ON SIMPLE LEFT SYMMETRIC ALGEBRAS OVER A SOLVABLE LIE ALGEBRA

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Abstract. The structure of complete left symmetric algebras and that of simple left symmetric algebras over a solvable Lie algebra have been studied by many authors (cf.[K], [SEG], [B]).

In [SHI] the structure of left symmetric algebras with a principal idempotent was studied.

In this paper, we shall study the structure of left symmetric algebras with a principal idempotent in I (resp. a principal nilpotent in II) and give some examples of simple left symmetric algebras over a solvable Lie algebra in III.

I.[A] Let $\mathcal{G}$ be a Lie algebra over a field K of characteristic 0, and A a left symmetric algebra over $\mathcal{G}$.

A symmetric bilinear form B of A is called of *Hessian type*([SHI]) if the following equality holds:

$$B(xy, z) + B(y, xz) = B(yx, z) + B(x, yz) \qquad (x, y, z \in A)$$

Denote by h the symmetric bilinear form of A defined by

$$h(x, y) = \mathrm{Tr}R(xy) \qquad (x, y \in A),$$

where $R(x)$ (resp. $L(x)$) denotes the right (resp. left) multiplication of A by x. Then h is of Hessian type. It is called *the canonical 2-form* of A.

Denote by $A^{\perp}$ the linear subspace of A defined by

$$A^{\perp} = \{x \in A; h(x, y) = 0 \quad (y \in A)\}.$$

A is called *non degenerate* if $A^{\perp} = \{0\}$.

Let u be an element of A. u is called *a principal idempotent* if

(1) $uu = u$, and

(2) u generates a left ideal $\langle u \rangle$ of A

For a principal idempotent u of A, denote by P a linear subspace of A defined by

$$P = \{x \in A; xu = 0\}.$$

Then P is a linear subspace of A containing $[P, P]$ and $A = \{u\} \oplus P$ as a linear space.

For $x, y \in P$, put

$$xy = x * y + B(x, y)u,$$

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where $x * y$ (resp. $B(x,y)u$) denote the P-component (resp. $\{u\}$ − component) of xy in $\{u\} \oplus P$.

By a direct calculation, we obtain the following.

Theorem 1 *Let $A = \{u\} \oplus P$ be a left symmetric algebra with a principal idempotent u. Then we have the following:*

(1) *$(P, *)$ is a left symmetric algebra.*

(2) *B is a symmetric bilinear form of $(P, *)$ of Hessian type.*

(3) *$D = L(u) \mid P$ is a derivation of $(P, *)$ stisfying the following relation:*

$$B(x,y) = B(Dx,y) + B(x,Dy), \quad (x,y \in P) \quad (1)$$

(4) *$h(u,u) = 1$, $h(u,P) = 0$, $h = h^* + B$ on P,*

where h (resp.h^*) denote the canonical 2-form of A (resp.P).

A pair (B, D) of a symmetric bilinear form B of Hessian type and a derivation D of $(P, *)$ satisfying the condition (1) is called *compatible of 1st kind.*

For a given compatible pair (B, D) of 1st kind of a left symmetric algebra $(P, *)$, define a binomial product on a linear space $A = \{u\} \oplus P$ as follows:

$$\begin{gathered} uu = u,\ ux = Dx,\ xu = 0, \\ xy = x * y + B(x,y)u \quad (x,y \in P). \end{gathered}$$

Then we can easily prove that the algebra $A(P, B, D)$ defined above is a left symmetric algebra $A = \{u\} \oplus P$ with a principal idempotent u.

Assume that the underlying Lie algebra $\mathcal{G}$ of a left symmetric algebra A is a solvable Lie algebra over the field C of all complex numbers. Then there exists an element u of A which generates a left ideal $\langle u \rangle$ of A, by Lie's theorem. Moreover we may assume that $uu = u$, or $uu = 0$. Thus we obtain the following.

Proposition 1 *Let A be a left symmetric algebra over a solvable Lie algebra over C. Assume that the radical $R(A) = \{0\}$. Then there exists a principal idempotent u of A and a compatible pair (B, D) of 1st kind satisfying the condition in Theorem 1.*

[B] Let $A(P, B, D)$ be a left symmetric algebra $\{u\} \oplus P$ with a principal idempotent u corresponding a compatible pair (B, D) of $(P, *)$ of 1st kind.

It is clear that $B = 0$ if and only if P is an ideal of $A = \{u\} \oplus P$.

Let A' be an ideal of A. If $u + x (x \in P)$ is contained in A', then $u = (u + x)u$ is an element of A'. Therefore we can easily prove the following.

Proposition 2 *Let A' be an ideal of a left symmetric algebra $A = A(P, B, D) = \{u\} \oplus P$ with a principal idempotent u. Then we have the following.*

(1) *$A' = \{u\} \oplus Q$ is an ideal of A if and only if Q is an ideal of $(P, *)$ satisfying $DP \subset Q$.*

(2) *A linear subspace Q of P is an ideal of A if and only if Q is an ideal of $(P, *)$ satisfying $DQ \subset Q$ and $B(P, Q) = 0$.*

Let $A_i(P_i, B_i, D_i)$ $(i = 1, 2)$ be a left symmetric algebra $\{u_i\} \oplus P_i$ with a principal idempotent u_i constructed by (P_i, B_i, D_i).

$P = P_1 \oplus P_2$ as an algebra. Let D (resp. B) be a derivation (resp. a symmetric bilinear form) of P defined as follows:

$$D \mid P_i = D_i,$$
$$B \mid P_i = B_i \text{ and } B(P_1, P_2) = 0.$$

Then it is clear that (B, D) is a compatible pair of P of 1st kind. The corresponding algebra $A = \{u\} \oplus P$ with a principal idempotent u is called *the algebra constructed by the direct sum* $P = P_1 \oplus P_2$.

A left symmetric algebra $A(P, B, D)$ with a principal idempotent u is called *decomposable* if there exist non trivial algebras P_i with a compatible pair $(B_i, D_i)(i = 1, 2)$ such that

$$(P, B, D) = (P_1, B_1, D_1) \oplus (P_2, B_2, D_2)$$

By the definition and the above proposition, we obtain the following.

Proposition 3 *Let $A(P, B, D)$ be the left symmetric algebra constructed by the direct sum of $(P_i, B_i, D_i)(i = 1, 2)$. If both $(P_i, B_i, D_i)(i = 1, 2)$ are simple, then A is simple.*

[C]

Proposition 4 *Let $\mathcal{G}$ be a solvable Lie algebra and $A = \{u\} \oplus P = A(P, B, D)$ a left symmetric algebra over $\mathcal{G}$ with a principal idempotent u corresponding to a compatible pair (B, D) of 1st kind of a left symmetric algebra P.*

If D is non singular and B is non degenerate, then A is simple and non degenerate.

In fact, if D is non singular, then we have $P = [A, A]$. Thus we have

$$\mathrm{Tr}R^*(x) = \mathrm{Tr}R(x) = 0 \ (x \in P),$$

where R^* denotes the right multiplication of $(P, *)$. Therefore $(P, *)$ is a complete algebra over a nilpotent Lie algebra $[\mathcal{G}, \mathcal{G}]$. Moreover if B is non degenerate, then A is non degenerate, by Theorm 1 (4), and simple, by Proposition2.

A symmetric bilinear form B of $(P, *)$ is called *a trace* form if the following relation holds:

$$B(x * y, z) = B(x, y * z) \ (x, y, z \in P).$$

Lemma 1 *If a left symmetric algebra $(P, *)$ is commutative, then*

(1) $(P, *)$ is associative,

(2) a symmetric bilinear form B of P of Hessian type is a trace form, and

(3) $P(B)^{\perp}$ is an ideal of P, where $P(B)^{\perp}$ denotes a linear subspace of P defined by

$$P(B)^{\perp} = \{x \in P; B(x, y) = 0 (y \in P)\}.$$

Proposition 5 *If $(P, *)$ is commutative and complete, then $P(B)^{\perp}$ is an ideal of $A(P, B, D)$. Therefore if $A(P, B, D)$ is simple, then the symmetric bilinear form B is non degenerate.*

A derivation D of P is called *split* if all eigen values of a linear endomorphism D of P are contained in the base field K.

For an eigen value λ of a split derivation D, denote by P_{λ} the linear subspace of P defined by

$$P_{\lambda} = \{x \in P; (D - \lambda id)^m x = 0, \text{ for some positive integer } m\}.$$

Then P is decomposed into the direct sum $P = \oplus_{\lambda} P_{\lambda}$ of weight spaces $\{P_{\lambda}\}_{\lambda \in \Lambda}$ satisfying $P_{\lambda} * P_{\mu} \subset P_{\lambda+\mu}$.

By Theorem 1,(3), we obtain the following.

Proposition 6 *Let $A(P, B, D)$ be a left symmetric algebra corresponding to a compatible pair of a zero algebra P of 1st kind with a split derivation D.*

If A is simple and indecomposable, then

(1) *B is non degenerate, and*

(2) *$P = P_{1/2}$ or $P = P_{\lambda} \oplus P_{1-\lambda}$ $(\lambda \in K, \lambda \neq 1/2, 0, 1)$.*

II.[A] Let $\mathcal{G}$ be a Lie algebra over K, and A a left symmetric algebra over $\mathcal{G}$.

An element v of A is called *principal nilpotent* if

(1) $R(v) = 0$, and

(2) $v \notin [A, A]$.

For a principal nilpotent v of A, there exists a linear subspace P of A of codimension 1 such that

(1) $P \supset [A, A]$, and

(2) $A = \{v\} \oplus P$ as a linear space.

For $x, y \in P$, put

$$xy = x * y + C(x, y)v,$$

where $x * y$ (resp. $C(x, y)v$) denotes the P-component (resp.$\{u\} -$ component) of xy.

Then we have

(1) $[x, y] = x * y - y * x,$

(2) $C(x, y) = C(y, x).$

Moreover we can easily prove the following.

Theorem 2 *Let $A = \{v\} \oplus P$ be a left symmetric algebra with a principal nilpotent v. Then we have the following:*

(1) *$(P, *)$ is a left symmetric algebra.*

(2) *C is a symmetric bilinear form of $(P, *)$ of Hessian type.*

(3) *$D = L(v) \mid P$ is a derivation of $(P, *)$ satisfying the following relation:*

$$C(Dx, y) + C(x, Dy) = 0 \ (x, y \in P). \qquad (2)$$

Conversely, let $(P, *)$ be a left symmetric algebra with a symmetric bilinear form C of Hessian type and a derivation D satisfying the above relation (2).

Define a bilinear product on a linear space $A = \{v\} \oplus P$ as follows:

$$vv = 0,\ vx = Dx,\ xv = 0, \text{ and}$$
$$xy = x * y + C(x, y)v \ (x, y \in P).$$

Then it is clear that the algebra $A(P, C, D)$ with the above multiplication is a left symmetric algebra with a principal nilpotent v. (C, D) is called *a compatible pair of P of 2th kind.*

Proposition 7 *Let $A(P, C, D)$ be a left symmetric algebra with a principal nilpotent v corresponding to a compatible pair (C, D) of 2nd kind. Then A is complete if and only if $(P, *)$ is complete.*

In fact, for $x \in P$, we have

$$\mathrm{Tr}R(x) = \mathrm{Tr}R^*(x),$$

where R (resp. R^*) denotes the right multiplication of A (resp. $(P, *)$). □

We can easily prove the following.

Proposition2' *Let $A(P, C, D)$ be a left symmetric algebra with a principal nilpotent v corresponding to a compatible pair (C, D) of 2nd kind.*

(1) *$A' = \{v\} \oplus Q$ is an ideal of $A = \{v\} \oplus P$ if and only if Q is an ideal of $(P, *)$ satisfying $DQ \subset Q$.*

(2) *A linear subspace Q of P is an ideal of A if and only if Q is an ideal of $(P, *)$ satisfying $DQ \subset Q$ and $C(P, Q) = 0$.*

Remark. There exists an ideal A' of $A(P, C, D)$ such that $v \notin A'$ and $A' \not\subset P$. (cf. Example (d). $A_1(P_1, C_1, D_1)$).

Let $A_i(P_i, C_i, D_i)$ $(i = 1, 2)$ be a left symmetric algebra $\{v_i\} \oplus P_i$ with a principal nilpotent v_i constructed by (P_i, C_i, D_i).

Put $P = P_1 \oplus P_2$ as an algebra. Denote by D (resp.C) a derivation (resp. a symmetric bilinear form) of P defined as follows:

$$D \mid P_i = D_i,$$
$$C \mid P_i = C_i \text{ and } C(P_1, P_2) = 0 \ (i = 1, 2).$$

Then it is clear that (C, D) is a compatible pair of P of 2nd kind. The corresponding algebra $A = \{v\} \oplus P$ with a principal nilpotent v is called *the algebra constructed by the direct sum* $P = P_1 \oplus P_2$.

A left symmetric algebra $A(P, C, D)$ with a principal nilpotent v is called *decomposable* if there exist non trival algebra P_i $(i = 1, 2)$ with a compatible pair (C_i, D_i) of 2nd kind such that $P = P_1 \oplus P_2$

Proposition3' *Let $A(P, C, D)$ be the algebra constructed by the direct sum (P_i, C_i, D_i) $(i = 1, 2)$. If both the algebra $A_i(P_i, C_i, D_i)$ are simple, then A is simple.*

Let $(P, *)$ be a complete left symmetric algebra over a nilpotent Lie algebra, and (C, D) a compatible pair of P of 2nd kind. Denote by $A(P, C, D)$ the left symmetric algebra with a principal nilpotent v corresponding to (C, D). Then, by Proposition 7, A is complete. Moreover, by Propsition 2', if $DP = P$ and C is non degenerate, then A is simple. Thus we obtain the following.

Proposition 8 *Let $(P, *)$ be a complete left symmetric algebra over a nilpotent Lie algebra, and $A(P, C, D)$ a left symmetric algebra with a principal nilpotent v corresponding to a compatible pair (C, D) of P of 2nd kind.*

If $DP = P$ and C is non degenerate, then A is complete and simple.

Corollary *Assume that $(P, *)$ is a zero algebra. Then the following statements are mutually equivalent:*

(1) *$A(P, C, D)$ is complete and simple.*

(2) *$DP = P$ and C is non degenerate.*

[B] Let $\mathcal{G}$ be a complex Lie algebra, (V, ρ) a $\mathcal{G}$-module corresponding to a Lie homomorphism ρ of $\mathcal{G}$ into the linear endomorphism ring $\mathrm{gl}(V)$ of V, and A a left symmetric algebra over $\mathcal{G}$.

We can easily prove the following.

Lemma 2 *For any $\alpha, \beta \in C$, $x, y \in \mathcal{G}$ and $u \in V$, we have*

$$(\rho(x) - (\alpha + \beta)\mathrm{id})^m (\rho(y)u) = \sum_{k=0}^{m} \binom{m}{k} \rho((\mathrm{ad}x - \beta\mathrm{id})^k y)\{(\rho(x) - \alpha\mathrm{id})^{m-k} u\}.$$

Corollary *For a left symmetric algebra A over $\mathcal{G}$. We have*

$$(3)\ (L(x) - (\alpha + \beta)\mathrm{id})^m (yz) = \sum_{k=0}^{m} \binom{m}{k} L((\mathrm{ad}x - \beta\mathrm{id})^k y)\{(L(x) - \alpha\mathrm{id})^{m-k} z\}$$

for any $\alpha, \beta \in C$, $x, y, z \in A$.

Assume that $\mathcal{G}$ is a solvable Lie algebra. Then, by Lie's theorem, there exists a base $\{x_1, x_2, \cdots, x_n\}$ of A such that $L(x)$ $(x \in A)$ is expressed as a upper triangular matrix with respect to the base $\{x_i\}$. The base $\{x_i\}$ of A is called *a canonical base* of a left symmetric algebra A. Denote by $(L(x)_{ij})_{1 \leq i,j \leq n}$ (resp. $(R(x)_{ij})_{1 \leq i,j \leq n}$) the matrix representation of $L(x)$ (resp. $R(x)$) with respect to the canonical base $\{x_i\}$.

Lemma 3 *Let $\langle v \rangle$ be a left ideal of A generated by v with $vv = 0$. Then we have* $[L(v), R(v)] = 0$.

In fact, since A is left symmetric, we have the following equality:

$$[L(x), R(x)] = R(x)R(y) + R(xy) \ (x, y \in A). \ (4)$$

For a left ideal $\langle v \rangle$, there exists a canonical base $\{x_i\}$ of A such that $x_1 = v$. With respect to the base $\{x_i\}$, denote by ϵ_i the $(1,1)$-component of $L(x_i)$. Then we have

$$R(v)_{ij} = \begin{cases} \varepsilon_{ij}, & \text{if } (i,j) = (1,j) \\ 0, & \text{otherwise.} \end{cases}$$

By the assumption that $vv = 0$, $\epsilon_1 = 0$ and $R(v)^2 = 0$. Thus, by (4), we obtain the desired equality. □

For a left ideal $\langle v \rangle$ with $vv = 0$, denote by $A^\alpha(v)$ (resp. $\mathcal{G}^\alpha(v)$) a linear subspace of $A(= \mathcal{G})$ defined by

$$A^\alpha(v) = \{x \in A; (L(v) - \alpha \mathrm{id})^m x = 0, \text{ for some positive integer } m\}$$

$$(\text{resp. } \mathcal{G}^\alpha(v) = \{x \in A; \ (\mathrm{ad}\, v - \alpha \, \mathrm{id})^m \, x = 0, \text{ for some positive integer } m\}) \, .$$

Proposition 9 *Let A be a left symmetric algebra over a complex solvable Lie algebra $\mathcal{G}$ and $\langle v \rangle$ a left ideal of A with $vv = 0$. Then we have*

(1) $A^\alpha(v) = \mathcal{G}^\alpha(v)$, *for any* $\alpha \in C$,

(2) $A(v)^\alpha A(v)^\beta \subset A^{\alpha+\beta}(v)$.

Therefore $A^0(v)(= \mathcal{G}^0(v))$ is a subalgebra of A containing v.

Proof. (1) Since $[L(v), R(v)] = 0$, by Lemma 4, we obtain the equality.
(2) By (1) and the equality (3), Lemma 3, Corollary, we obtain the inclusion (2). □

Now we quote the following ([K],[SEG]).

Lemma 4 *Let A be a left symmetric algebra over a Lie algebra $\mathcal{G}$. Then the following statements* (1), (2) *and* (3) *(resp.* (4) *and* (5)*) are mutually equivalent:*

(1) *A is complete,*

(2) $R(x)$ $(x \in A)$ *is nilpotent,*

(3) $\mathrm{Tr}R(x) = 0$ $(x \in A)$.

(4) $L(x)$ $(x \in A)$ *is nilpotent,*

(5) $R(x)$ $(x \in A)$ *is nilpotent and* $\mathcal{G}$ *is a nilpotent Lie algebra.*

Proposition 10 *Let A be a left symmetric algebra over a complex solvable Lie algebra $\mathcal{G}$, $\langle v \rangle$ a left ideal of A with $vv = 0$, and $A = \oplus_\alpha A^\alpha(v)$ the weight space decomposition of A with respect to $L(v)$.*

If $A^0(v)$ is a proper subalgebra of A and $R(v) \mid A^0(v) = 0$, then v is a principal nilpotent of A.

In fact, since $\langle v \rangle$ is a left ideal of A, we have

$$A^\alpha(v)v = 0 \ (\alpha \notin 0),$$

by Proposition 9. Therefore $R(v) \mid A^0(v) = 0$ implies that $R(v) = 0$.

Moreover since $A^0(v)$ $(= \mathcal{G}^0(v)) \neq A$, $L(v)$ is not nilpotent, that is, $v \notin [\mathcal{G}, \mathcal{G}]$. Thus v is a principal nilpotent of A. □

Corollary 1 *If $A^0(v)$ is a complete, proper subalgebra of A over a nilpotent Lie algebra, then v is a principal nilpotent of A.*

In fact, if $A^0(v)$ is a complete algebra over a nilpotent Lie algebra, then, by Lemma 5, $L(x) \mid A^0$ $(x \in A^0(v))$ is nilpotent. Therefore we have $R(v) \mid A^0(v) = 0$. □

Corollary 2 *If* $\dim A^0(v) = 1$*, then A is complete and v is a principal nilpotent of A.*

In fact, if $\dim A^0(v) = 1$, then v is a principal nilpotent, by Corollary 1. Moreover any element x of $A^\alpha(v)$ $(\alpha \neq 0)$ is contained in the derived Lie algebra $[\mathcal{G}, \mathcal{G}]$. Therefore we have $\mathrm{Tr}R(x) = 0$ $(x \in A)$. □

Corollary 3 *Assume that A is complete. If $A^0(v)$ is a proper subalgebra of A and the underlying Lie algebra of $A^0(v)$ is nilpotent, then v is a principal nilpotent of A.*

In fact, since A is complete, $A^0(v)$ is also complete, by lemma 4. If the underlying Lie algebra of a complete subalgebra $A^0(v)$ is nilpotent, then $L(x) \mid A^0(v)$ $(x \in A^0(v))$ is nilpotent, by Lemma 4. Therefore $R(v) \mid A^0(v) = 0$. □

[C] Let $A_i(P_i, C_i, D_i)$ $(i = 1, 2)$ be a left symmetric algebra with a principal nilpotent v_i constructed by a compatible pair (C_i, D_i) of P_i of 2nd kind, and $A(P, C, D)$ the left symmetric algebra with a pincipal nilpotent v constructed by the direct sum $P = P_1 \oplus P_2$. Then $\{v\} \oplus Q_1 \oplus Q_2$ $(Q_i \subset P_i)$ is a left ideal of A if and only if $\{v_i\} \oplus Q_i$ $(i = 1, 2)$ is a left ideal of A_i. Moreover we have

$$\begin{aligned} &\mathrm{Tr}R(x_i) = \mathrm{Tr}R_i^*(x_i), \text{ and} \\ &h(x_i, y_i) = h_i^*(x_i, y_i)\ (x_i, y_i \in P_i), \end{aligned}$$

where R (resp. R_i^*) denotes the right multiplication of A (resp. $(P_i, *)$) and h (resp. h_i^*) denotes the canonical 2-form of A (resp. $(P_i, *)$). Thus we obtain the following.

Proposition 11 *Let $A_i(P_i, C_i, D_i)$ $(i = 1, 2)$ be a left symmetric algebra with a principal nilpotent v_i, and $A(P, C, D)$ the left symmetric algebra with a principal nilpotent v construted by the direct sum $P = P_1 \oplus P_2$.*

(1) *The radical $R(A)$ of A is expressed as*

$$R(A) = \{v\} \oplus Q_1 \oplus Q_2,$$

where Q_i is a D_i-invariant left ideal of $(P_i, *)$ such that $R(A_i) = \{v_i\} \oplus Q_i$ $(i = 1, 2)$.

(2) *$(P_i, *)$ is non degenerate if and only if*

$$R(A_i) = A_i^{\perp} = \{v_i\},$$

where $A_i^{\perp}$ denotes the orthogonal complement of A_i with respect to the canonical 2-form h_i of A_i $(i = 1, 2)$.

Corollary *Assume that $(P_1, *)$ is complete and $(P_2, *)$ is non degenerate. Then we have the following:*

(1) *$R(A) = \{v\} \oplus P_1$, and*

(2) *if $D_2 \neq 0$, then $R(A) = \{v\} \oplus P_2$ is not an ideal of A.*

Remark. Using the above corollary, we can construct a left symmetric algebra A over a solvable Lie algebra whose radical $R(A)$ is not an ideal of A. Therefore a theroem stated in my paper[M1] is a fault (cf. Example (d)).

III. Let $(P, *)$ be a left symmetric algebra over K. We shall give some examples of compatibale pair (B, D) (resp. (C, D)) of 1st kind (resp. 2nd kind).

(a) $P\{x_1, x_2, \cdots, x_m\}$: a zero algebra with a base $\{x_1, x_2, \cdots, x_m\}$.

(1) $$Dx_i = \tfrac{1}{2}x_i,\ B(x_i, x_j) = \delta_{ij}$$

(2) $$Dx_i = \tfrac{1}{2}x_i,$$

$$B(x_i, x_j) = \begin{cases} 0, & \text{if } i + j \neq m + 1, \\ 1, & \text{if } i + j = m + 1. \end{cases}$$

(3) $$Dx_i = \tfrac{1}{2}x_i + x_{i-1},$$

$$B(x_i, x_j) = \begin{cases} 0, & \text{if } i+j \neq m+1, \\ (-1)^{i-1}, & \text{if } i+j = m+1, \end{cases}$$

where m is an odd integer and $x_0 = 0$.

The corresponding algebra $A(P, B, D)$ are simple and non degenerate([SHI],[B] for(1)).

(b) $P\{x_i, y_i\}_{1 \le i \le m}$: a zero algebra with a base $\{x_i, y_i\}_{1 \le i \le m}$.

(1) $Dx_i = \lambda x_i$, $Dy_i = (1-\lambda)y_i$ $(\lambda \in K, \lambda \neq \frac{1}{2}, 0, 1)$
$B(x_i, x_j) = 0$, $B(y_i, y_j) = 0$, $B(x_i, y_j) = \delta_{ij}$,

(2) $Dx_i = \lambda x_i + x_{i-1}$, $Dy_i = (1-\lambda)y_i + y_{i-1}$ $(\lambda \in K, \lambda \neq \frac{1}{2}, 0, 1)$,
$B(x_i, x_j) = 0$, $B(y_i, y_j) = 0$,

$$B(x_i, y_j) = \begin{cases} 0, & \text{if } i+j \neq m+1, \\ (-1)^{i-1}, & \text{if } i+j = m+1. \end{cases}$$

(3) $Dx_i = \lambda x_i$, $Dy_i = -\lambda y_i$ $(\lambda \in K, \lambda \neq 0)$,
$C(x_i, x_j) = 0$, $C(y_i, y_j) = 0$, $C(x_i, y_j) = \delta_{ij}$,

(4) $Dx_i = \lambda x_i + x_{i-1}$, $Dy_i = -\lambda y_i + y_{i-1}$ $(\lambda \in K, \lambda \neq 0)$,
$C(x_i, x_j) = 0$, $C(y_i, y_j) = 0$,

$$C(x_i, y_j) = \begin{cases} 0, & \text{if } i+j \neq m+1, \\ (-1)^{i-1}, & \text{if } i+j = m+1, \end{cases}$$

where m is an odd integer and $x_0 = y_0 = 0$, for (2) and (4).

Both the algebra $A(P, B, D)$ (1) and (2) are simple and non degenerate. Both the algebra $A(P, B, D)$ (3) and (4) are complete and simple ([B] for (1),(3)).

(c) $P\{w_1, w_2, w_3\}$: a complete algebra with the following multiplication table.

P	w_1	w_2	w_3
w_1	0	0	0
w_2	0	0	$2w_1$
w_3	0	w_1	0

(1) $Dw_1 = -\beta w_1$, $Dw_2 = \beta w_2$, $Dw_3 = -2\beta w_3$,$(\beta \in K, \beta \neq 0)$

C	w_1	w_2	w_3
w_1	0	1	0
w_2	1	0	0
w_3	0	0	0

(2) $Dw_1 = w_1$, $Dw_2 = \beta w_2$, $Dw_3 = (1-\beta)w_3$,$(\beta \in K, \beta \neq 0, 1)$

B	w_1	w_2	w_3
w_1	0	0	0
w_2	0	0	1
w_3	0	1	0

(3) $Dw_1 = (1-\beta)w_1$, $Dw_2 = \beta w_2$, $Dw_3 = (1-2\beta)w_3$,$(\beta \in K, \beta \neq 0, 1)$

B	w_1	w_2	w_3
w_1	0	1	0
w_2	1	0	0
w_3	0	0	0

The algebra $A(P, B, D)(1)$ is complete and simple (cf. [B]). The algebra $A(P, B, D)(2)$ is degenerate, and the radical $R(A)$ of A ($= A^{\perp} = \{w_1\}$) is an ideal of A. The algebra $A(P, B, D)(3)$ is simple, $R(A) = 0$, and $A^{\perp} = \{w_3\}$.

Let $(P, *)$ be a left symmetric algebra, and h^* the canonical 2-form of $(P, *)$.
For any derivation D of $(P, *)$, we have

$$[D, R^*(x)] = R^*(Dx), \text{ ie. } \mathrm{Tr} R^* = 0.$$

Therefore we obtain the following.

Lemma For any derivation D of $(P, *)$, (h^*, D) is a compatible pair of 2nd kind.

(d)

(1) $P_1\{x, y_i\}_{1\leq i\leq r}$ $(r \geq 2)$: a non degenerate algebra with a base $\{x, y_i\}_{1\leq i\leq r}$ having the following muliplication :
$x * x = x$,
$x * y_i = \frac{1}{2}y_i$,
$y_i * x = 0$,
$y_i * y_j = \delta_{i\ r-j+1}$.

$D_1x = 0$,
$D_1y_i = \alpha_i y_i$,
$\alpha_i + \alpha_{r-i+1} = 0$,

$$\alpha_i = \begin{cases} 0, & \text{if } r\text{: odd and } i = \frac{r+1}{2}\ , \\ nonzero, & \text{otherwise.} \end{cases}$$

$C_1 =$ the canoncial 2-form h_1 of P_1.

(2) $P_2\{z_i\}_{1\leq i\leq 2s}$: a zero algebra with a base $\{z_i\}_{1\leq i\leq 2s}$.
$D_2 z_i = \beta_i z_i$,
$\beta_i + \beta_{2s-i+1} = 0$ $(\beta_i \neq 0)$,
$C_2(z_i, z_j) = \delta_{i\ 2s-j+1}$.

(3) $(P_3\{w_1, w_2, w_3\}, C_3, D_3)$ is the algebra with a compatible pair (C_3, D_3) of 2nd kind described in (c), (1).

Denote by A_{ij} (resp. A_{123}) the simple left symmetric algebra with a principal nilpotent v constructed by the direct sum $P_i \oplus P_j$ (resp. $P_1 \oplus (P_2 \oplus P_3)$) $(1 \leq i \neq j \leq 3)$. Then we obtain the following :

(1) A_{23} is complete,

(2) the radical $R(A_{12})$ (resp. A_{13}, A_{123}) is a left ideal $\{v\} \oplus P_2$ (resp. $\{v\} \oplus P_3$, $\{v\} \oplus (P_2 \oplus P_3)$) of A_{12} (resp. A_{13}, A_{123}) which is not an ideal.

(3) $R(A_{12}) = A_{12}^{\perp}$ (resp. $R(A_{13}) = A_{13}^{\perp}$, $R(A_{123}) = A_{123}^{\perp}$).

(e) Let A be a left symmetric algebra of dimension $3r$ with a base $\{u_i, x_i, y_i\}_{1 \leq i \leq r}$ and the following multiplication table:

(1) $u_iu_j = \delta_{ij}u_j$, $u_ix_j = \alpha_{ij}x_j$, $u_iy_j = \beta_{ij}y_j$, $x_ju_i = y_ju_i = 0$,
$x_ix_j = y_iy_j = 0$, $x_iy_j = y_jx_i = \delta_{ij}u_j$
$\alpha_{ij}, \beta_{ij} \neq 0$, $\alpha_{ij} + \beta_{ij} = \delta_{ij}$.

(2) $u_iu_j = 0$, $u_ix_j = \alpha_{ij}x_j$, $u_iy_j = \beta_{ij}y_j$, $x_ju_i = y_ju_i = 0$,
$x_ix_j = y_iy_j = 0$, $x_iy_j = y_jx_i = \delta_{ij}u_j$
$\alpha_{ij}, \beta_{ij} \neq 0$, $\alpha_{ij} + \beta_{ij} = 0$, $\det(\alpha_{ij})_{i,j=1,2,\cdots,r} \neq 0$

The algebra (1) (resp. (2)) is simple and non degenerate (resp. simple and complete).

(f) Let A_1 (resp. A_2, A_3) be a left symmetric algebra of dimension 8 with a base $\{v_1, v_2, x_i, y_i\}_{1 \leq i \leq 3}$ and the following multiplication:
$v_iv_j = 0$, $v_ix_j = \alpha_{ij}x_j$, $v_ix_j = -\alpha_{ij}y_j$
$x_iv_j = y_iv_j = 0$, $x_ix_j = y_iy_j = 0$

(1) $\alpha_{ij} \neq 0$, except for $(i,j) = (2,3)$,

$$x_iy_j = y_jx_i = \begin{cases} v_1 + v_2, & \text{if } i = j = 1, 2, \\ v_1, & \text{if } i = j = 3, \\ 0, & \text{otherwise.} \end{cases}$$

(2) $\alpha_{ij} \neq 0$, except for $(i,j) = (2,2), (2,3)$

$$x_iy_j = y_jx_i = \begin{cases} v_1 + v_2, & \text{if } i = j = 1, \\ v_1, & \text{if } i = j = 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

(3) $\alpha_{ij} \neq 0$, except for $(i,j) = (2,1), (1,3)$

$$x_iy_j = y_jx_i = \begin{cases} v_1, & \text{if } i = j = 1, \\ v_1 + v_2, & \text{if } i = j = 2, \\ v_2, & \text{if } i = j = 3, \\ 0, & \text{otherwise.} \end{cases}$$

Then A_1 (resp. A_2, A_3) is a simple complete algebra([B]).

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