x)\frac{4}{1+t}d\mu_m(t)\\
&= af(x) + bg(x) + (a+b)c\int_{(0,\infty)} \left({}_{\frac{1+t}{2}}f : t\,{}_{\frac{1+t}{2t}}g + \frac{c+tc}{4}\mathbf{I}\right)(x)\frac{4}{1+t}d\mu_m(t)\\
&= (f\sigma_m g)(x) + (a+b)c + c\int_{(0,\infty)} d\mu_m(t) = (f\sigma_m g)(x) + c. \qquad \square
\end{aligned}$$

This theorem shows that convex functional means are reduced to the case for positive functionals by translation. Recall that the order $m \le n$ as operator means is defined by $AmB \le AnB$ for all positive operators A and B. By the definition via integral representation, we immediately have a map $m \mapsto \sigma_m$ is order-preserving for positive functionals, and hence for all functionals in Γ:

Corollary 4.9. *If $m \le n$ as operator means, then $f\sigma_m g \le f\sigma_n g$ for all $(f,g) \in \Gamma^2$.*

Now we have a formula for means of constant functions:

Theorem 4.10. *If r and s are real numbers and m is an operator mean, then*

$$(r\mathbf{I})\sigma_m(s\mathbf{I}) = \left(\int_{[0,\infty]} \frac{1}{1+t}d\mu_m(t)\right) r\mathbf{I} + \left(\int_{[0,\infty]} \frac{t}{1+t}d\mu_m(t)\right) s\mathbf{I}.$$

In particular, if m is symmetric, then $(r\mathbf{I})\sigma_m(s\mathbf{I}) = \frac{r+s}{2}\mathbf{I}$.

Proof. By Lemma 4.7, we have

$$\begin{aligned}(r\mathbf{I})\sigma_m(s\mathbf{I})(x) &= ar\mathbf{I}(x) + bs\mathbf{I}(x) + \int_{(0,\infty)} \left(\tfrac{1+t}{2}(r\mathbf{I}) : t\tfrac{1+t}{2t}(s\mathbf{I})\right)(x)\frac{4}{1+t}d\mu_m(t) \\ &= ar + bs + \int_{(0,\infty)} (r\mathbf{I}) : (ts\mathbf{I})(x)\frac{4}{1+t}d\mu_m(t) \\ &= ar + bs + \int_{(0,\infty)} \frac{r+ts}{1+t}d\mu_m(t) \\ &= \left(\int_{[0,\infty]} \frac{1}{1+t}d\mu_m(t)\right) r + \left(\int_{[0,\infty]} \frac{t}{1+t}d\mu_m(t)\right) s. \qquad \square\end{aligned}$$

By the above, we can define the *weight* $\mathbf{W}(\sigma_m)$ by

$$\mathbf{W}(\sigma_m) = \int_{[0,\infty]} \frac{t}{1+t}d\mu_m(t).$$

In fact, we can confirm that the nonsymmetric degree for the weighted arithmetic, geometric or harmonic mean is equal to its weight respectively.

5. Quadratic functionals.

To observe the difference between operator means and convex functional ones, we confirm basic properties for quadratic functionals f in Γ and discuss when a convex functional mean is associated by some operator mean. Note that $f(0) = 0$, $f(x) = f(-x)$ and hence f is nonnegative by

$$f(x) = \frac{f(x) + f(-x)}{2} \geq f(0) = 0.$$

Now we characterize its *local boundedness*, which is rather 'boundedness' for functionals:

Lemma 5.1. *Every quadratic functional $f \in \Gamma$ is locally bounded in the sense $\sup_{\|x\|=1} f(x) < \infty$ if and only if f is continuous at 0. In this case, f is finite.*

Proof. Suppose f is not locally bounded. Then there exist a sequence of unit vectors x_n with $f(x_n) > n^3$. Putting $y_n = \frac{1}{n}x_n$, we have $y_n \to 0$ by $\|y_n\| = 1/n$ while $f(y_n) \to \infty$ by

$$f(y_n) = \frac{1}{n^2}f(x_n) > n.$$

Conversely suppose there exists $M > 0$ with $\sup_{x\neq 0} f(x/\|x\|) \leq M$. Then

$$0 \leq f(x) = \|x\|^2 f\left(\frac{x}{\|x\|}\right) \leq M\|x\|^2,$$

and hence $x_n \to 0$ implies $f(x_n) \to 0 = f(0)$. $\square$

Here we give a proper example in an infinite space which is not locally bounded. Let $H = \ell^2$ with an orthonormal basis $\{e_k\}$. Putting

$$f\left(\sum_{n=1}^{\infty} \gamma_n e_n\right) = \sum_{n=1}^{\infty} n^3|\gamma_n|^2,$$

we have f is not locally bounded and hence not continuous at 0. In fact, $x_n = (1/n)e_n \to 0$ while $f(x_n) \to \infty$.

Note that the parallel addition is quadratic preserving:

Lemma 5.2. *If f and g are quadratic functionals in Γ, then so is $f : g$.*

Proof. Since f and g are quadratic, then

$$ {}_2(f*g)(\gamma x) = \inf_{y+z=2x} f(\gamma y) + g(\gamma z) = |\gamma|^2 \inf_{y+z=2x} f(y) + g(z) = |\gamma|^2 \ {}_2(f*g)(x). $$

Taking the conjugation twice, we have $4(f:g)(\gamma x) = 4|\gamma|^2(f:g)(x)$. □

A quadratic functional $f \in \Gamma$ is called a *PL functional* if f satisfies

parallelogram law: $$f(x+y) + f(x-y) = 2\Big(f(x) + f(y)\Big).$$

For example, consider $f(x) = \|x\|_p^2$ for the p-(semi)norm on H. Then f is PL functional if and only if $p = 2$.

It is easy to see the conjugate preserves PL functionals:

Lemma 5.3. *If f is a PL functional, then so is f^*.*

Proof. For a PL functional $f \in \Gamma$, we have

$$ \begin{aligned} f^*(x+y) + f^*(x-y) &= \sup_{v,w} \mathrm{Re}\ \langle x+y, v\rangle - f(v) + \langle x-y, w\rangle - f(w) \\ &= \sup_{z_1,z_2} \mathrm{Re}\ \langle x+y, z_1+z_2\rangle + \langle x-y, z_1-z_2\rangle - (f(z_1+z_2) + f(z_1-z_2)) \\ &= 2\sup_{z_1,z_2} \mathrm{Re}\ \langle x, z_1\rangle + \langle y, z_2\rangle - f(z_1) - f(z_2) = 2\left(f^*(x) + f^*(y)\right). \end{aligned} $$ □

In general, locally bounded PL functionals have good properties:

Lemma 5.4. *A locally bounded PL functional $f \in \Gamma$ is finite, positive and continuous.*

Proof. It suffices to show the continuity. The parallelogram law shows

$$ f(x \pm 2y) + f(x) = 2\left(f(x \pm y) + f(y)\right), $$

Considering the difference of both sides, we have

$$ f(x+2y) - f(x-2y) = 2\left(f(x+y) - f(x-y)\right). $$

Then, it follows inductively that

$$ f(x+y) - f(x-y) = \frac{1}{2}\left(f(x+2y) - f(f-2y)\right) = \cdots = \frac{1}{2^k}\left(f(x+2^k y) - f(f-2^k y)\right). $$

For a fixed vector x, any small number $\varepsilon > 0$ and the bound $M = \sup_{\|z\|=1} f(z)$, there exists $K > 0$ with

$$ \frac{M}{2^K}(\|x\|+1) < \frac{\varepsilon}{2}. $$

For a sequence $y_n \to 0$, there exists a number N with $\|y_n\| < 1/2^K$ for all $n \geq N$. Then

$$ \begin{aligned} |f(x+y_n) - f(x-y_n)| &= \frac{1}{2^K}|f(x+2^K y_n) - f(x-2^K y_n)| \\ &\leq \frac{1}{2^K}\left(|f(x+2^K y_n)| + |f(x-2^K y_n)|\right) \leq \frac{M}{2^K}\left(\|x+2^K y_n\| + \|x-2^K y_n\|\right) \\ &\leq \frac{2M}{2^K}\left(\|x\| + 2^K\|y_n\|\right) < \frac{2M}{2^K}\left(\|x\|+1\right) < \varepsilon. \end{aligned} $$

Thus $f(x+y_n) - f(x-y_n) \to 0$ as $n \to \infty$. Moreover, by the continuity at 0,

$$ f(x+y_n) + f(x-y_n) = 2\left(f(x) + f(y_n)\right) \to 2f(x), $$

and consequently $f(x+y_n) \to f(x)$, which shows the continuity. □

As usual, we can identfy a continuous PL functional with a quadratic one for positive operators:

Lemma 5.5. *If f is a locally bounded PL functional in Γ, then it is the quadratic functional f_A for some positive operator A on H.*

Proof. Note that f is continuous. So, similarly to the von Neumann-Jordan theorem, we can define a continuous positive sesquilinear form Φ by the *polarization identity*

$$4\Phi(x, y) = f(x + y) - f(x - y) + if(x + iy) - if(x - iy).$$

Thereby there exists a bounded linear positive operator A as its Radon-Nikodym type derivative with $f = f_A$. $\square$

One of the outstanding properties of the parallel addition is *LBPL preserving* in the sense that $f : g$ is a locally bounded PL functional in Γ if f and g are so:

Lemma 5.6. *The parallel addition are LBPL preserving.*

Proof. By Lemma 5.3, the parallel addition is also PL preserving. The local boundedness follows from Theorem 3.4 and Lemma 5.1. $\square$

Theorem 5.7. *Convex functional means associated by operator ones are LBPL preserving.*

Finally, like the Kubo-Ando theory, we attempt to define convex functional means axiomatically. Let Σ_0 be the class of convex functional means $f\sigma g \in \Gamma$ for $(f, g) \in \Gamma^2$ which satisfies the monotonicity, semi-continuity, homegeneity, transformer inequality and normalization. This class Σ_0 is worth considering by the following result for constant functionals corresponding to Theorem 4.10:

Theorem 5.8. *$F = (\alpha\mathbf{I})\sigma(\beta\mathbf{I})$ is a constant functional for $\sigma \in \Sigma_0$.*

Proof. By the transformer equality for invertible T, we have

$$F = (\alpha\mathbf{I})\sigma(\beta\mathbf{I}) = ({}_T(\alpha\mathbf{I}))\,\sigma\,({}_T(\beta\mathbf{I})) = {}_T((\alpha\mathbf{I})\sigma(\beta\mathbf{I})).$$

For all nonzero vectors x and y, we can take an invertible operator T with $Tx = y$, and consequently $F(x) = C$ for some $C \in \mathbb{R}$ for all nonzero vectors x. Suppose $F \neq C\mathbf{I}$. Then, by $F \in \Gamma$, we have $F(0) = C$ and $F(x) = \infty$ for $x \neq 0$. Then, by the homogeneity,

$$F_n(x) \equiv \left(\frac{\alpha}{n}\mathbf{I}\right) \sigma \left(\frac{\beta}{n}\mathbf{I}\right)(x) = \frac{1}{n}F(x) = \infty$$

for all nonzero x while the semicontinuity implies $F_n \downarrow 0\mathbf{I}$, which is the contradiction. $\square$

To connect convex functional means to operator ones, we consider a subclass Σ of Σ_0 in which σ is LBPL preserving. Let Γ_0 be the locally bounded PL functionals in Γ. Restricting ourselves to Γ_0, we have a final theorem like the Kubo-Ando theory (Here the order is defined as in Corollary 4.9) :

Theorem 5.9. *The map, $m \to \sigma_m\Big|_{\Gamma_0}$, gives an order-preserving bijection from the class of operator means onto $\Sigma\Big|_{\Gamma_0}$.*

Proof. Let $\sigma \in \Sigma$. Since σ is LBPL preserving, then $f_A\sigma f_B$ is also locally bounded PL functional and hence equal to f_C for some positive operator C, say $A\ m\ B$. Then the binary operation m on positive operators satisfies also the monotonicity, the semi-continuity, the transformer inequality and the normalization in the operator sense, that is, m is an operator mean by the Kubo-Ando theory [11]. The injectivity is clear since all functionals belong to Γ_0 here. $\square$

Finally, we pay attention to the relation to Atteia and Raïssouli's means [3]: Their *harmonic mean* $f\tau_h g$ is defind using the conjugates by

$$f\tau_h g = \left(\frac{f^* + g^*}{2}\right)^*,$$

which is also available in a complex space in [16]. On the other hand, our harmonic mean $f\sigma_! g$ is $2(f : g)$, so that, they coincide. Thereby their convex functional means which are based on this harmonic one belongs to our class Σ at least when we discuss functionals in Γ_0. For example, recall that their geometric mean τ in [3] is defined also as the arithmetico-harmonic mean:

Theorem 5.10. *If f and g belong to Γ_0, then $f\tau g = f\sigma_\sharp g$.*

Proof. The mean $f\tau g$ is defined as $\operatorname*{s-lim}_{n\to\infty} f\gamma_n g = \operatorname*{s-lim}_{n\to\infty} f\gamma_n^* g$ by the following sequence:

$$f\gamma_0 g = f\sigma_\nabla g,\ f\gamma_0^* g = f\sigma_! g,\ f\gamma_n g = (f\gamma_{n-1}g)\sigma_\nabla(f\gamma_{n-1}^* g),\ f\gamma_n^* g = (f\gamma_{n-1}g)\sigma_!(f\gamma_{n-1}^* g).$$

Clearly, we have γ_n and γ_n^* are convex functional means associated with the corresponding operator means ∇_n and $!_n$. Thus they have the integral representation with the corresponding measure σ_{∇_n} and $\sigma_{!_n}$. Moreover they are determined only for operators. Since $f_A\tau f_B = f_{A\sharp B} = f_A\sigma_\sharp f_B$, we have they coinside. $\square$

In [16], the geometrico-harmonic mean $f\tau_{gh}g$ also intorduced, which is equal to σ_{gh} associated with the geometrico-harmonic operator mean gh for quadratic convex functionals with the same domain in Γ_0. Clearly other means in [3] coincide with our ones for convex functionals with the same domain. But the following example shows that our means can be defined even if $\operatorname{dom} f \neq \operatorname{dom} g$. Thus we have an exact extension for their convex functional means, even for the geometric mean:

Example. Let C and D be one-dimensional subspaces orthogonal each other. Then $E = C + D$ is the closed (2-dimensional) subspace. Thereby, similarly to the paragraph before Lemma 3.1, we have and hence

$$\mathbf{1}_C : \mathbf{1}_D = \frac{1}{4}\mathbf{1}_{(C+D)/2} = \mathbf{1}_{E/2} = \mathbf{1}_E,$$

so that

$$g_1 \equiv \mathbf{1}_C\tau_h\mathbf{1}_D = \mathbf{1}_E$$

by $s\mathbf{1}_C = \mathbf{1}_C$ for all $s > 0$. On the other hand, it is easy to see

$$\mathbf{1}_C + \mathbf{1}_D = \mathbf{1}_{C\cap D} = \mathbf{1}_{\{0\}} \quad \text{and hence} \quad f_1 \equiv \mathbf{1}_C\tau_a\mathbf{1}_D = \mathbf{1}_{\{0\}}$$

for the aritmetic mean τ_a. Thereby

$$g_2 \equiv f_1\tau_h g_1 = \mathbf{1}_{\{0\}+E} = \mathbf{1}_E = g_1 \quad \text{and} \quad f_2 \equiv f_1\tau_a g_2 = \mathbf{1}_{\{0\}\cap E} = \mathbf{1}_{\{0\}} = f_1.$$

Thus this procedure shows $f_n = f_1 \neq g_1 = g_n$ for all n, and hence they cannot coincide and their geometric mean cannot be defined.

On the other hand, since

$$\tfrac{1+t}{2}\mathbf{1}_C : t\tfrac{1+t}{2t}\mathbf{1}_D = \mathbf{1}_{\frac{2}{1+t}C} : \mathbf{1}_{\frac{2t}{1+t}D} = \mathbf{1}_C : \mathbf{1}_D = \mathbf{1}_E$$

for all $t > 0$, then our geometric mean $\sigma_\sharp$ is

$$(\mathbf{1}_C\sigma_\sharp\mathbf{1}_D)(x) = \int_{(0,\infty)} \mathbf{1}_E(x)\frac{4}{1+t}d\mu_\sharp(t) = \int_{(0,\infty)} \mathbf{1}_E(x)d\mu_\sharp(t) = \mathbf{1}_E.$$

Moreover this computation shows that

$$\mathbf{1}_C\sigma_m\mathbf{1}_D = \mathbf{1}_E$$

if the measure μ_m for an operator mean m satisfies $\mu_m(\{0\}) = \mu_m(\{\infty\}) = 0$ like the geometric operator mean $\sharp$.

Acknowledgements. The author would like to express his hearty thanks to Prof. Raïssouli for his giving us useful information.

References

[1] W.N.Anderson and R.J.Duffin: *Series and parallel addition of matrices*, J. Math. Anal. Appl., **26**(1969), 576–594.

[2] T.Ando: Topics on operator inequalities, Hokkaido Univ. Lecture Note, 1978.

[3] M.Atteia and M.Raïssouli: *Self dual operators on convex functionals geometric mean and square root of convex functionals*, J. Convex Anal., **8**(2001), 223–240.

[4] H.Attouch: Variational convergence for functions and operators, Pitman, 1984.

[5] P.A.Fillmore and J.P.Williams: *On operator ranges*, Adv. in Math., **7**(1971), 254–281.

[6] J.I.Fujii: *Arithmetico-geometric mean of operators*, Math. Japon., **23** (1978), 667–669.

[7] J.I.Fujii: *Operator concave functions and means of positive linear functionals*, Math. Japon., **25** (1980), 453–461.

[8] J.I.Fujii: *On Izumino's view of operator means*, Math. Japon., **33** (1988), 671-675.

[9] P.R.Halmos: A Hilbert Space Problem Book, 2nd ed., Springer-Verlag, 1982.

[10] J-B.Hiriart-Urruty and C.Lemaréchal: Convex analysis and minimization algorithms I, II, Springer-Verlag, 1993.

[11] F.Kubo and T.Ando: *Means of positive linear operators*, Math. Ann., **248** (1980) 205–224.

[12] K.Löwner: *Über monotone Matrixfunctionen*, Math. Z., **38**(1934), 177–216.

[13] U.Mosco: Convergence of convex sets and of solutions of variational inequalities, Adv. in Math., **3**(1969), 510–585.

[14] W.Pusz and S.L.Woronowicz: *Functional calculus for sesquilinear forms and the purification map*, Rep. on Math. Phys., **8** (1975), 159–170.

[15] W.Pusz and S.L.Woronowicz: *Form convex functions and the WYDL and other inequalities*, Let. in Math. Phys., **2**(1978), 505–512.

[16] M.Raïssouli and M.Chergui: *Arithmetico-Geometric and Geometrico-harmonic means of two convex functionals*, Sci. Math. Japon., **55-3**(2002), 485–492.

[17] R.T.Rockafellar and R. J-B. Wets: Variational analysis, Springer-Verlag, 1998.

Department of Arts and Sciences (Information Science), Osaka Kyoiku University, Asahi-gaoka, Kashiwara, Osaka 582-8582, Japan

E-mail address: `fujii@@cc.osaka-kyoiku.ac.jp`