ON KP-RADICAL IN BCI-ALGEBRAS

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ABSTRACT. The aim of this paper is to study and investigate some properties of kp-radical in BCI-algebras. Moreover, some properties on BCI-homomorphism are obtained.

1. Introduction

In [2],the notion of nil radicals in BCI-algebras was introduced and various properties were developed in [3]. Further, several results on nil ideals were obtained in [4;5]. In [6]. Abujabl etc. introduced the concept of k-radical in BCI-algebras and studied some propesties. In this paper, we introduce the concept of kp-radical in BCI-algebras and investigate some properties.

By a BCI-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$.

(II) $(x * (x * y)) * y = 0$.

(III) $x * x = 0$

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A binary relation $\leq$ on X can be defined by $x \leq y$ if and only if $x * y = 0$. For a BCI-algebra X, the BCK-part of X is $X+ = \{x \in X \mid 0 * x = 0\}$. The p-radical of a BCI-algebra is the BCK-part of X, a BCI-algebra is called p-semisimple if its p-radical is $\{0\}$.

A nonempty subset S of a BCI-algebra X is called a subalgebra if, for any x and y in $S, x * y \in S$. A non-empty subset I of a BCI-algebra X is called an ideal of X if $0 \in I$, and if $x * y \in I, y \in I$ imply that $x \in I$. An ideal I of a BCI-algebra X is a closed ideal of X if $0 * x \in I$ for every $x \in I$. A mapping $f : X \to Y$ of BCI - algebras is called homomorpism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. For any element x, y in a BCI-algebra X, we use $x * y^k$ to denote the element $(\cdots((x * y) * y)\cdots) * y$, where y occurs k times.

Let S be a nonempty subset of a BCI-algebra X. For any positive integer k, $\sqrt[k]{S} = \{x \in X \mid 0 * x^k \in S\}$ is called the k-radical of S. It's clear that $0 \in \sqrt[k]{S}$. But it is not necessary that $S = \sqrt[k]{S}$. Let I be an ideal of a BCI-algebra X. Then I is called a k-semiprime ideal of X if $I = \sqrt[k]{I}$ for any positive integer k. Some properties of k-radical were studied in [6].

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2. Main Results

Definition 2.1. Let S be a non-empty subset of a BCI-algebra X. Then, for any positive integer k, we define $\sqrt[kp]{S} = \{x \in X \mid 0 * (0 * x^k) \in S\}$,which is called the kp-radical of S.

Theorem 2.2. Let X be a BCI-algebra and I an ideal of X. Then $\sqrt[kp]{I}$ is an ideal of X.

Proof. It's clear that $0 \in \sqrt[kp]{I}$. Let y and $x * y \in \sqrt[kp]{I}$, then $0 * (0 * y^k) \in I$ and $0*(0*(x*y)^k) \in I$. Hence $(0*(0*x^k))*(0*(0*y^k)) = 0*((0*x^k)*(0*y^k)) = 0*(0*(x*y)^k) \in I$. Since I is an ideal of X, then $0 * (0 * x^k) \in I$, and that $x \in \sqrt[kp]{I}$. Hence I is an ideal of X.

Proposition 2.3. Let S be a subalgebra of a BCI-algebra X, then $\sqrt[kp]{S}$ is a subalgebra of X.

Proof. Let $x, y \in \sqrt[kp]{S}$, then $0 * (0 * x^k) \in S$ and $0 * (0 * y^k) \in S$. Hence $0 * (0 * (x * y)^k) = (0 * (0 * x^k)) * (0 * (0 * y^k)) \in S$ because S is a subalgera of X. So $x * y \in \sqrt[kp]{S}$, and that $\sqrt[kp]{S}$ is a subalgebra of X.

Proposition 2.4. If S is a subalgebra of a BCI-algebra X and $x \in \sqrt[kp]{S}$, then $0 * x \in \sqrt[kp]{S}$.

Proof. Let $x \in \sqrt[kp]{S}$, then $0 * (0 * x^k) \in S$. Hence $0 * (0 * (0 * x^k)) \in S$ because S is a subalgebra of X. Therefore $0 * (0 * (0 * x)^k) = 0 * (0 * (0 * x^k)) \in S$, and that $0 * x \in \sqrt[kp]{S}$.

Proposition 2.5. If A is a closed ideal of a BCI-algebra X, then $\sqrt[kp]{A}$ is a closed ideal of X.

It's immediate consepuence of Proposition 2.3 and 2.4.

Theorem 2.6. If I and J are ideals of a BCI-algebra X, then $\sqrt[kp]{I \cap J} = \sqrt[kp]{I} \cap \sqrt[kp]{J}$.

Proof. Let $x \in \sqrt[kp]{I \cap J}$, then $0*(0*x^k) \in I \cap J$. Thus $0*(0*x^k) \in I$ and $0*(0*x^k) \in J$, and that $x \in \sqrt[kp]{I}$ and $x \in \sqrt[kp]{J}$. Hence $x \in \sqrt[kp]{I} \cap \sqrt[kp]{J}$. Thus $\sqrt[kp]{I \cap J} = \sqrt[kp]{I} \cap \sqrt[kp]{J}$. Conversely, let $x \in \sqrt[kp]{I} \cap \sqrt[kp]{J}$ then $x \in \sqrt[kp]{I}$ and $x \in \sqrt[kp]{J}$, and that $0 * (0 * x^k) \in I$ and $0*(0*x^k) \in J$. Hence $0*(0*x^k) \in I \cap J$ and that $x \in \sqrt[kp]{I \cap J}$. Thus $\sqrt[kp]{I} \cap \sqrt[kp]{J} \subseteq \sqrt[kp]{I \cap J}$. Therefore $\sqrt[kp]{I} \cap \sqrt[kp]{J} = \sqrt[kp]{I \cap J}$.

Theorem 2.7. Let I and J be two ideal of a BCI-algebra X, then $\sqrt[kp]{I \cup J} = \sqrt[kp]{I} \cup \sqrt[kp]{J}$.

Proof. If $x \in \sqrt[kp]{I \cup J}$, then $0 * (0 * x^k) \in I \cup J$. Hence $0 * (0 * x^k) \in I$ or $0 * (0 * x^k) \in J$, and that $0 * (0 * x^k) \in I \cup J$. Thus $\sqrt[kp]{I \cup J} = \sqrt[kp]{I} \cup \sqrt[kp]{J}$. Now, Let $x \in \sqrt[kp]{I} \cup \sqrt[kp]{J}$, then $x \in \sqrt[kp]{I}$ or $x \in \sqrt[kp]{J}$. Hence $0 * (0 * x^k) \in I$ or $0 * (0 * x^k) \in J$, and that $0 * (0 * x^k) \in I \cup J$. Hence $x \in \sqrt[kp]{I \cup J}$, and that $\sqrt[kp]{I} \cup \sqrt[kp]{J} \subseteq \sqrt[kp]{I \cup J}$. Therefore $\sqrt[kp]{I \cup J} = \sqrt[kp]{I} \cup \sqrt[kp]{J}$.

Definition 2.8. An ideal I of a BCI-algebra X is called a kp-semiprime ideal if $I = \sqrt[kp]{I}$.

Theorem 2.9. If I and J are kp-semiprime ideals of a BCI-algebra X, then $I \cap J$ is a kp-semiprime ideal.

Proof. From Theorem 2.6, we obtain $\sqrt[kp]{I \cap J} = \sqrt[kp]{I} \cap \sqrt[kp]{J}$. But $\sqrt[kp]{I} = I$ and $\sqrt[kp]{J} = J$ because I and J are kp-semiprime ideals. Thus $\sqrt[kp]{I \cap J} = I \cap J$, and that $I \cap J$ is a kp-semiprime ideal.

Theorem 2.10. If I and J are kp-semiprime ideals of a BCI- algebra X and $I \cup J$ is an ideal, then $I \cup J$ is a kp-semiprime ideal.

Proof. From Theorem 2.7, we obtain $\sqrt[kp]{I \cap J} = \sqrt[kp]{I} \cup \sqrt[kp]{J}$. But $\sqrt[kp]{I} = I$ and $\sqrt[kp]{J} = J$ because I and J are kp-semiprime ideals. Thus $\sqrt[kp]{I \cup J} = I \cup J$. and that $I \cup J$ is a kp-semiprime ideal.

Lemma 2.11. If $f : X \to Y$ is a homomorphism of BCI-algebras and I is an ideal of Y, then $f^{-1}(I)$ is an ideal of x.

Obviously.

Theorem 2.12. If $f : X \to Y$ is a homomorphism of BCI-algebras, then, for every ideal I of Y, $f^{-1}(\sqrt[kp]{I})$ is an ideal containing $\sqrt[kp]{f^{-1}(I)}$.

Proof. If I is an ideal of Y, then $\sqrt[kp]{I}$ is an ideal of Y by Theorem 2.2 and $f^{-1}(\sqrt[kp]{I})$ is an ideal of X by Lemma 2.11. Now Let $x \in \sqrt[kp]{f^{-1}(I)}$, then $0 * (0 * x^k) \in f^{-1}(I)$. Thus $f(0*(0*x^k)) \in I$, and that $0*(0*f(x)^k) \in I$. Thus $f(x) \in \sqrt[kp]{I} \subseteq Y$. Hence $x \in f^{-1}(\sqrt[kp]{I})$. Therefore, $\sqrt[kp]{f^{-1}(I)} \subseteq f^{-1}(\sqrt[kp]{I})$.

Theorem 2.13. Let $f : X \to Y$ be an onto homomorphism of BCI-algebras. If I is an ideal of X such that $kerf \subseteq I$, then $f^{-1}(f(I)) = I$.

Proof. Clearly, $I \subseteq f^{-1}(f(I))$. Now assume $x \in f^{-1}(f(I))$. Then $f(x) = f(y)$ for some $y \in I$. So $f(x) * f(y) = 0$, and that $f(x * y) = f(x) * f(y) = 0$. Hence $x * y \in kerf \subseteq I$. Thus $x * y \in I$ and $y \in I$. Hence $x \in I$ because I is an ideal of X. So $f^{-1}(f(I)) \subseteq I$. Therefore $f^{-1}(f(I)) = I$.

Theorem 2.14. If $f : X \to Y$ is a homomorpism of BCI-algebras and I is an ideal of X, then

(i) $f(\sqrt[kp]{I}) \subseteq \sqrt[kp]{(f(I))}$.

(ii) If $kerf \subseteq I$, then $\sqrt[kp]{f(I)} = f(\sqrt[kp]{I})$.

Proof. (i) If $y \in f(\sqrt[kp]{I})$, then there exists $x \in \sqrt[kp]{I}$ such that $y = f(x)$. Hence $x \in X$ and $0 * (0 * x^k) \in I$. Thus $y = f(x) \in f(X)$ and $f(0 * (0 * x^k)) \in f(I)$. Hence $0 * (0 * f(x)^k) = 0 * (0 * f(x^k)) = f(0 * (0 * x^k)) \in f(I)$. So $y \in f(x)$ and $0 * (0 * y^k) \in f(I)$. Therefore $y \in \sqrt[kp]{f(I)}$, and that $f(\sqrt[kp]{I}) \subseteq \sqrt[kp]{f}(I)$.

(ii) Let $x \in \sqrt[kp]{f(I)}$, then $x = f(y) \in f(I)$ for some $y \in X$ and $0 * (0 * x^k) \in f(I)$. Thus $0 * (0 * f(y)^k) \in f(I)$. And so $f(0 * (0 * y^k)) \in f(I)$. Therefore $0 * (0 * y^k) \in f^{-1}(f(I))$. But $kerf \subseteq I$, and by Theorem 2.13, we have $I = f^{-1}(f(I))$. Hence $0 * (0 * y^k) \in I$, and that $y \in \sqrt[kp]{I}$. Thus $x = f(y) \in f(\sqrt[kp]{I})$. Hence $\sqrt[kp]{f(I)} \subseteq f(\sqrt[kp]{I})$. Using (i), we obtain $\sqrt[kp]{f(I)} = f(\sqrt[kp]{I})$.

Let X and Y be BCI-algebras. Define $*$ on $X * Y = \{(x, y) \mid x \in X, y \in Y\}$ by $(x * y) * (u, v) = (x * u, y * v)$ for every $(x, y), (u, v) \in X * Y$. Then $(X * Y, *, (0, 0))$ is a BCI-algebra.

Theorem 2.15. Let S and T be nonempty subsets of BCI-algebras X and Y, respectively, then $\sqrt[kp]{S} * \sqrt[kp]{T} = \sqrt[kp]{S * T}$.

Proof. Let S and T be nonempty subsets of BCI-algebras X and Y, respectively, then $\sqrt[kp]{S * T} = \{(s, t) \in X * Y \mid (0, 0) * ((0, 0) * (s, t)^k) \in S * T\} = \{(s, t) \in X * Y \mid (0 * (0 * s^k), 0 * (0 * t^k)) \in S * T\} = \{(s, t) \in X * Y \mid 0 * (0 * s^k) \in S, 0 * (0 * t^k) \in T\} = \{(s, t) \in X * Y \mid s \in \sqrt[kp]{S}, t \in \sqrt[kp]{T}\} = \sqrt[kp]{S} * \sqrt[kp]{T}$.

Theorem 2.16. Let I and J be ideals of BCI-algebras X and Y, respectively, then $X * Y / \sqrt[kp]{I * J} \cong X / \sqrt[kp]{I} * Y / \sqrt[kp]{J}$.

Proof. If I and J are ideals of X and Y, respectively, then $\sqrt[kp]{I}$ and $\sqrt[kp]{J}$ are ideals of X and Y, respectively. By [4; Theorem 8], $\sqrt[kp]{I} * \sqrt[kp]{J}$ is an ideals of $X * Y$. Consider the natural homomorphisms

$$\Pi_x : X \to \frac{X}{\sqrt[kp]{I}} \qquad by \qquad x \mapsto \overline{x}, \;\; \forall \;\; x \in X$$

$$\Pi_y : Y \to \frac{Y}{\sqrt[kp]{J}} \qquad by \qquad y \mapsto \overline{y}, \ \ \forall \ \ y \in Y$$

. Clearly, $\overline{x * x'} = \Pi_x(x * x') = \Pi_x(x) * \Pi_x(x') = \overline{x} * \overline{x'}$ and $\overline{y * y'} = \Pi_y(y * y') = \Pi_y(y) * \Pi_y(y') = \overline{y} * \overline{y'}$ for $x, x' \in X$ and $y, y' \in Y$. Define $f : X * Y \to \frac{X}{\sqrt[kp]{I}} * \frac{Y}{\sqrt[kp]{I}}$ by $(x, y) \mapsto (\Pi_x(x), \Pi_y(y)) = (\overline{x}, \overline{y})$ for every $(x, y) \in X * Y$.

Let $(x, y), (u, v) \in X*Y$ such that $(x, y) = (u, v)$, then $(\overline{x}, \overline{y}) = (\overline{u}, \overline{v})$ and that $f((x, y)) = f((u, v))$. Therefore, f is a well-defined map. Also, $f((x, y) * (u, v)) = f((x * u, y * v)) = (\overline{x * u}, \overline{y * v}) = (\overline{x} * \overline{u}, \overline{y} * \overline{v}) = (\overline{x}, \overline{y}) * (\overline{u}, \overline{v}) = f((x, y)) * f((u, v))$.

Therefore, f is a homomorphism. Clearly, f is an onto map. By the homomorphism theorem, we have $\frac{X*Y}{kerf} \cong \frac{X}{\sqrt[kp]{I}} * \frac{Y}{\sqrt[kp]{J}}$. Furthermore, $kerf = \{(x, y) \in X * Y \mid f((x, y)) = (\overline{0}, \overline{0})\} = \{(x, y) \in X * Y \mid (\overline{x}, \overline{y}) = (\overline{0}, \overline{0})\} = \{(x, y) \in X * Y \mid \overline{x} = \overline{0}, \overline{y} = \overline{0}\} = \{(x, y) \in X * Y \mid \overline{x} \in \sqrt[kp]{I}, y \in \sqrt[kp]{J}\} = \sqrt[kp]{I} * \sqrt[kp]{J} = \sqrt[kp]{I * J}$ by Theorem 2.15.

Therefore, $\frac{X*Y}{\sqrt[kp]{I*J}} \cong \frac{X}{\sqrt[kp]{I}} * \frac{Y}{\sqrt[kp]{J}}$.

Theorem 2.17. An ideal I of a BCI-algebra X is a kp-semiprime ideal if and only if X/I has no non-zero nilpotent elements of index k.

Proof. Let I be a kp-semiprime ideal and let $\overline{a} \in X/I = X/\sqrt[kp]{I}$ be a nilpotent element of index k. Then $\overline{0} * (\overline{0} * \overline{a}^k) = \overline{0}$ and so $\overline{0 * (0 * a^k)} = \overline{0}$. Thus $0 * (0 * a^k) \in I$. Therefore, $a \in \sqrt[kp]{I}$. But $I = \sqrt[kp]{I}$ because I is a kp-semiprime ideal. Hence $a \in I$ and so $\overline{a} = \overline{0}$. Therefore, any nilpotent elements of index k in X/I is zero.

Conversely, we have $\sqrt[kp]{I} \subseteq I$. Let $\overline{a} \in X/I$ be a nilpotent element of index k. Then $\overline{a} = \overline{0}$, and hence, $a \in I$. Moreover, $\overline{0} * (\overline{0} * \overline{a^k}) = 0$ and so $\overline{0 * (0 * a^k)} = \overline{0}$. Thus $0 * (0 * a^k) \in I$ and so $a \in \sqrt[kp]{I}$. Hence $I \subseteq \sqrt[kp]{I}$, and that I is a kp-semiprime ideal.

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