# ASYMPTOTIC ESTIMATION THEORY FOR TIME SERIES REGRESSION MODELS WITH MULTIPLE CHANGE POINTS 

Takayuki Shiohama<br>Received June 7, 2002; revised June 14, 2002


#### Abstract

This paper discusses the problem of estimating multiple change points in the trend function of a time series regression model where the residual process is a circular ARMA model, and the trend function satisfies a sort of Grenander's conditions. First, the asymptotic representation of the likelihood ratio between contiguous hypothesis is given. Then the limiting distributions of the maximum likelihood estimator (MLE) and the Bayes estimator (BE) for the regression coefficients and change points are derived. It is seen that the BE is asymptotically efficient, and that the MLE is not so generally.


## 1. Introduction

The problem of testing and estimating change point in linear regression model attracted much attention from both econometrics and statistics researchers. For example, Bai (1997) studied the least squares estimation of change point in multiple regression where he showed the consistency, rate of convergence and the asymptotic distribution of an estimator. For testing structural changes, we refer the recent contributions of Andrews (1993), Andrews and Ploberger (1996) and Hidalgo and Robinson (1996).

In comparison, the literature addressing the issue of multiple structural changes is relatively sparse. Garcia and Perron (1996) studied the Wald test for two changes in a dynamic time series. Liu, Wu and Zidek (1997) studied multiple shifts in linear regression model estimated by least squares and obtained the consistency and the rate of convergence of the estimated break dates. Bai and Perron (1998) extended their results allowing for general forms of serial correlation and heterosckedasticity in the errors.

From statistical point of view, it is very important to investigate the asymptotically efficient estimators. A number of authors considered the consistency property and the rate of convergence for estimated change points, but the asymptotically efficient estimators were not as well studied in the literature. For a diffusion type process, the problem of detecting multiple changes was studied by Kutoyants (1984), and he obtained that the Bayes estimator is asymptotically efficient. In ARMA context, Shiohama, Taniguchi and Puri (2002) investigated a regression model with trend functions, and obtained the consistency and the limiting distributions of Bayes estimators ( BE ) and maximum likelihood estimators (MLE). They also showed that the BE is asymptotically efficient.

In this paper we develop the asymptotic theory for estimators of multiple change points in time series regressions. The results include consistency, asymptotic distributions and asymptotic efficiency. To show these we use the general results given by Ibragimov and Has'minski (1981).

This paper is organized as follows. Section 2 specifies the model and describes assumptions. Also in Section 2, the asymptotic representation of the likelihood ratio process

[^0]between contiguous hypothesis is given. Section 3 defines the MLE and BE for unknown parameters, and states the asymptotics of these estimators. A numerical example is illustrated in Section 4. Technical materials are collected in Section 5.

## 2. Regression model with multiple changes

Consider the following time series regression models

$$
\begin{align*}
y_{t} & =\left\{\begin{array}{cc}
\boldsymbol{\beta}_{1}^{\prime} \boldsymbol{z}_{t}+u_{t} & t=1, \ldots,\left[\tau_{1} n\right] \\
\boldsymbol{\beta}_{2}^{\prime} \boldsymbol{z}_{t}+u_{t}, & t=\left[\tau_{1} n\right]+1 \ldots,\left[\tau_{2} n\right] \\
\vdots & \\
\boldsymbol{\beta}_{m+1}^{\prime} \boldsymbol{z}_{t}+u_{t} & t=\left[\tau_{m} n\right]+1 \ldots, n
\end{array}\right.  \tag{2.1}\\
& =\left\{\boldsymbol{\beta}_{1}^{\prime} \chi\left(t / n \leq \tau_{1}\right)+\cdots+\boldsymbol{\beta}_{m+1}^{\prime} \chi\left(t / n>\tau_{m}\right)\right\} \boldsymbol{z}_{t}+u_{t} \\
& =r_{t}\left(\boldsymbol{\beta}_{1}, \ldots, \boldsymbol{\beta}_{m+1}, \boldsymbol{\tau}\right)+u_{t}, \quad(\text { say }), \quad t=1, \ldots, n
\end{align*}
$$

where $\boldsymbol{\beta}_{j}=\left(\beta_{j 1}, \ldots, \beta_{j q}\right)^{\prime}, j=1, \ldots, m+1$ are unknown parameter vectors, $\boldsymbol{\tau}=\left(\tau_{1}, \ldots, \tau_{m}\right)^{\prime}$ is a vector of unknown change points, $\chi$ is the indicator function, and the residual process $\left\{u_{t}\right\}$ is a circular Gaussian ARMA process with $E\left(u_{t}\right)=0$ and spectral density $f(\lambda)$. The spectral density $f(\lambda)$ is assumed to be bounded and bounded away from zero. The regression function $\boldsymbol{z}_{t}=\left(z_{t 1}, \ldots, z_{t q}\right)^{\prime}$ is nonrandom and observable. Let

$$
a_{k j}^{n}(h)= \begin{cases}\sum_{t=1}^{n-h} z_{k, t+h}, z_{j t}, & h=0,1, \ldots  \tag{2.2}\\ \sum_{t=1-h}^{n} z_{k, t+h} z_{j t}, & h=-1,-2, \ldots\end{cases}
$$

The following Grenander's conditions are assumed to hold:
Assumption 2.1.
(G.1) $a_{k k}^{n}(0)=O(n), k=1, \ldots, q$ and $\sum_{t=l}^{l+\rho} z_{t k}^{2}=O(\rho)$, for any $l=1, \ldots, n$ and, $k=$ $1, \ldots, q$.
(G.2) $\lim _{n \rightarrow \infty} z_{n+1, k}^{2} / a_{k k}^{n}(0)=0, k=1, \ldots, q$.
(G.3) $\lim _{n \rightarrow \infty} a_{k j}^{n}(h) /\left\{a_{k k}^{n}(0) a_{j j}^{n}(0)\right\}^{1 / 2}=r_{k j}(h)$ exists for every $k, j=1, \ldots, q$ and $h \in \mathbb{Z}$.

Denote by $\boldsymbol{R}(h)$ the $q \times q$ matrix $\left(r_{k j}(h)\right)$.
(G.4) $\boldsymbol{R}(0)$ is nonsingular.

From (G.3) there exists a Hermitian matrix function $\boldsymbol{M}(\lambda)=\left(M_{k j}(\lambda)\right)$ with positive semidefinite increments such that

$$
\begin{equation*}
\boldsymbol{R}(h)=\int_{-\pi}^{\pi} e^{i h \lambda} d \boldsymbol{M}(\lambda) \tag{2.3}
\end{equation*}
$$

Suppose that the stretch of series from model (2.1) $\boldsymbol{y}_{n}=\left(y_{1}, \cdots, y_{n}\right)^{\prime}$ is available. Denote the covariance matrix of $\boldsymbol{u}_{n}=\left(u_{1}, \cdots, u_{n}\right)^{\prime}$ by $\Sigma_{n}$ and let $\boldsymbol{t}_{n}=\left(r_{1}, \ldots, r_{n}\right)^{\prime}$ with $r_{t}=r_{t}\left(\boldsymbol{\beta}_{1}, \ldots, \boldsymbol{\beta}_{m+1}, \boldsymbol{\tau}\right)$. Then the likelihood function based on $\boldsymbol{y}_{n}$ is given by
(2.4) $L_{n}\left(\boldsymbol{\beta}_{1}, \cdots, \boldsymbol{\beta}_{m+1}, \boldsymbol{\tau}\right)=\frac{1}{(2 \pi)^{n / 2}\left|\Sigma_{n}\right|^{1 / 2}} \exp \left[-\frac{1}{2}\left(\boldsymbol{y}_{n}-\boldsymbol{t}_{n}\right)^{\prime} \Sigma_{n}^{-1}\left(\boldsymbol{y}_{n}-\boldsymbol{t}_{n}\right)\right]$.

Since we assume that $\left\{u_{t}\right\}$ is a circular ARMA process, it is seen that $\Sigma_{n}$ has the following representation

$$
\Sigma_{n}=\boldsymbol{U}_{n}^{*} \operatorname{diag}\left\{2 \pi f\left(\lambda_{1}\right), \cdots, 2 \pi f\left(\lambda_{n}\right)\right\} \boldsymbol{U}_{n}
$$

where $\boldsymbol{U}_{n}=\left\{n^{-1 / 2} \exp (2 \pi i t s / n) ; t, s=1, \ldots, n\right\}$ and $\lambda_{k}=2 \pi k / n($ see Anderson (1977)). Write

$$
F_{n}\left(\lambda_{k}\right)=\frac{1}{\sqrt{2 \pi n}} \sum_{t=1}^{n}\left(y_{t}-r_{t}\right) e^{-i t \lambda_{k}}
$$

Then the likelihood function (2.4) is rewritten as

$$
\begin{equation*}
L_{n}\left(\boldsymbol{\beta}_{1}, \cdots \boldsymbol{\beta}_{m+1}, \boldsymbol{\tau}\right)=\frac{1}{(2 \pi)^{n}\left\{\prod_{k=1}^{n} f\left(\lambda_{k}\right)\right\}^{1 / 2}} \exp \left[-\frac{1}{2} \sum_{k=1}^{n} f\left(\lambda_{k}\right)^{-1}\left|F_{n}\left(\lambda_{k}\right)\right|^{2}\right] \tag{2.5}
\end{equation*}
$$

Define the local sequence for the parameters

$$
\begin{equation*}
\boldsymbol{\beta}_{i}^{(n)}=\boldsymbol{\beta}_{i}+\boldsymbol{D}_{n}^{-1} \boldsymbol{b}_{i}, \quad \text { and } \quad \boldsymbol{\tau}^{(n)}=\boldsymbol{\tau}+n^{-1} \boldsymbol{\rho} \tag{2.6}
\end{equation*}
$$

where $\boldsymbol{D}_{n}=\operatorname{diag}\left(\sqrt{a_{11}^{n}(0)}, \ldots, \sqrt{a_{q q}^{n}(0)}\right), \boldsymbol{b}_{i} \in \mathbb{R}^{q}$ for $i=1, \ldots, m+1$, and $\boldsymbol{\rho}=\left(\rho_{1}, \ldots, \rho_{m}\right)^{\prime} \in$ $\mathbb{R}^{m}$. For notational convenience, in what follows, we use the the following convention $\tau_{0}=0, \tau_{m+1}=1$ and $\rho_{m+1}=0$. Under the local sequence (2.6) the likelihood ratio process is represented as

$$
\begin{align*}
& Z_{n}\left(\boldsymbol{b}_{1}, \cdots, \boldsymbol{b}_{m+1}, \boldsymbol{\rho}\right)  \tag{2.7}\\
\equiv & \frac{L_{n}\left(\boldsymbol{\beta}_{1}^{(n)}, \cdots, \boldsymbol{\beta}_{m+1}^{(n)}, \boldsymbol{\tau}^{(n)}\right)}{L_{n}\left(\boldsymbol{\beta}_{1}, \cdots, \boldsymbol{\beta}_{m+1}, \boldsymbol{\tau}\right)} \\
= & \exp \left[-\frac{1}{2 \sqrt{n}} \sum_{k=1}^{n} f\left(\lambda_{k}\right)^{-1 / 2}\left\{d_{n}\left(\lambda_{k}\right) A\left(\lambda_{k}\right)+\overline{d_{n}\left(\lambda_{k}\right)} \overline{A\left(\lambda_{k}\right)}\right\}\right. \\
& \left.-\frac{1}{2 n} \sum_{k=1}^{n}\left|A\left(\lambda_{k}\right)\right|^{2}\right]
\end{align*}
$$

where $d_{n}\left(\lambda_{k}\right)=(2 \pi n)^{-1 / 2} \sum_{t=1}^{n} u_{t} e^{i t \lambda_{k}}$ and

$$
\begin{aligned}
& A\left(\lambda_{k}\right) \\
&= \frac{1}{\sqrt{2 \pi f\left(\lambda_{k}\right)}}\left\{\sum_{j=1}^{m} \sum_{s=\left[\tau_{j} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]}\left(\boldsymbol{\beta}_{j+1}-\boldsymbol{\beta}_{j}\right)^{\prime} \boldsymbol{z}_{s} e^{-i s \lambda_{k}}-\sum_{j=1}^{m+1} \sum_{s=\left[\tau_{j-1} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{b}_{j}^{\prime} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} e^{-i s \lambda_{k}}\right\} \\
&=A_{1}\left(\lambda_{k}\right)+A_{2}\left(\lambda_{k}\right) \quad \text { (say). }
\end{aligned}
$$

The asymptotic representation of $Z_{n}\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\rho}\right)$ is given as follows.

Theorem 2.1. Suppose that Assuption 2.1 holds. Then for all $\left(\boldsymbol{\beta}_{1}^{\prime}, \ldots, \boldsymbol{\beta}_{m+1}^{\prime}, \boldsymbol{\tau}\right) \in \boldsymbol{\Theta} \subset$ $\mathbb{R}^{q \times(m+1)+m}$, the log-likelihood function ratio has the asymptotic representation

$$
\begin{aligned}
& \log Z_{n}\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\rho}\right) \\
= & \sum_{j=1}^{m}\left(\boldsymbol{\beta}_{j+1}-\boldsymbol{\beta}_{j}\right)^{\prime} W_{j 1}+\sum_{j=2}^{m} \sqrt{\tau_{j}-\tau_{j-1}} \boldsymbol{b}_{j}^{\prime} W_{j 2}+\sqrt{\tau_{1}} \boldsymbol{b}_{1}^{\prime} W_{12}+\sqrt{1-\tau_{m}} \boldsymbol{b}_{m+1}^{\prime} W_{m+1,2} \\
& -\frac{1}{8 \pi^{2}} \sum_{j_{1}, j_{2}=1}^{m}\left(\boldsymbol{\beta}_{j_{1}+1}-\boldsymbol{\beta}_{j_{1}}\right)^{\prime} \int_{-\pi}^{\pi} \boldsymbol{A}\left(\tau_{j_{1}}, \rho_{j_{1}} ; \lambda\right) \boldsymbol{A}\left(\tau_{j_{2}}, \rho_{j_{2}} ; \lambda\right)^{*} f(\lambda)^{-1} d \lambda\left(\boldsymbol{\beta}_{j_{2}+1}-\boldsymbol{\beta}_{j_{2}}\right) \\
& -\sum_{j=1}^{m+1} \frac{\tau_{j}-\tau_{j-1}}{4 \pi} \boldsymbol{b}_{j}^{\prime} \int_{-\pi}^{\pi} f(\lambda)^{-1} d \boldsymbol{M}(\lambda) \boldsymbol{b}_{j} \delta_{j}-\frac{1}{4 \pi} \sum_{j=1}^{m} \min \left(\tau_{j+1}-\tau_{j}, \tau_{j}-\tau_{j-1}\right) \\
& \times\left[\boldsymbol{b}_{j+1}^{\prime} \int_{-\pi}^{\pi} \frac{1}{2 \pi} \sum_{l=1-\left[\rho_{j}\right]}^{\infty} \Gamma(l) e^{-i l \lambda} d \boldsymbol{M}(\lambda) \boldsymbol{b}_{j}+\boldsymbol{b}_{j}^{\prime} \int_{-\pi}^{\pi} \frac{1}{2 \pi} \sum_{l=-\infty}^{\left[\rho_{j}\right]-1} \Gamma(l) e^{-i l \lambda} d \boldsymbol{M}(\lambda) \boldsymbol{b}_{j+1}\right]+o_{p}(1) \\
= & \log Z\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\rho}\right)+o_{p}(1), \quad \text { (say) }
\end{aligned}
$$

where $\boldsymbol{A}\left(\tau_{j}, \rho_{j} ; \lambda\right)=\sum_{s=\left[\tau_{j} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{z}_{s} e^{i s \lambda}, \delta_{j}=1$ for $j=2, \ldots, m, \delta_{1}=\delta_{m+1}=2$,

$$
\begin{aligned}
& W_{j 1}=-\frac{1}{2 \pi} \int_{-\pi}^{\pi} \sum_{s=\left[\tau_{j} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{z}_{s} e^{i s \lambda} f(\lambda)^{-1} d Z_{u}(\lambda) \quad \text { for } j=1, \ldots, m \\
& W_{j 2}=\int_{-\pi}^{\pi}\left(2 \pi \sqrt{\tau_{j}-\tau_{j-1}}\right)^{-1} \sum_{s=\left[\tau_{j-1} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} e^{i s \lambda} f(\lambda)^{-1} d Z_{u}(\lambda) \text { for } j=2, \ldots, m \\
& W_{12}=\int_{-\pi}^{\pi}\left(2 \pi \sqrt{\tau_{1}}\right)^{-1} \sum_{s=1}^{\left[\tau_{1} n+\rho_{1}\right]} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} e^{i s \lambda}\left(1+e^{i n \lambda}\right) f(\lambda) d Z_{u}(\lambda)
\end{aligned}
$$

and

$$
W_{m+1,2}=\int_{-\pi}^{\pi}\left(2 \pi \sqrt{1-\tau_{m}}\right)^{-1} \sum_{s=\left[\tau_{m} n\right]+1}^{m} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} e^{i s \lambda}\left(1+e^{-i n \lambda}\right) f(\lambda) d Z_{u}(\lambda)
$$

Let $\boldsymbol{W}_{1}=\left(W_{11}^{\prime}|\cdots| W_{m 1}^{\prime}\right)^{\prime}$ and $\boldsymbol{W}_{2}=\left(W_{12}^{\prime}|\cdots| W_{m+1,2}^{\prime}\right)^{\prime}$. Then

$$
\binom{\boldsymbol{W}_{1}}{\boldsymbol{W}_{2}} \underset{D}{\rightarrow} N\left(\mathbf{0},\left(\begin{array}{cc}
\boldsymbol{V}_{1} & \mathbf{0} \\
\mathbf{0} & \boldsymbol{V}_{2}
\end{array}\right)\right)
$$

where $\boldsymbol{V}_{1}$ is $(q m) \times(q m)$ matrix with $(i, j)$ th block

$$
\frac{1}{4 \pi^{2}} \int_{-\pi}^{\pi} \boldsymbol{A}\left(\tau_{i}, \rho_{i} ; \lambda\right) \boldsymbol{A}\left(\tau_{j}, \rho_{j} ; \lambda\right)^{*} f(\lambda)^{-1} d \lambda, \quad \text { for } i, j=1, \ldots, m
$$

and $\boldsymbol{V}_{2}$ is $(q(m+1)) \times(q(m+1))$ matrix with

$$
\boldsymbol{V}_{2}=\left(\begin{array}{ccccc}
2 A & B & & & 0 \\
B & A & B & & \\
& \ddots & \ddots & \ddots & \\
0 & & B & A & B \\
B & 2 A
\end{array}\right)
$$

where

$$
A=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(\lambda)^{-1} d \boldsymbol{M}(\lambda)
$$

and

$$
\int_{-\pi}^{\pi}\left[\frac{1}{2 \pi} \sum_{l=1-\left[\rho_{j}\right]}^{\infty} \Gamma(l) e^{-i l \lambda}+\overline{\frac{1}{2 \pi} \sum_{l=1-\left[\rho_{j}\right]}^{\infty} \Gamma(l) e^{-i l \lambda}}\right] d \boldsymbol{M}(\lambda) .
$$

Next we present some fundamental lemmas which are useful in the estimation of multiple change points.

Lemma 2.1. Suppose that Assumption 2.1 holds. Then for some $p>2$ and for any compact set $\mathscr{C} \in \boldsymbol{\Theta}$, we have

$$
\sup _{\beta_{1}, \ldots, \beta_{m+1}, \tau \in \mathscr{C}} E Z_{n}^{1 / p}\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\rho}\right) \leq \exp \left\{-g\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\rho}\right)\right\}
$$

where

$$
g\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\rho}\right)=C \sum_{j=1}^{m}\left|\rho_{j}\right|+\sum_{i, j=1}^{m+1} \boldsymbol{b}_{i}^{\prime} \boldsymbol{K} \boldsymbol{b}_{j}
$$

with some positive definite matrix $\boldsymbol{K}$ and $C>0$.
Lemma 2.2. Suppose that Assumption 2.1 holds. Then for any compact set $\mathscr{C} \subset \boldsymbol{\Theta}$, there exist some integer $\kappa(\mathscr{C})=\kappa, B(\mathscr{C})=B$ such that for any integer $p>1$

$$
\begin{aligned}
& \sup _{\beta_{1}, \ldots, \beta_{m+1}, \tau \in \mathscr{C}, b_{j}<H, \rho_{j}<H}\left[\sum_{j=1}^{m+1}\left\|\boldsymbol{b}_{j}^{(2)}-\boldsymbol{b}_{j}^{(1)}\right\|^{\kappa}+\left\|\boldsymbol{\rho}^{(2)}-\boldsymbol{\rho}^{(1)}\right\|^{\kappa}\right]^{-1} \\
\times & E\left[Z_{n}^{1 / 2 p}\left(\boldsymbol{b}_{1}^{(2)}, \ldots, \boldsymbol{b}_{m+1}^{(2)}, \boldsymbol{\rho}^{(2)}\right)-Z_{n}^{1 / 2 p}\left(\boldsymbol{b}_{1}^{(1)}, \ldots, \boldsymbol{b}_{m+1}^{(1)}, \boldsymbol{\rho}^{(1)}\right)\right]^{2 p} \leq B(1+H)^{\kappa} .
\end{aligned}
$$

## 3. Estimation theory

In this section we state the asymptotic behavior of MLE and BE. The limiting distributions of these estimators are different and it is shown that the $B E$ is asymptotically optimal. To show this we use the general results by Ibragimov and Has'minski (1981).

First, we need to introduce a loss function $l(y), y \in \mathbb{R}^{d}$ which is

1. nonnegative, continuous at point 0 and $l(0)=0$, but is not identically 0 ;
2. symmetric: $l(y)=l(-y)$;
3. the set $\{y: l(y)<c\}$ are convex for all $c>0$.

We denote by $\boldsymbol{W}_{p}$ the class of loss function satisfying 1-3 with polynomial majorants. The example of such function is $w(y)=|y|^{p}, p>0$.

The MLE $\hat{\boldsymbol{\theta}}^{(M L)^{\prime}}=\left(\hat{\boldsymbol{b}}_{1}^{(M L)^{\prime}}, \ldots, \hat{\boldsymbol{b}}_{m+1}^{(M L)^{\prime}}, \hat{\boldsymbol{\tau}}^{(M L)^{\prime}}\right.$ ) and BE (for quadratic loss function) $\tilde{\boldsymbol{\theta}}^{(B)^{\prime}}=\left(\tilde{\boldsymbol{b}}_{1}^{(B)^{\prime}}, \ldots, \tilde{\boldsymbol{b}}_{m+1}^{(B)^{\prime}}, \tilde{\boldsymbol{\tau}}^{(B)^{\prime}}\right)$ are defined by the usual relations

$$
\begin{equation*}
L\left(\hat{\boldsymbol{b}}_{1}^{(M L)}, \ldots, \hat{\boldsymbol{\theta}}_{m+1}^{(M L)}, \hat{\boldsymbol{\tau}}^{(M L)}\right)=\max _{b_{1}, \ldots, b_{m+1}, \tau \in \boldsymbol{\Theta}} L\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\tau}\right) \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\tilde{\boldsymbol{\theta}}^{(B)}=\int_{\boldsymbol{\Theta}} \boldsymbol{\theta} q\left(\boldsymbol{\theta} \mid Y_{n}\right) d \boldsymbol{\theta}, \quad q\left(\boldsymbol{\theta} \mid Y_{n}\right)=\frac{\pi(\boldsymbol{\theta}) L_{n}(\boldsymbol{\theta})}{\int_{\boldsymbol{\Theta}} \pi(\boldsymbol{v}) L_{n}(\boldsymbol{v}) d \boldsymbol{v}}, \tag{3.2}
\end{equation*}
$$

respectively, where $\pi(\boldsymbol{\theta})$ is a prior density on $\boldsymbol{\Theta}$. We suppose that the prior density is a bounded, positive and continuous function possessing a polynomial majorant on $\boldsymbol{\Theta}$. Further, let us define two random variables $\hat{\boldsymbol{u}}$ and $\tilde{\boldsymbol{u}}$, for $Z(\boldsymbol{u}), \boldsymbol{u}=\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\tau}\right)$, by relations

$$
\begin{aligned}
& Z(\hat{\boldsymbol{u}})=\sup _{\boldsymbol{u} \in \mathbb{R} a \times(m+1)+m} Z(\boldsymbol{u}) \\
& \tilde{\boldsymbol{u}}=\frac{\int_{\mathbb{R} Q \times(m+1)+m} \boldsymbol{u} Z(\boldsymbol{u}) d \boldsymbol{u}}{\int_{\mathbb{R} 9 \times(m+1)+m} Z(\boldsymbol{v}) d \boldsymbol{v}} .
\end{aligned}
$$

Our main results are stated as follows
Theorem 3.1. Let the parameter set $\Theta$ be an open subset of $\mathbb{R}^{q \times(m+1)+m}$. Then the $M L E$ is uniformly on $\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\tau}\right) \in \boldsymbol{\Theta}$, consistent

$$
P \lim _{n \rightarrow \infty} \hat{\boldsymbol{\theta}}^{M L}=\boldsymbol{\theta}
$$

and converges in distribution

$$
\mathscr{L}_{\theta}\left\{\mathscr{A}_{n}\left(\hat{\boldsymbol{\theta}}^{(M L)}-\boldsymbol{\theta}\right)\right\} \underset{D}{\longrightarrow} \mathscr{L}(\hat{\boldsymbol{u}}),
$$

where $\mathscr{A}_{n}=\operatorname{diag}(\underbrace{\boldsymbol{D}_{n}, \ldots, \boldsymbol{D}_{n}}_{m+1}, \underbrace{n \ldots, n}_{m})$. For any continuous loss function $w \in W_{p}$, we have

$$
\lim _{n t o \infty} E_{\theta} w\left(\mathscr{A}_{n}\left(\hat{\boldsymbol{\theta}}^{(M L)}-\boldsymbol{\theta}\right)\right)=E w(\hat{\boldsymbol{u}}) .
$$

Proof. The proof follows from Theorem 2.1, Lemmas 2.1 and 2.2 of this paper and Theorem 1.10.1 of Ibragimov and Has'minski (1981).

The asymptotic behavior of BE is in the following theorem.
Theorem 3.2. The Bayes estimator $\tilde{\boldsymbol{\theta}}^{(B)}$, uniformly on $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, is consistent

$$
P \lim _{n \rightarrow \infty} \tilde{\boldsymbol{\theta}}^{(B)}=\boldsymbol{\theta}
$$

and converges in distribution

$$
\mathscr{L}_{\boldsymbol{\theta}}\left\{\mathscr{A}_{n}\left(\tilde{\boldsymbol{\theta}}^{(B)}-\boldsymbol{\theta}\right)\right\} \underset{D}{\longrightarrow} \mathscr{L}(\tilde{\boldsymbol{u}}) .
$$

For any continuous loss function $w \in \boldsymbol{W}_{p}$, we have

$$
\lim _{n \rightarrow \infty} E_{\theta} w\left(\mathscr{A}_{n}\left(\tilde{\boldsymbol{\theta}}^{(B)}-\boldsymbol{\theta}\right)\right)=E w(\tilde{\boldsymbol{u}}) .
$$

Proof. The properties of the likelihood ratio $Z_{n}\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\rho}\right)$ established in Theorem 2.1, Lemmas 2.1 and 2.2 allow us to refer to Theorem 1.10.2 of Ibragimov and Has'minski
(1981).

Remark. It can be seen that the BE is asymptotically efficient and satisfies

$$
E\|\hat{\boldsymbol{u}}\| \geq E\|\tilde{\boldsymbol{u}}\|
$$

Of cource the MLE is not asymptotically efficient generally.

## 4. A NUMERICAL EXPERIMENT

In this section we briefly report results from Monte Carlo simulations. The data are generated from two mean breaks with the AR(1) process

$$
y_{t}= \begin{cases}u_{t} & t=1, \ldots\left[\tau_{1} n\right]  \tag{4.1}\\ 2+u_{t} & t=\left[\tau_{1} n\right]+1, \ldots,\left[\tau_{2} n\right] . \\ u_{t} & t=\left[\tau_{2} n\right]+1, \ldots, n, \quad n=100\end{cases}
$$

where $u_{t}=0.7 u_{t-1}+\varepsilon_{t}$ and $\varepsilon_{t} \sim$ i.i.d. $N(0,1)$. The change points $\left(\tau_{1}, \tau_{2}\right)$ are chosen to be $(1 / 3,2 / 3)$. For simplicity, we assume that the parameters except change points are known. The average estimates of change points and the squared root of mean square error of MLE and BE are computed based on 100 repetitions, and are reported in Table 1. The histograms are displayed in Figure 1.

From these simulations, we point out that two changes are well detected by both MLE and BE, whereas the mean square error of BE is smaller than that of MLE. This can be explained by the difference of the shape of distributions. We can see that the distribution of MLE has fatter tails. These simulation results are consistent with the theoretical results of previous section.

Table 1.
Average estimates and RMSE of $\boldsymbol{\tau}$ when $\left(\tau_{1}, \tau_{2}\right)=(1 / 3,2 / 3), n=100$.

|  | $\hat{\tau}_{1}^{(M L)}$ | $\hat{\tau}_{2}^{(M L)}$ | $\tilde{\tau}_{1}^{(B)}$ | $\tilde{\tau}_{2}^{(B)}$ |
| :--- | :--- | :--- | :--- | :--- |
| Mean | 0.3222 | 0.6739 | 0.3172 | 0.6786 |
| RMSE | 0.0882 | 0.0809 | 0.0622 | 0.0598 |

## 5. Proofs

In this section we just give the proof of Theorem 2.1, because the proofs for Lemmas 2.1 and 2.2 are similar to those of Shiohama et al. (2002). The details are given in Shiohama (2002), which can be obtained from the author.

Proof of Theorem 2.1. We have from (2.7),

$$
\begin{align*}
& \log Z_{n}\left(\boldsymbol{b}_{1}, \ldots, \boldsymbol{b}_{m+1}, \boldsymbol{\rho}\right)  \tag{5.1}\\
= & -\frac{1}{2 \sqrt{n}} \sum_{k=1}^{n} f\left(\lambda_{k}\right)^{-1 / 2}\left\{d_{n}\left(\lambda_{k}\right) A\left(\lambda_{k}\right)+\overline{d_{n}\left(\lambda_{k}\right) A\left(\lambda_{k}\right)}\right\}-\frac{1}{2 n} \sum_{k=1}^{n}\left|A\left(\lambda_{k}\right)\right|^{2} . \\
= & D_{1}+D_{2}+D_{3} \quad \text { (say). }
\end{align*}
$$



Figure 1. Histograms for estimated change points.

The first term $D_{1}$ can be evaluated as

$$
\begin{align*}
D_{1}= & -\frac{1}{2 \sqrt{n}} \sum_{k=1}^{n} f\left(\lambda_{k}\right)^{-1 / 2}\left\{d_{n}\left(\lambda_{k}\right) A\left(\lambda_{k}\right)\right\}  \tag{5.2}\\
= & -\frac{1}{4 \pi n} \sum_{k=1}^{n} f\left(\lambda_{k}\right)^{-1} \sum_{j=1}^{m} \sum_{t=1}^{n} \sum_{s=\left[\tau_{j} n\right]+1}^{\left[\tau_{j} n\right]+\rho_{j}}\left(\boldsymbol{\beta}_{j+1}-\boldsymbol{\beta}_{j}\right)^{\prime} \boldsymbol{z}_{s} u_{t} e^{i(t-s) \lambda_{k}} \\
& +\frac{1}{4 \pi n} \sum_{k=1}^{n} f\left(\lambda_{k}\right)^{-1} \sum_{j=1}^{m+1} \sum_{t=1}^{n} \sum_{s=\left[\tau_{j-1} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{b}_{j}^{\prime} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} u_{t} e^{i(t-s) \lambda_{k}} \\
= & D_{11}+D_{12} \quad \text { (say). }
\end{align*}
$$

Here we write the spectral density $f(\lambda)$ in the form

$$
f(\lambda)=\frac{1}{2 \pi} \sum_{l=-\infty}^{\infty} R_{f}(\lambda) e^{-i l \lambda}
$$

where $R_{f}$ 's satisfy $\sum_{l=-\infty}^{\infty}|l|^{p}\left|R_{f}(l)\right|<\infty$ for any given $p \in \mathbb{Z}$. Then, from Theorem 3.8.3 of Brillinger (1975) we may write

$$
\begin{equation*}
f(\lambda)^{-1}=\frac{1}{2 \pi} \sum_{l=-\infty}^{\infty} \Gamma(l) e^{-i l \lambda} \tag{5.3}
\end{equation*}
$$

where $\Gamma(l)$ 's satisfy for any given $p \in \mathbb{Z}$

$$
\sum_{l=-\infty}^{\infty}|l|^{p}|\Gamma(l)|<\infty
$$

Then $D_{11}$ becomes

$$
D_{11}=-\frac{1}{4 \pi n} \frac{1}{2 \pi} \sum_{k=1}^{n} \sum_{l=-\infty}^{\infty} \Gamma(l) \sum_{j=1}^{m} \sum_{t=1}^{n} \sum_{s=\left[\tau_{j} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]}\left(\boldsymbol{\beta}_{j+1}-\boldsymbol{\beta}_{j}\right)^{\prime} \boldsymbol{z}_{s} u_{t} e^{i(t-s-l) \lambda_{k}} .
$$

It is well known that

$$
\sum_{k=1}^{n} e^{i(t-s-l) \lambda_{k}}= \begin{cases}n & \text { if } t-s-l=0(\bmod n)  \tag{5.4}\\ 0 & \text { otherwise }\end{cases}
$$

Since $-\left[\tau_{j} n+\rho_{j}\right] \leq t-s \leq\left[\left(1-\tau_{j}\right) n\right]$, for each $j$ and $\Gamma(l)$ satisfies $\sum_{l}|l|^{p}|\Gamma(l)|<\infty$ for any given $p$, we have

$$
\begin{equation*}
\sum_{|l| \geq n}|\Gamma(l)| \leq n^{-p} \sum_{|l| \geq n}(l)^{p}|\Gamma(l)|=o\left(n^{-p}\right) \tag{5.5}
\end{equation*}
$$

Hence we have only to evaluate $D_{11}$ for $t-s-l=0$. Then, it is easy to see

$$
\begin{aligned}
D_{11} & =-\frac{1}{4 \pi} \frac{1}{2 \pi} \sum_{l=-\infty}^{\infty} \Gamma(l) \sum_{j=1}^{m} \sum_{t=1}^{n} \sum_{s=\left[\tau_{j} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]}\left(\boldsymbol{\beta}_{j+1}-\boldsymbol{\beta}_{j}\right)^{\prime} \boldsymbol{z}_{s} u_{t} n^{-1} \sum_{k=1}^{n} e^{i(t-s-l) \lambda_{k}} \\
& \simeq-\frac{1}{8 \pi^{2}} \sum_{j=1}^{m} \sum_{l=-\infty}^{\infty} \Gamma(l) \sum_{s=\left[\tau_{j} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]}\left(\boldsymbol{\beta}_{j+1}-\boldsymbol{\beta}_{j}\right)^{\prime} \boldsymbol{z}_{s} u_{s+l} \equiv \tilde{D}_{11} \quad \text { (say) }
\end{aligned}
$$

Hence we have

$$
\begin{align*}
& \tilde{D}_{11}=-\frac{1}{8 \pi^{2}} \sum_{j=1}^{m} \sum_{l=-\infty}^{\infty} \Gamma(l)\left(\boldsymbol{\beta}_{j+1}-\boldsymbol{\beta}_{j}\right)^{\prime} \sum_{s=\left[\tau_{j} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{z}_{s} \int_{-\pi}^{\pi} e^{i l \lambda} e^{i s \lambda} d Z_{u}(\lambda)  \tag{5.6}\\
= & -\frac{1}{4 \pi} \sum_{j=1}^{m}\left(\boldsymbol{\beta}_{j+1}-\boldsymbol{\beta}_{j}\right)^{\prime} \int_{-\pi}^{\pi} \sum_{s=\left[\tau_{j} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{z}_{s} e^{i s \lambda} f(\lambda)^{-1} d Z_{u}(\lambda) \\
= & \frac{1}{2} \sum_{j=1}^{m}\left(\boldsymbol{\beta}_{j+1}-\boldsymbol{\beta}_{j}\right)^{\prime} W_{j 1} \quad \text { (say). }
\end{align*}
$$

For the random variables $W_{j 1}, j=1, \ldots, m$, let $\sum_{s=\left[\tau_{j} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{z}_{s} e^{i s \lambda}=\boldsymbol{A}\left(\tau_{j}, \rho_{j} ; \lambda\right)$, We observe that

$$
E\left(W_{i 1} W_{j 1}^{*}\right) \underset{n \rightarrow \infty}{\longrightarrow} \frac{1}{4 \pi^{2}} \int_{-\pi}^{\pi} \boldsymbol{A}\left(\tau_{i}, \rho_{i} ; \lambda\right) \boldsymbol{A}\left(\tau_{j}, \rho_{j} ; \lambda\right)^{*} f(\lambda)^{-1} d \lambda, \quad i, j=1, \ldots, m
$$

Recalling that $\left\{u_{t}\right\}$ is Gaussian, we have

$$
\begin{equation*}
\boldsymbol{W}_{1} \underset{D}{\longrightarrow} N\left(\mathbf{0}, \boldsymbol{V}_{1}\right) \tag{5.7}
\end{equation*}
$$

Next, we turn to evaluate $D_{12}$ in (5.2). By using (5.3)

$$
\begin{equation*}
D_{12}=\frac{1}{4 \pi n} \frac{1}{2 \pi} \sum_{k=1}^{n} \sum_{l=-\infty}^{\infty} \Gamma(l) e^{-i l \lambda_{k}} \sum_{j=1}^{m+1} \sum_{t=1}^{n} \sum_{s=\left[\tau_{j-1} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{b}_{j}^{\prime} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} u_{t} e^{i(t-s) \lambda_{k}} \tag{5.8}
\end{equation*}
$$

Since $n-\left[\tau_{j-1} n\right]-1 \geq t-s \geq 1-\left[\tau_{j} n\right]$, for each $j$ and (5.5), we have only to evaluate $D_{12}$ for

$$
t-s-l= \begin{cases}0, n, & \text { for } j=1 \\ 0, & \text { for } j=2, \cdots, m \\ 0,-n & \text { for } j=m+1\end{cases}
$$

Then $D_{12}$ becomes

$$
\begin{aligned}
D_{12} \simeq & \frac{1}{8 \pi^{2}} \sum_{j=2}^{m} \sum_{l=-\infty}^{\infty} \Gamma(l) \boldsymbol{b}_{j}^{\prime} \sum_{s=\left[\tau_{j-1} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} u_{s+l} \\
& +\frac{1}{8 \pi^{2}} \sum_{l=-\infty}^{\infty} \Gamma(l) \boldsymbol{b}_{1}^{\prime} \sum_{s=1}^{\left[\tau_{1} n+\rho_{1}\right]} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s}\left(u_{s+l}+u_{s+l+n}\right) \\
& +\frac{1}{8 \pi^{2}} \sum_{l=-\infty}^{\infty} \Gamma(l) \boldsymbol{b}_{m+1}^{\prime} \sum_{s=\left[\tau_{m} n\right]+1}^{n} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s}\left(u_{s+l}+u_{s+l-n}\right) \\
= & \tilde{D}_{12}^{\dagger}+\tilde{D}_{12}^{\dagger \dagger}+\tilde{D}_{12}^{\dagger \dagger \dagger} \quad \text { (say). }
\end{aligned}
$$

Similarly as in $\tilde{D}_{11}$

$$
\begin{align*}
& =\frac{1}{8 \pi^{2}} \sum_{j=2}^{m} \sum_{l=-\infty}^{\infty} \Gamma(l) \boldsymbol{b}_{j}^{\prime} \sum_{s=\left[\tau_{j-1} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \int_{-\pi}^{\pi} e^{i s \lambda} e^{i l \lambda} d Z_{u}(\lambda) \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s}  \tag{5.9}\\
& =\frac{1}{4 \pi} \sum_{j=2}^{m} \boldsymbol{b}_{j}^{\prime} \int_{-\pi}^{\pi} \sum_{s=\left[\tau_{j-1} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} e^{i s \lambda} f(\lambda)^{-1} d Z_{u}(\lambda) \\
& =\sum_{j=2}^{m} \frac{\sqrt{\tau_{j}-\tau_{j-1}}}{2} \boldsymbol{b}_{j}^{\prime} \int_{-\pi}^{\pi}\left(2 \pi \sqrt{\tau_{j}-\tau_{j-1}}\right)^{-1} \sum_{s=\left[\tau_{j-1} n\right]+1}^{\left[\tau_{j} n+\rho_{j}\right]} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} e^{i s \lambda} f(\lambda)^{-1} d Z_{u}(\lambda) \\
& \left.=\sum_{j=2}^{m} \frac{\sqrt{\tau_{j}-\tau_{j-1}}}{2} \boldsymbol{b}_{j}^{\prime} W_{j 2} \quad \text { (say }\right) .
\end{align*}
$$

where

$$
\begin{equation*}
W_{j 2} \underset{D}{\longrightarrow} N\left(\mathbf{0}, \frac{1}{2 \pi} \int_{-\pi}^{\pi} f(\lambda)^{-1} d \boldsymbol{M}(\lambda)\right), \quad \text { for } j=2, \ldots, m \tag{5.10}
\end{equation*}
$$

which follows from Assumption2.1. Analogously, we get

$$
\begin{equation*}
\tilde{D}_{12}^{\dagger \dagger}=\frac{\sqrt{\tau_{1}}}{2} \boldsymbol{b}_{1}^{\prime} W_{12} \quad \text { and } \quad \tilde{D}_{12}^{\dagger \dagger}=\frac{\sqrt{1-\tau_{m}}}{2} \boldsymbol{b}_{m+1}^{\prime} W_{m+1,2} \tag{5.11}
\end{equation*}
$$

where

$$
\begin{align*}
& W_{12}=\int_{-\pi}^{\pi}\left(2 \pi \sqrt{\tau_{1}}\right)^{-1} \sum_{s=1}^{\left[\tau_{1} n+\rho_{1}\right]} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} e^{i s \lambda}\left(1+e^{i n \lambda}\right) f(\lambda) d Z_{u}(\lambda)  \tag{5.12}\\
& \text { and } \\
& W_{m+1,2}=\int_{-\pi}^{\pi}\left(2 \pi \sqrt{1-\tau_{m}}\right)^{-1} \sum_{s=\left[\tau_{m} n\right]+1}^{m} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s} e^{i s \lambda}\left(1+e^{-i n \lambda}\right) f(\lambda) d Z_{u}(\lambda) .
\end{align*}
$$

Recalling $u_{t}$ is Gaussian, $W_{12}$ and $W_{m+1,2}$ are asymptotically normal with mean $\mathbf{0}$ and covariance matrix

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{-\pi}^{\pi} 2 f(\lambda)^{-1} d \boldsymbol{M}(\lambda) \tag{5.13}
\end{equation*}
$$

Similar arguments for evaluating $D_{11}$ and $D_{12}$ yield

$$
\begin{align*}
& D_{2} \simeq \frac{1}{2} \sum_{j=1}^{m}\left(\boldsymbol{\beta}_{j+1}-\boldsymbol{\beta}_{j}\right)^{\prime} W_{j 1}  \tag{5.14}\\
+ & \sum_{j=2}^{m} \frac{\sqrt{\tau_{j}-\tau_{j-1}}}{2} b_{j}^{\prime} W_{j 2}+\frac{\sqrt{\tau}_{1}}{2} \boldsymbol{b}_{1}^{\prime} W_{12}+\frac{\sqrt{1-\tau_{m}}}{2} \boldsymbol{b}_{m+1}^{\prime} W_{m+1,2} .
\end{align*}
$$

The last term in (5.1) becomes

$$
\begin{aligned}
D_{3} & =-\frac{1}{2 n} \sum_{k=1}^{n}\left|A\left(\lambda_{k}\right)\right|^{2} \\
& =-\frac{1}{2 n} \sum_{k=1}^{n}\left\{A_{1}\left(\lambda_{k}\right) \overline{A_{1}\left(\lambda_{k}\right)}+A_{1}\left(\lambda_{k}\right) \overline{A_{2}\left(\lambda_{k}\right)}+A_{2}\left(\lambda_{k}\right) \overline{A_{1}\left(\lambda_{k}\right)}+A_{2}\left(\lambda_{k}\right) \overline{A_{2}\left(\lambda_{k}\right)}\right\} \\
& =D_{31}+D_{32}+D_{33}+D_{34} \quad \text { (say). }
\end{aligned}
$$

We have

$$
\begin{aligned}
& D_{31} \\
= & -\frac{1}{4 n \pi} \sum_{k=1}^{n} f\left(\lambda_{k}\right)^{-1}\left(\sum_{j_{1}=1}^{m} \sum_{s_{1}=\left[\tau_{j_{1}} n\right]+1}^{\left[\tau_{j_{1}} n+\rho_{j_{1}}\right]}\left(\boldsymbol{\beta}_{j_{1}+1}-\boldsymbol{\beta}_{j_{1}}\right)^{\prime} \boldsymbol{z}_{s_{1}} e^{i s_{1} \lambda_{k}}\right) \\
& \times\left(\sum_{j_{2}=1}^{m} \sum_{s_{2}=\left[\tau_{j_{2}} n\right]+1}^{\left[\tau_{j_{2}} n+\rho_{j_{2}}\right]} \boldsymbol{z}_{s_{2}}^{\prime}\left(\boldsymbol{\beta}_{j_{2}+1}-\boldsymbol{\beta}_{j_{2}}\right) e^{-i s_{2} \lambda_{k}}\right) \\
= & -\frac{1}{8 \pi^{2}} \sum_{j_{1}, j_{2}=1}^{m}\left(\boldsymbol{\beta}_{j_{1}+1}-\boldsymbol{\beta}_{j_{1}}\right)^{\prime} \int_{-\pi}^{\pi} \boldsymbol{A}\left(\tau_{j_{1}}, \rho_{j_{1}} ; \lambda\right) \boldsymbol{A}\left(\tau_{j_{2}}, \rho_{j_{2}} ; \lambda\right)^{*} f(\lambda)^{-1} d \lambda\left(\boldsymbol{\beta}_{j_{2}+1}-\boldsymbol{\beta}_{j_{2}}\right)+o(1)
\end{aligned}
$$

As for $D_{34}$,

$$
\begin{aligned}
& D_{34} \\
&=-\frac{1}{4 n \pi} \sum_{k=1}^{n} f\left(\lambda_{k}\right)^{-1}\left(\sum_{j_{1}=1}^{m+1} \sum_{s_{1}=\left[\tau_{j_{1}-1} n\right]+1}^{\left[\tau_{j_{1}} n+\rho_{j_{1}}\right]} \boldsymbol{b}_{j_{1}}^{\prime} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s_{1}} e^{i s_{1} \lambda_{k}}\right)\left(\sum_{j_{2}=1}^{m+1} \sum_{s_{2}=\left[\tau_{j_{2}-1} n\right]+1}^{\left[\tau_{j_{2}} n+\rho_{j_{2}}\right]} \boldsymbol{z}_{s_{2}}^{\prime} \boldsymbol{D}_{n}^{-1} \boldsymbol{b}_{j_{2}} e^{-i s_{2} \lambda_{k}}\right) \\
&=-\frac{1}{4 \pi} \frac{1}{2 \pi} \sum_{l=-\infty}^{\infty} \Gamma(l) \\
& \times \sum_{j_{1}, j_{2}=1}^{m+1} \boldsymbol{b}_{j_{1}}^{\prime} \sum_{s_{1}=\left[\tau_{j_{1}-1} n\right]+1}^{\left[\tau_{j_{1}} n+\rho_{j_{1}}\right]} \sum_{s_{2}=\left[\tau_{j_{2}-1} n\right]+1}^{\left[\tau_{j_{2}} n+\rho_{j_{2}}\right]} \boldsymbol{D}_{n}^{-1} \boldsymbol{z}_{s_{1}} \boldsymbol{z}_{s_{2}}^{\prime} \boldsymbol{D}_{n}^{-1} \boldsymbol{b}_{j_{2}} \frac{1}{n} \sum_{k=1}^{n} e^{i\left(s_{1}-s_{2}-l\right) \lambda_{k}} .
\end{aligned}
$$

Since (5.5), it is seen that we have only to evaluate $j_{1}=j_{2}$ and $\left|j_{1}-j_{2}\right|=1$. Hence this yields

$$
\begin{aligned}
& D_{34} \\
= & -\sum_{j=1}^{m+1} \frac{\left(\tau_{j}-\tau_{j-1}\right)}{4 \pi} \boldsymbol{b}_{j}^{\prime} \int_{-\pi}^{\pi} f(\lambda)^{-1} d \boldsymbol{M}(\lambda) \boldsymbol{b}_{j} \delta_{j}-\frac{1}{4 \pi} \sum_{j=1}^{m} \min \left(\tau_{j+1}-\tau_{j}, \tau_{j}-\tau_{j-1}\right) \\
& \times\left[\boldsymbol{b}_{j+1}^{\prime} \int_{-\pi}^{\pi} \frac{1}{2 \pi} \sum_{l=1-\left[\rho_{j}\right]}^{\infty} \Gamma(l) e^{-i l \lambda} d \boldsymbol{M}(\lambda) \boldsymbol{b}_{j}+\boldsymbol{b}_{j}^{\prime} \int_{-\pi}^{\pi} \frac{1}{2 \pi} \sum_{l=-\infty}^{\left[\rho_{j}\right]-1} \Gamma(l) e^{-i l \lambda} d \boldsymbol{M}(\lambda) \boldsymbol{b}_{j+1}\right]+o(1)
\end{aligned}
$$

As for $D_{32}$, we have

$$
\begin{aligned}
& D_{32} \\
= & -\frac{1}{4 n \pi} \sum_{k=1}^{n} f\left(\lambda_{k}\right)^{-1} \\
& \times\left(\sum_{j_{1}=1}^{m} \sum_{s_{1}=\left[\tau_{j_{1}} n\right]+1}^{\left[\tau_{j_{1}} n+\rho_{j_{1}}\right]}\left(\boldsymbol{\beta}_{j_{1}+1}-\boldsymbol{\beta}_{j_{1}}\right)^{\prime} \boldsymbol{z}_{s_{1}} e^{i s_{1} \lambda_{k}}\right)\left(\sum_{j_{2}=1}^{m+1} \sum_{s_{2}=\left[\tau_{j_{2}-1} n\right]+1}^{\left[\tau_{j_{2}} n+\rho_{j_{2}}\right]} \boldsymbol{z}_{s_{2}}^{\prime} \boldsymbol{D}_{n}^{-1} \boldsymbol{b}_{j_{2}} e^{-i s_{2} \lambda_{k}}\right) \\
= & \frac{1}{8 \pi^{2}} \sum_{l=-\infty}^{\infty} \Gamma(l) \sum_{j_{1}=1}^{m} \sum_{j_{2}=1}^{m+1} \sum_{s_{1}=\left[\tau_{j_{1}} n\right]+1}^{\left[\tau_{j_{1}} n+\rho_{j_{1}}\right]} \sum_{s_{2}=\left[\tau_{j_{2}-1} n\right]+1}^{\left[\tau_{\left.j_{2} n \rho_{j_{2}}\right]}\right.}\left(\boldsymbol{\beta}_{j_{1}+1}-\boldsymbol{\beta}_{j_{1}}\right)^{\prime} \boldsymbol{z}_{s_{1}} \boldsymbol{z}_{s_{2}}^{\prime} \boldsymbol{D}_{n}^{-1} \boldsymbol{b}_{j_{2}} \\
& \times \frac{1}{n} \sum_{k=1}^{n} e^{i\left(s_{1}-s_{2}-l\right) \lambda_{k}} \\
= & \frac{1}{8 \pi} \sum_{l=-\infty}^{\infty} \Gamma(l) \sum_{j_{1}=1}^{m} \sum_{j_{2}=1}^{m+1} \sum_{s=\left[\tau_{j_{2}-1} n\right]+1}^{\left[\tau_{j_{2}} n \rho_{j_{2}}\right]}\left(\boldsymbol{\beta}_{j_{1}+1}-\boldsymbol{\beta}_{j_{1}}\right)^{\prime} \boldsymbol{z}_{s+l} \boldsymbol{z}_{s}^{\prime} \boldsymbol{D}_{n}^{-1} \boldsymbol{b}_{j_{2}} \\
= & \min _{1 \leq i \leq q} O\left(a_{i i}^{n}(0)^{-1 / 2}\right)
\end{aligned}
$$

where we use (G.1) and (5.4) to get the result. The asymptotic representation fot $D_{33}$ is obtained similarly as for $D_{32}$, which gives $D_{33}=\min _{1 \leq i \leq q} O\left(a_{i i}^{n}(0)^{-1 / 2}\right)$. The joint asympotic normality and the covariance structure of $\left(\boldsymbol{W}_{1}, \boldsymbol{W}_{2}\right)^{-}$follows from the above evaluation.

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Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Machikaneyama-cho 1-3, Toyonaka, Osaka 560-8531, JAPAN

Email address: shiohama@sigmath.es.osaka-u.ac.jp


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