THE CLASS OF B-ALGEBRAS COINCIDES WITH THE CLASS OF GROUPS

MICHIRO KONDO AND YOUNG BAE JUN

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ABSTRACT. In this short note, we show that every B-algebra is group-derived.

1. INTRODUCTION

The notion of BCK-algebras was proposed by Y. Imai and K. Iséki in 1966. In the same year, K. Iséki [1] introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra. It is so important to try to generalize the algebraic structures. J. Neggers and H. S. Kim [3] introduced a class of algebras which is related to several classes of algebras such as $BCK/BCI/BCH$-algebras. They call it a *B-algebra*, and they proved that every group $(X; \circ, 0)$ determines a B-algebra $(X; *, 0)$, which is called the *group-derived B-algebra*. In connection with converse, they also proved that every B-algebra X with additional condition is group-derived. But, in this paper, we show that the additional condition is superfluous, and hence we conclude that the class of B-algebras coincides with the class of groups.

2. PRELIMINARIES

A *B-algebra* is a nonempty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,
(II) $x * 0 = x$,
(III) $(x * y) * z = x * (z * (0 * y))$

for all x, y, z in X.

Example 2.1. (Neggers and Kim [3, Examples 2.1 and 2.2]) (1) Let $X = \{0, 1, 2\}$ be a set with the following Cayley table:

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then $(X; *, 0)$ is a B-algebra.

(2) Let X be the set of all real numbers except for a nonnegative integer $-n$. Define a binary operation $*$ on X by

$$x * y := \frac{n(x-y)}{n+y}.$$

Then $(X; *, 0)$ is a B-algebra.

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Theorem 2.2. (Neggers and Kim [4, Theorem 3.1]) *Let* $(X; \circ, 0)$ *be a group. If we define* $x * y = x \circ y^{-1}$, *then* $(X; *, 0)$ *is a* B*-algebra.*

The above Theorem 2.2 shows that every group $(X; \circ, 0)$ determines a B-algebra $(X; *, 0)$, which is called the *group-derived* B*-algebra.*

3. Main Results

J. Neggers and H. S. Kim proved that every B-algebra X with additional condition is group-derived, that is,

Theorem 3.1. (Neggers and Kim [4, Theorem 3.3]) *Every* B*-algebra* X *with additional condition*

(C) *The mapping* $\phi : X \to X$, $x \mapsto 0 * x$, *is surjection*

is group-derived.

But we show that, in the following theorem, the condition (C) is superfluous. Before suggesting the main theorem, we need the following lemma.

Lemma 3.2. *Every* B*-algebra* X *satisfies the identity* $0 * (0 * x) = x$ *for all* $x \in X$.

Proof. For any $x \in X$, we have

$$\begin{aligned} 0 * (0 * x) &= (x * x) * (0 * x) && \text{by (I)} \\ &= x * \big((0 * x) * (0 * x)\big) && \text{by (III)} \\ &= x * 0 = x, && \text{by (II)} \end{aligned}$$

proving the lemma. □

Theorem 3.3. *Every* B*-algbera is group-derived.*

Proof. Let $(X; *, 0)$ be a B-algebra. Define an operation "$\circ$" on X by $x \circ y = x * (0 * y)$ for all $x, y \in X$. Then $(X; \circ, 0)$ is a group. In fact, $x \circ 0 = x * (0 * 0) = x * 0 = x$ and $0 \circ x = 0 * (0 * x) = x$ by Lemma 3.2. Therefore 0 acts like the identity element on X. Also, $x \circ (0 * x) = x * \big(0 * (0 * x)\big) = x * x = 0$ and $(0 * x) \circ x = (0 * x) * (0 * x) = 0$, which shows that $0 * x$ behaves like a multiplicative inverse of the element x with respect to the operation "$\circ$". Finally the following calculations show that X satisfies the associative law under the operation "$\circ$".

$$\begin{aligned} (x \circ y) \circ z &= \big(x * (0 * y)\big) * (0 * z) && \\ &= x * \big((0 * z) * (0 * (0 * y))\big) && \text{by (III)} \\ &= x * \big((0 * z) * y\big) && \text{by Lemma 3.2} \\ &= x * \big(0 * (y * (0 * z))\big) && \text{by (III)} \\ &= x * \big(0 * (y \circ z)\big) = x \circ (y \circ z). && \end{aligned}$$

Noticing that $x * y = x * \big(0 * (0 * y)\big) = x * (0 * y^{-1}) = x \circ y^{-1}$ for all $x, y \in X$, we know that $(X; *, 0)$ is group-derived. This completes the proof. □

Combining Theorems 2.2 and 3.3, we conclude that the class of B-algebras coincides with the class of groups.

References

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M. Kondo: School of Information Environment, Tokyo Denki University,
Inzai, 270-1382, Japan.
E-mail: kondo@cis.shimane-u.ac.jp

Y. B. Jun: Department of Mathematics Education, Gyeongsang National University, Chinju (Jinju) 660-701, Korea.
E-mail: ybjun@nongae.gsnu.ac.jp