Primitive recursive analogues of the least regular cardinal and the least weakly inaccessible cardinal *†

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Abstract. In this paper, we establish primitive recursive analogues of certain regular cardinals including the least weakly inaccessible cardinal. In order to embody our ideas, we utilize certain ordinal representation systems, which are based on the ideas of M.Rathjen and W.Buchholz.

1 Introduction For more than twenty years, several proof theorists have investigated proof theoretic ordinals of set theoretic systems based on **KP** (Kripke-Platek set theory). In order to obtain such ordinals, they deal with several large cardinals and admissible ordinals. For example, they first consider a certain cardinal κ and an admissible ordinal α corresponding to κ and a system $\mathbf{KP}^*$ which is based on **KP** and characterized by α, and next they define certain sets of ordinals and functions by employing κ (which are called "Skolem hulls and collapsing functions") and, by using such sets and functions, establish a primitive recursive structure which generates the proof theoretic ordinal of $\mathbf{KP}^*$.

Observing their inventions such as Skolem hulls, collapsing functions, and primitive recursive structures above, we can expect a possibility of existence of primitive recursive ordinals corresponding to large cardinals (and admissible ordinals) which are employed for establishing the proof theoretic ordinals.

In this paper, we propose two primitive recursive ordinals which can be expected to correspond to the least regular cardinal and the least weakly inaccessible cardinal. These ordinals are obtained ¿from primitive recursive structures $\mathcal{T}(\Omega)$ and $\mathcal{T}(I)$, which are called "EORS"s (elementary ordinal representation systems) and established by certain Skolem hulls and collapsing functions. In order to establish $\mathcal{T}(\Omega)$ and $\mathcal{T}(I)$, we refer ideas of M.Rathjen in [Ra98], [Ra99] and that of W.Buchholz in [Bu93], which were employed for establishing the proof theoretic ordinals of $\mathbf{KP}\omega$ and **KPi** (see also [Po98]).

In Section 2, we define certain Skolem hulls and collapsing functions and define an EORS $\mathcal{T}(\Omega)$. Thus, we define a primitive recursive analogue of the least regular cardinal greater than ω by using $\mathcal{T}(\Omega)$. In Section 3, we define an EORS $\mathcal{T}(I)$ by using Skolem hulls and collapsing functions defined in [Bu93], and define a set of ordinals obtained from $\mathcal{T}(I)$ to be a set of primitive recursive ordinals corresponding to regular cardinals less than or equal to the least weakly inaccessible cardinal. Then we show that an ordinal $\psi_I^I(\Omega_1)$ which is characterized by $\mathcal{T}(I)$ has a property similar to that of the least weakly inaccessible cardinal, that is, $\psi_I^I(\Omega_1)$ is an element of the set above as well as the limit of elements of the set which are less than $\psi_I^I(\Omega_1)$.

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In the last section, we compare countable ordinals denoted by $\mathcal{T}(\Omega)$ with those denoted by $\mathcal{T}(I)$, in particular, the primitive recursive analogue of the least regular ordinal which is defined in Section 2 with that defined in Section 3.

2 A primitive recursive analogue of the least regular cardinal In this section, we define a candidate of a primitive recursive ordinal corresponding to the least weakly inaccessible cardinal. For this discussion, we define certain Skolem hulls and collapsing functions (see [Bu93], [Ra98] and [Po98]). We also refer to [Ra90], [Ra95], [Bu92] and [Bu93] to show lemmas in this section.

By $+$ we denote ordinary (noncommutative) ordinal addition. An ordinal α is called an *additive principal number* if α is closed under $+$. We let AP denote the class of all additive principal numbers. We also let φ denote the Veblen function, which is defined by: for any ordinals α, β, $\varphi\alpha\beta$ is the β^{th} additive principal number γ such that $\forall\xi < \alpha(\varphi\xi\gamma = \gamma)$. Note that $\varphi 0\alpha$ is often denoted by ω^α and $\varphi 1\alpha$ by ε_α. We also let ω denote the least infinite ordinal, and Ω the least uncountable ordinal, which is the least regular cardinal.

Definition 2.1 For each ordinal α and β, we define $C^\Omega(\alpha,\ \beta)$ as well as ψ^α_Ω by recursion on α:
$(\Omega 1)$ $\beta \cup \{0, \Omega\} \subset C^\Omega(\alpha, \beta)$.
$(\Omega 2)$ $\gamma = \gamma_1 + \gamma_2$ & $\gamma_1, \gamma_2 \in C^\Omega(\alpha, \beta)$ $\Rightarrow$ $\gamma \in C^\Omega(\alpha, \beta)$.
$(\Omega 3)$ $\gamma = \varphi\gamma_1\gamma_2$ & $\gamma_1, \gamma_2 \in C^\Omega(\alpha, \beta)$ $\Rightarrow$ $\gamma \in C^\Omega(\alpha, \beta)$.
$(\Omega 4)$ $\gamma = \psi^\xi_\Omega$ & $\xi \in C^\Omega(\alpha, \beta)$ & $\xi < \alpha$ & $\xi \in C^\Omega(\xi, \gamma)$ $\Rightarrow$ $\gamma \in C^\Omega(\alpha, \beta)$.
$\psi^\alpha_\Omega \simeq \min\{\rho < \Omega :\ C^\Omega(\alpha, \rho) \cap \Omega = \rho\}$.

Lemma 2.2 (1) For any $a \le a'$ and $\beta \le \beta'$, $C^\Omega(\alpha, \beta) \subset C^\Omega(\alpha', \beta')$.
(2) If β is limit, then $C^\Omega(\alpha, \beta) = \bigcup_{\delta<\beta} C^\Omega(\alpha, \delta)$.
(3) If α is limit, then $C^\Omega(\alpha, \beta) = \bigcup_{\gamma<\alpha} C^\Omega(\gamma, \beta)$.

Proof. (1) and (2) are trivial. For (3), it suffices to show that

$$\forall\xi\ (\xi \in C^\Omega(\alpha, \beta)\ \Rightarrow\ \exists\gamma < \alpha(\xi \in C^\Omega(\gamma, \beta)))$$

by induction on ξ. This can be shown easily. □

Lemma 2.3 ψ^α_Ω is defined and $\psi^\alpha_\Omega < \Omega$.

Proof. Let $\{\eta_n\}_{n<\omega}$ be a sequence of ordinals defined by:

$$\eta_0 = \sup(C^\Omega(\alpha, 0) \cap \Omega) \qquad \text{and} \qquad \eta_{n+1} = \sup(C^\Omega(\alpha, \eta_n) \cap \Omega),$$

and let η^* be $\sup\{\eta_n : n < \omega\}$. Since the cardinality of $C^\Omega(\alpha, 0)$ is ω, $\eta_0 < \Omega$ by the regularity of Ω. By repetition of this argument, one obtains $\eta_n < \Omega$ and $\eta^* < \Omega$. So, by the definition of η^* and Le.2.2,

$$C^\Omega(\alpha, \eta^*) \cap \Omega = \bigcup_n C^\Omega(\alpha, \eta_n) \cap \Omega = \eta^*.$$

Thus, ψ^α_Ω is an ordinal less than or equal to $\eta^* < \Omega$. □

Lemma 2.4 Each ordinal of the form ψ^α_Ω is strongly critical, that is, for each α, ψ^α_Ω is closed under φ.

Proof. $\psi^\alpha_\Omega = C^\Omega(\alpha, \psi^\alpha_\Omega) \cap \Omega$ follows from Def.3.1, and both of Ω and $C^\Omega(\alpha, \psi^\alpha_\Omega)$ are closed under the operations $+$ and φ. So, ψ^α_Ω is also closed under these operations. □

Definition 2.5 (1) $\gamma =_{\rm nf} \gamma_1 + \ldots + \gamma_n :\Leftrightarrow\ \gamma = \gamma_1 + \ldots + \gamma_n$ and $\gamma > \gamma_1 \geq \cdots \geq \gamma_n$ and $\gamma_1, \cdots, \gamma_n \in AP$, where AP stands for $\{\omega^\alpha :\ \alpha$ is an ordinal$\}$.
(2) $\gamma =_{\rm nf} \varphi\alpha\beta :\Leftrightarrow\ \gamma = \varphi\alpha\beta\ \&\ \alpha, \beta < \gamma$.
(3) $\gamma =_{\rm nf} \psi^\alpha_\Omega :\Leftrightarrow\ \gamma = \psi^\alpha_\Omega\ \&\ \alpha \in C^\Omega(\alpha, \gamma)$.

Lemma 2.6 (1) If $\alpha \in C^\Omega(\alpha, \psi^\alpha_\Omega)$ and $\alpha < \beta$, then $\psi^\alpha_\Omega < \psi^\beta_\Omega$.
(2) Let $\mu =_{\rm nf} \psi^\alpha_\Omega$ and $\nu =_{\rm nf} \psi^\beta_\Omega$. Then, $\mu < \nu$ iff $\alpha < \beta$.
(3) If $\mu =_{\rm nf} \psi^\alpha_\Omega$, $\nu =_{\rm nf} \psi^\beta_\Omega$ and if $\mu = \nu$, then $\alpha = \beta$.

Proof. We can obtain (3) from (2), and (2) from (1). So, we show (1) as follows. By Def.2.1 and Le.2.2.(1), $\psi^\beta_\Omega \subset C^\Omega(\alpha, \psi^\beta_\Omega) \cap \Omega \subset C^\Omega(\beta, \psi^\beta_\Omega) \cap \Omega = \psi^\beta_\Omega$. So, by the definition of ψ^α_Ω, $\psi^\alpha_\Omega \leq \psi^\beta_\Omega$. Thus, $\alpha \in C^\Omega(\alpha, \psi^\alpha_\Omega) \subset C^\Omega(\beta, \psi^\beta_\Omega)$. Since we also have $\alpha < \beta$ and $\psi^\alpha_\Omega < \Omega$, $\psi^\alpha_\Omega \in C^\Omega(\beta, \psi^\beta_\Omega) \cap \Omega = \nu$. □

Definition 2.7 We define the set $\mathcal{T}(\Omega)$ of ordinals and the *rank* $r(\gamma)$ $(< \omega)$ of each element γ of $\mathcal{T}(\Omega)$, as follows:
$(\mathcal{T}_\Omega 1)$ $0, I \in \mathcal{T}(\Omega)$ and $r(0) = r(I) = 0$.
$(\mathcal{T}_\Omega 2)$ If $\gamma =_{\rm nf} \gamma_1 + \cdots + \gamma_n$ and $\gamma_1, \cdots, \gamma_n \in \mathcal{T}(\Omega)$, then $\gamma \in \mathcal{T}(\Omega)$ and $r(\gamma) = \max\{r(\gamma_1), \cdots, r(\gamma_n)\} + 1$.
$(\mathcal{T}_\Omega 3)$ If $\gamma =_{\rm nf} \varphi\alpha\beta\ \&\ \alpha, \beta \in \mathcal{T}(\Omega)\ \&\ (\gamma < \Omega$ or $\alpha = 0)$, then $\gamma \in \mathcal{T}(\Omega)\ \&\ r(\gamma) = \max\{r(\alpha), r(\beta)\} + 1$.
$(\mathcal{T}_\Omega 4)$ If $\gamma =_{\rm nf} \psi^\alpha_\Omega$ and $\alpha \in \mathcal{T}(\Omega)$, then $\gamma \in \mathcal{T}(\Omega)$ and $r(\gamma) = r(\alpha) + 1$.

We omit proofs of the following lemmas 2.8 and 2.9. We will show the more generalized versions of them in the next section (see Le.3.10 and Le.3.11).

Lemma 2.8 Every element of $\mathcal{T}(\Omega)$ has a unique expression,

Lemma 2.9 $\mathcal{T}(\Omega) = C^\Omega(\varepsilon_{\Omega+1}, 0) \cap \varepsilon_{\Omega+1}$, in particular, $\psi^{\varepsilon_{\Omega+1}}_\Omega = \mathcal{T}(\Omega) \cap \Omega$.

Remark 2.10 One can consider $\mathcal{T}(\Omega)$, $=_{\rm nf}$ and several properties of the sets $C^\Omega(\alpha, \beta)$ to be formalized as expressions by finite strings of suitable symbols (for example, see [Ra91; Sec.2]). Indeed, we can regard $\mathcal{T}(\Omega)$ as a primitive recursive order structure.

Now we define primitive recursive analogues of regular cardinals by using $\mathcal{T}(\Omega)$.

Definition 2.11 A primitive recursive ordinal γ is called a *proof-theoretically regular ordinal based on* $\mathcal{T}(\Omega)$ if γ is an element of $\mathcal{T}(\Omega)$ and of the form ψ^κ_Ω, where κ is a regular cardinal.

Proposition 2.12 ψ^Ω_Ω is the least proof-theoretically regular ordinal based on $\mathcal{T}(\Omega)$. In fact, $\mathcal{T}(\Omega)$ contains only one proof-theoretically regular ordinal based on $\mathcal{T}(\Omega)$.

Proof. It is straightforward since no element of $\mathcal{T}(\Omega)$ besides Ω which is a regular cardinal [1]. □

By Le.2.4, $\psi^0_\Omega \geq \Gamma_0$ which is the least strongly critical ordinal. (In fact, one can easily check $\psi^0_\Omega = \Gamma_0$.) This means ψ^Ω_Ω is much larger than Γ_0. In Section 4, the reader shall know that $\mathcal{T}(\Omega)$ is an EORS for $\mathbf{KP}\omega$, that is, $\psi^{\varepsilon_{\Omega+1}}_\Omega$ is equal to the proof theoretic ordinal of $\mathbf{KP}\omega$. Therefore, we can consider ψ^Ω_Ω to be an ordinal related to the least admissible ordinal $> \omega$ as well as the least regular cardinal.

[1] In this paper, we consider each regular cardinal to be uncountable. So, we do not regard ω $(= \varphi 00)$ as a regular cardinal.

3 A primitive recursive analogue of the least weakly inaccessible cardinal In this section, we define a candidate for a primitive recursive ordinal corresponding to the least weakly inaccessible cardinal. For this discussion, we introduce Skolem hulls and collapsing functions defined in [Bu93] and refer to ideas in [Bu92], [Bu93] and [Ra98].

By I we always denote the least weakly inaccessible cardinal. For each ordinal α, we let Ω_α denote the α^{th} cardinal. For example, $\Omega_0 = \omega$, $\Omega_1 = \Omega$ and $\Omega_I = I$. Also we let κ and π denote elements of $\mathcal{R}^I := \{I\} \cup \{\Omega_{\alpha+1} : \alpha < I\}$.

Definition 3.1 ([Bu93]: Def.4.2) For each ordinal α and β, we define a set $C^I(\alpha,\ \beta)$ as well as a function ψ_I^α by recursion on α:
(I1) $\beta \cup \{0, I\} \subset C^I(\alpha, \beta)$.
(I2) $\gamma = \gamma_1 + \gamma_2$ & $\gamma_1, \gamma_2 \in C^I(\alpha,\beta)$ $\Rightarrow$ $\gamma \in C^I(\alpha,\beta)$.
(I3) $\gamma = \varphi\gamma_1\gamma_2$ & $\gamma_1, \gamma_2 \in C^I(\alpha,\beta)$ $\Rightarrow$ $\gamma \in C^I(\alpha,\beta)$.
(I4) $\gamma = \Omega_{\gamma_1}$ & $\gamma_1 \in C^I(\alpha,\beta)$ $\Rightarrow$ $\gamma \in C^I(\alpha,\beta)$.
(I5) $\gamma = \psi_I^\xi(\kappa)$ & $\xi, \kappa \in C^I(\alpha,\beta)$ & $\xi < \alpha$ & $\xi \in C^I(\xi,\gamma)$ $\Rightarrow$ $\gamma \in C^I(\alpha,\beta)$.
$\psi_I^\alpha(\kappa) \simeq \min\{\rho < \kappa :\ C^I(\alpha,\rho) \cap \kappa = \rho \wedge \kappa \in C^I(\alpha,\rho)\}$.

One can show properties of C^I and ψ_I similarly to the previous section. (We omit proofs of lemmas below whenever they are proved quite similarly to the previous section.)

Lemma 3.2 (1) For any $\alpha \le \alpha'$ and $\beta \le \beta'$, $C^I(\alpha,\beta) \subset C^I(\alpha',\beta')$.
(2) If β is limit, then $C^I(\alpha,\beta) = \bigcup_{\delta<\beta} C^I(\alpha,\delta)$. If α is limit, then $C^I(\alpha,\beta) = \bigcup_{\gamma<\alpha} C^I(\gamma,\beta)$.

Lemma 3.3 If $\kappa \in C^I(\alpha,\kappa)$, then $\psi_I^\alpha(\kappa)$ is defined and $\psi_I^\alpha(\kappa) < \kappa$.

Proof. By Le.3.2, $\kappa \in C^I(\alpha,\kappa)$ implies that there exists an $\eta < \kappa$ with $\kappa \in C^I(\alpha,\eta)$. So, by considering a sequence $\{\eta_n\}_{n<\omega}$ with $\eta_0 = \sup(C^I(\alpha,\eta)\cap\kappa)$ and $\eta_{n+1} = \sup(C^I(\alpha,\eta_n)\cap\kappa)$, we can show this lemma in a way similar to the proof of Le.2.3. □

Remark 3.4 (1) If $\kappa < I$, then κ is of the form $\Omega_{\xi+1}$ and $\xi < \kappa$, and hence, $\xi \in C^I(\alpha,\kappa)$ and hence $\kappa \in C^I(\alpha,\kappa)$. Thus, $\psi_I^\alpha(\kappa)$ is defined whenever $\kappa < I$.
(2) Le.3.3 implies that $\kappa \in C^I(\alpha,\psi_I^\alpha(\kappa))$ whenever $\kappa \in C^I(\alpha,\kappa)$. One can also see that $\psi_I^\alpha(\kappa)$ is *not* regular, by seeing the proofs of Le.3.3 and Le.2.3.

Lemma 3.5 Each ordinal of the form $\psi_I^\alpha(\kappa)$ is strongly critical.

Definition 3.6 (1) $\gamma =_{\text{nf}} \Omega_\sigma :\Leftrightarrow\ \gamma = \Omega_\sigma$ & $\sigma < \gamma$.
(2) $\gamma =_{\text{nf}} \psi_I^\alpha(\kappa) :\Leftrightarrow\ \gamma = \psi_I^\alpha(\kappa)$ & $\alpha \in C^I(\alpha,\gamma)$.

Lemma 3.7 If $\gamma =_{\text{nf}} \psi_I^\alpha(\kappa)$, $\delta =_{\text{nf}} \psi_I^\beta(\pi)$ and $\gamma = \delta$, then $\alpha = \beta$ and $\kappa = \pi$.

Lemma 3.8 Let $\mu =_{\text{nf}} \psi_I^\alpha(\kappa)$ and $\nu =_{\text{nf}} \psi_I^\beta(\pi)$. Then, $\mu < \nu$ iff all the following properties hold.
(i) If $\kappa < \pi$, then $\kappa < \nu$.
(ii) If $\kappa = \pi$, then $\alpha < \beta$.
(iii) If $\kappa > \pi$, then $\mu < \pi$.

Definition 3.9 ([Ra98]: Def.3.3) We define the set $\mathcal{T}(I)$ of ordinals and the *rank* $r(\gamma)$ $(< \omega)$ of each element γ of $\mathcal{T}(I)$, as follows:
($\mathcal{T}_I$1) $0, I \in \mathcal{T}(I)$ and $r(0) = r(I) = 0$.

$(\mathcal{T}_I 2)$ If $\gamma =_{\rm nf} \gamma_1 + \cdots + \gamma_n$ & $\gamma_1, \cdots, \gamma_n \in \mathcal{T}(I)$, then $\gamma \in \mathcal{T}(I)$ & $r(\gamma) = \max\{r(\gamma_1), \cdots, r(\gamma_n)\} + 1$.
$(\mathcal{T}_I 3)$ If $\gamma =_{\rm nf} \varphi\alpha\beta$ & $\alpha, \beta \in \mathcal{T}(I)$ & ($\gamma < I$ or $\alpha = 0$), then $\gamma \in \mathcal{T}(I)$ & $r(\gamma) = \max\{r(\alpha), r(\beta)\} + 1$.
$(\mathcal{T}_I 4)$ If $\gamma =_{\rm nf} \Omega_\alpha < I$, $\alpha > 0$ & $\alpha \in \mathcal{T}(I)$, then $\gamma \in \mathcal{T}(I)$ & $r(\gamma) = r(\alpha) + 1$.
$(\mathcal{T}_I 5)$ If $\gamma =_{\rm nf} \psi_I^\alpha(\kappa)$ & $\kappa, \alpha \in \mathcal{T}(I)$, then $\gamma \in \mathcal{T}(I)$ & $r(\gamma) = \max\{r(\kappa), r(\alpha)\} + 1$.

Lemma 3.10 (1) For each $i = 1, \cdots, 4$, any element which is constructed by $(\mathcal{T}_I i)$ at the last step can not be constructed by $(\mathcal{T}_I j)$ at the last step for any $j = i+1, \cdots, 5$.
(2) Every element of $\mathcal{T}(I)$ has a unique expression,

Proof. (1) We check this lemma, as follows.
(i) Clearly, we can obtain 0 and I only by $(\mathcal{T}_I 1)$.
(ii) While AP does not contain any element constructed by $(\mathcal{T}_I 2)$ (at the last step), AP contains every element constructed by each of $(\mathcal{T}_I 3) \sim (\mathcal{T}_I 5)$ (at the last step).
(iii) Any element constructed by $(\mathcal{T}_I 2)$ or $(\mathcal{T}_I 3)$ is not strongly critical. However, by Le.3.5, every element constructed by $(\mathcal{T}_I 4)$ or $(\mathcal{T}_I 5)$ is strongly critical.
(iv) Let γ be an element constructed by $(\mathcal{T}_I 4)$. Then, γ is a regular cardinal less than I. So, by Re.3.4.(2), we can not obtain γ by $(\mathcal{T}_I 5)$.
(2) The results follows from (1) immediately above and Le.3.7. □

Lemma 3.11 (1) $\mathcal{T}(I) = C^I(\varepsilon_{I+1}, 0) \cap \varepsilon_{I+1}$.
(2) For each $\alpha \leq \varepsilon_{I+1}$, $C^I(\alpha, \psi_I^\alpha(\Omega_1)) = C^I(\alpha, 0)$.
(3) In particular, $\psi_I^{\varepsilon_{I+1}}(\Omega_1) = \mathcal{T}(I) \cap \Omega_1$.

Proof. (1) $\supset$: By induction on the definition of $C(\varepsilon_{I+1}, 0)$. $\subset$: By induction on the rank of each element of $\mathcal{T}(I)$.
(2) By induction on α. Assume that, for each $\xi < \alpha$, $C^I(\xi, \psi_I^\xi(\Omega_1)) = C^I(\xi, 0)$. It suffices to show $\psi_I^\alpha(\Omega_1) \subset C^I(\alpha, 0)$.

Claim 1: $\forall \gamma \in C^I(\alpha, 0) \cap \Omega_1$ $(\gamma \subset C^I(\alpha, 0))$.
(*Proof of Claim 1*: (Case 1) Let γ be a strongly critical ordinal less than Ω_1. Then, γ has the form $\psi_I^\eta(\pi)$ for some $\eta < \alpha$ and some π. Since $\pi > \Omega_1$ implies $\Omega_1 \in C^I(\eta, \psi_I^\eta(\pi)) \cap \pi = \gamma$, we have $\pi = \Omega_1$ from $\gamma < \Omega_1$. Therefore, since $C^I(\eta, \psi_I^\eta(\Omega_1)) = C^I(\eta, 0)$ by induction hypothesis, $\gamma = \psi_I^\eta(\Omega_1) \subset C^I(\eta, 0) \subset C^I(\alpha, 0)$.
(Case 2) Let γ be an ordinal less than Ω_1. Then, let $\gamma_0 = \max\{\{0\} \cup SC(\gamma)\}$, where $SC(\gamma)$ denotes a set defined by:
(i) $SC(0) :=$ the empty set;
(ii) $SC(\gamma) := \{\gamma\}$ if γ is a strongly critical ordinal;
(iii) $SC(\gamma + \delta) := SC(\gamma) \cup SC(\delta)$;
(iv) $SC(\varphi\gamma\delta) := SC(\gamma) \cup SC(\delta)$.
Then, by (Case 1), $\gamma_0 \cup \{\gamma_0\} \subset C^I(\alpha, 0)$. So, $\gamma \subset \gamma^* \subset C^I(\alpha, 0)$, where $\gamma^* = \min\{\eta :$ η is a strongly critical ordinal larger than $\gamma_0\}$. □)

By Claim 1, $C^I(\alpha, 0) \cap \Omega_1$ is an ordinal. Let $\beta = C^I(\alpha, 0) \cap \Omega_1$. Then, by induction on the definition of $C^I(\alpha, \beta)$, we can easily show

$$\forall \gamma \in C^I(\alpha, \beta) \cap \Omega_1 \ (\gamma \in C^I(\alpha, 0) \cap \Omega_1).$$

So, $\beta \subset C^I(\alpha, \beta) \cap \Omega_1 \subset C^I(\alpha, 0) \cap \Omega_1 = \beta$. Therefore, since $\Omega_1 \in C^I(\alpha, \beta)$, we have $\psi_I^\alpha(\Omega_1) \leq \beta$ by the definition of $\psi_I^\alpha(\Omega_1)$. So, we have $\psi_I^\alpha(\Omega_1) \subset C^I(\alpha, 0)$.

(3) By (1) and (2) above, one obtain $\psi_I^{\varepsilon_{I+1}}(\Omega_1) = C^I(\varepsilon_{I+1}, \psi_I^{\varepsilon_{I+1}}(\Omega_1)) \cap \Omega_1 = C^I(\varepsilon_{I+1}, 0) \cap \Omega_1 = \mathcal{T}(I) \cap \Omega_1$. □

In a way similar to Re.2.10, we can regard $\mathcal{T}(I)$ as a primitive recursive order structure.

In a way similar to Def.2.11, we define primitive recursive analogues of certain regular cardinals, as follows. Let Reg denote the class of all regular cardinals.

Definition 3.12 A primitive recursive ordinal γ is called a *proof-theoretically regular ordinal based on* $\mathcal{T}(I)$ if γ is an element of $\mathcal{T}(I)$ and of the form $\psi_I^\kappa(\Omega_1)$ with $\kappa \in Reg$.

Note that the condition of a regular cardinal ι to be an inaccessible cardinal is that $\iota = \sup(Reg \cap \iota)$. (We should also remark that, by the regularity of ι, this condition implies that ι is the ι^{th} regular cardinal.) Thus, similarly to the condition above, we can define primitive recursive analogues of inaccessible cardinals by using $\mathcal{T}(I)$, as follows.

Definition 3.13 A primitive recursive ordinal γ is called a *proof-theoretically inaccessible ordinal based on* $\mathcal{T}(I)$ if γ is an element of $Reg(\mathcal{T}(I))$ as well as the supremum of $Reg(\mathcal{T}(I)) \cap \gamma$, where $Reg(\mathcal{T}(I))$ denotes the set of p.t.r.o.s (proof-theoretically regular ordinals) under $\mathcal{T}(I)$.

Theorem 3.14 $\psi_I^I(\Omega_1)$ is the least proof-theoretically inaccessible ordinal based on $\mathcal{T}(I)$.

Proof. (I) We first show $\psi_I^I(\Omega_1) = \sup(Reg(\mathcal{T}(I)) \cap \psi_I^I(\Omega_1))$. By Le.3.8, this property follows from $\psi_I^I(\Omega_1) = \sup\{\ \psi_I^\kappa(\Omega_1) \in \mathcal{T}(I) :\ \kappa \in Reg \cap I\ \}$. Moreover, by Le.3.8, it is clear that $\psi_I^I(\Omega_1) \geq \sup\{\psi_I^\kappa(\Omega_1) \in \mathcal{T}(I) :\ \kappa \in Reg \cap I\}$. So, we show that

$$(*) \qquad \forall \gamma\ (\ \gamma < \psi_I^I(\Omega_1)\ \Rightarrow\ \exists \alpha < I\ (\ \psi_I^{\Omega_{\alpha+1}}(\Omega_1) \in \mathcal{T}(I)\ \&\ \gamma < \psi_I^{\Omega_{\alpha+1}}(\Omega_1)\ )\ )$$

by induction on the rank of γ. (Note that Le.3.11.(2) enables us to use induction on the rank of γ to show $(*)$.)
(i) If γ is constructed by $(\mathcal{T}_I 1) \sim (\mathcal{T}_I 3)$, $(*)$ follows from Le.3.5.
(ii) There is not a case where γ is constructed by $(\mathcal{T}_I 4)$, since $\psi_I^I(\Omega_1) < \Omega_1$ by Le.3.3.
(iii) Suppose that γ is constructed by $(\mathcal{T}_I 5)$. So, we assume $\gamma =_{\text{nf}} \psi_I^\delta(\kappa)$. Note that $\kappa \geq \Omega_1$.

Claim 2: For any $\psi_I^\xi(\pi), \psi_I^\zeta(\Omega_1) \in \mathcal{T}(I)$, if $\psi_I^\xi(\pi) < \psi_I^\zeta(\Omega_1)$, then $\pi = \Omega_1$ and $\xi < \zeta$.
(*Proof of Claim 2*: Suppose that $\pi > \Omega_1$. Since $\psi_I^\xi(\pi) = C^I(\xi, \psi_I^\xi(\pi)) \cap \pi$ and $\Omega_1 \in C^I(\xi, \psi_I^\xi(\pi))$, it holds that $\Omega_1 < \psi_I^\xi(\pi)$. On the other hand, $\psi_I^\xi(\pi) < \psi_I^\zeta(\Omega_1) < \Omega_1$. It is contradict. So, $\pi = \Omega$, and hence, by Le.3.8, $\xi < \zeta$. □)

By Claim 2, $\kappa = \Omega_1$ and $\delta < I$. Since I is the I^{th} regular cardinal, $\delta < \Omega_{\delta+1} < I$. Thus, we must only show $\psi_I^{\Omega_{\delta+1}}(\Omega_1) \in \mathcal{T}(I)$ and $\gamma < \psi_I^{\Omega_{\delta+1}}(\Omega_1) < \psi_I^I(\Omega_1)$.

Claim 3: $\psi_I^{\Omega_{\delta+1}}(\Omega_1) \in \mathcal{T}(I)$ and $\gamma < \psi_I^{\Omega_{\delta+1}}(\Omega_1) < \psi_I^I(\Omega_1)$.
(*Proof of Claim 3*: By Def.3.1 and Le.2.2.(1), one obtains

$$\psi_I^{\Omega_{\delta+1}}(\Omega_1) \subset C^I(\delta, \psi_I^{\Omega_{\delta+1}}(\Omega_1)) \cap \Omega_1 \subset C^I(\Omega_{\delta+1}, \psi_I^{\Omega_{\delta+1}}(\Omega_1)) \cap \Omega_1 = \psi_I^{\Omega_{\delta+1}}(\Omega_1).$$

So, by the definition of γ $(= \psi_I^\delta(\Omega_1))$, $\gamma \leq \psi_I^{\Omega_{\delta+1}}(\Omega_1)$. $\gamma \in \mathcal{T}(I)$ implies $\delta \in C^I(\delta, \gamma)$, and hence, $\delta \in C^I(\delta, \psi_I^{\Omega_{\delta+1}}(\Omega_1))$. This means $\Omega_{\delta+1} \in C^I(\Omega_{\delta+1}, \psi_I^{\Omega_{\delta+1}}(\Omega_1))$, and hence,

$\psi_I^{\Omega_{\delta+1}}(\Omega_1) \in \mathcal{T}(I)$. Also we obtain $\gamma \in C^I(\Omega_{\delta+1}, \psi_I^{\Omega_{\delta+1}}(\Omega_1)) \cap \Omega_1$ and hence $\gamma < \psi_I^{\Omega_{\delta+1}}(\Omega_1)$. We also have $\psi_I^{\Omega_{\delta+1}}(\Omega_1) < \psi_I^I(\Omega_1)$ by Le.3.8. $\square$)

(II) We next show that for any $\psi_I^\pi(\Omega_1) \in Reg(\mathcal{T}(I)) \cap \psi_I^I(\Omega_1)$,

$$\psi_I^\pi(\Omega_1) \neq \sup\{\psi_I^\kappa(\Omega_1) \in \mathcal{T}(I) : \ \kappa \in Reg \cap \pi\}.$$

Since $\pi \in Reg \cap I$, π is the form of $\Omega_{\alpha+1}$ for some α. Since $\psi_I^\pi(\Omega_1) \in \mathcal{T}(I)$, $\pi \in C^I(\pi, \psi_I^\pi(\Omega_1))$, and hence, $\alpha \in C^I(\pi, \psi_I^\pi(\Omega_1))$, and hence, $\psi_I^{\Omega_\alpha}(\Omega_1) \in C^I(\pi, \psi_I^\pi(\Omega_1))$. So, $\psi_I^{\Omega_\alpha}(\Omega_1) < \psi_I^\pi(\Omega_1)$. On the other hand, for each $\psi_I^\kappa(\Omega_1) \in \mathcal{T}(I)$, one obtains $\kappa \in Reg \cap \pi \ \Rightarrow \ \psi_I^\kappa(\Omega_1) \leq \psi_I^{\Omega_\alpha}(\Omega_1)$ in a way similar to the proof of Claim 3. So, we have the result. $\square$

In order to convince ourselves of the similarity of this ordinal and the least weakly inaccessible cardinal we should establish the following property:

Conjecture $\psi_I^I(\Omega_1)$ is the $\psi_I^I(\Omega_1)^{\text{th}}$ element of $Reg(\mathcal{T}(I))$.

4 A relationship between proof-theoretically regular ordinals based on $\mathcal{T}(\Omega)$ and those based on $\mathcal{T}(I)$ In this section, we compare proof-theoretically regular ordinals based on $\mathcal{T}(\Omega)$ with those based on $\mathcal{T}(I)$. Finally, we summarize certain regular cardinals, admissible ordinals, and proof-theoretically regular ordinals.

Definition 4.1 We define a mapping $*$ from $\mathcal{T}(\Omega)$ into $\mathcal{T}(I)$ by recursion on the rank of each element of $\mathcal{T}(\Omega)$.
(i) $0^* := 0$ and $\Omega^* := \Omega_1$.
Let γ be an element of $\mathcal{T}(\Omega)$.
(ii) If $\gamma =_{\text{nf}} \gamma_1 + \cdots + \gamma_n$, then $\gamma^* := \gamma_1^* + \cdots + \gamma_n^*$.
(iii) If $\gamma =_{\text{nf}} \varphi\alpha\beta$, then $\gamma^* := \varphi\alpha^*\beta^*$.
(iv) If $\gamma =_{\text{nf}} \psi_\Omega^\alpha$, then $\gamma^* := \psi_I^{\alpha^*}(\Omega_1)$.

Lemma 4.2 $*$ is well-defined, that is, for each $\gamma \in \mathcal{T}(\Omega)$, γ^* is uniquely determined as an element of $\mathcal{T}(I)$. Moreover, $*$ strictly preserves the order of $\mathcal{T}(\Omega)$, that is, for each $\gamma, \delta \in \mathcal{T}(\Omega)$, $\gamma < \delta \ \Rightarrow \ \gamma^* < \delta^*$.

Sketch of Proof. By Le.2.8, each element of $\mathcal{T}(\Omega)$ is uniquely determined by 0, Ω, $+$, ψ and ψ_Ω. So, for each $\gamma \in \mathcal{T}(\Omega)$, γ^* is uniquely expressed by operators in $\mathcal{T}(I)$. So, it suffices to show that $\forall\gamma \in \mathcal{T}(\Omega)$ $(\gamma^* \in \mathcal{T}(I))$ and that $*$ strictly preserves the order on $\mathcal{T}(\Omega)$. So, we define a 2-ary predicate $P(x, y)$ by:

$$P(x, y) :\Leftrightarrow \ (x^*, y^* \in \mathcal{T}(\Omega) \ \& \ (x < y \Rightarrow x^* < y^*)),$$

and show the predicate $P(x, y)$ by induction on $\omega \cdot \max\{r(x), r(y)\} + \min\{r(x), r(y)\}$. Let γ denote an element of $\{x, y\}$ whose rank is not less than the other's rank, and δ denote the other. One can check $P(\gamma, \delta)$ and $P(\delta, \gamma)$ in the cases where $\gamma = 0$, $\gamma = \Omega$, $\gamma =_{\text{nf}} \gamma_1 + \gamma_2$, $\gamma =_{\text{nf}} \psi\gamma_1\gamma_2$ and $\gamma =_{\text{nf}} \psi_\Omega^{\gamma_1}$. However, for simplicity, we deal with only the most critical part, that is, we show only $\gamma^* \in \mathcal{T}(I)$ in the case where $\gamma =_{\text{nf}} \psi_\Omega^{\gamma_1}$ in what follows.

Claim 4: Let $\alpha, \beta \in \mathcal{T}(\Omega)$ with $r(\alpha), r(\beta) < r(\gamma)$ and with $\beta \in C^\Omega(\beta, \psi_\Omega^\beta)$. Then,

$$\alpha \in C^\Omega(\beta, \psi_\Omega^\beta) \ \Rightarrow \ \alpha^* \in C^I(\beta^*, \psi_I^{\beta^*}(\Omega_1)).$$

(*Proof of Claim 4*: By induction on $r(\alpha)$. We deal with only the case where $\alpha =_{\rm nf} \psi^{\xi}_{\Omega_1}$, since the other cases are trivial. In this case, since $\alpha \in C^{\Omega}(\beta, \psi^{\beta}_{\Omega})$, it holds that $\psi^{\xi}_{\Omega} < \psi^{\beta}_{\Omega}$ or that $(\xi \in C^{\Omega}(\beta, \psi^{\beta}_{\Omega})\ \&\ \xi < \beta)$. $\psi^{\xi}_{\Omega} < \psi^{\beta}_{\Omega}$ implies $\xi < \beta$ from Le.2.6, and hence, $\xi \in C^{\Omega}(\xi, \psi^{\xi}_{\Omega}) \subset C^{\Omega}(\beta, \psi^{\beta}_{\Omega})$. Therefore, it follows $\xi^* < \beta^*$ from the main induction hypothesis, and $\xi^* \in C^I(\beta^*, \psi^{\beta^*}_I(\Omega_1))$ ¿from the induction hypothesis of this claim. So, we have the result. □)

Since $\gamma_1 \in C^{\Omega}(\gamma_1, \psi^{\gamma_1}_{\Omega})$, it follows $\gamma_1^* \in C^I(\gamma_1^*, \psi^{\gamma_1^*}_I(\Omega_1))$ ¿from Claim 4. This implies $\gamma^* = \psi^{\gamma_1^*}_I(\Omega_1) \in \mathcal{T}(I)$ by the main induction hypothesis. □

Lemma 4.3 For each $\gamma < \psi^{\varepsilon_{\Omega_1+1}}_I(\Omega_1)$, there exists $\xi \in \mathcal{T}(\Omega) \cap \Omega$ such that $\xi^* = \gamma$.

Proof. By induction on $r(\gamma)$. The cases where $\gamma =_{\rm nf} 0$, $\gamma =_{\rm nf} \gamma_1 + \gamma_2$ and $\gamma =_{\rm nf} \varphi\gamma_1\gamma_2$ are trivial. There is not the case where $\gamma =_{\rm nf} \Omega_{\gamma_1}$ and $\gamma_1 > 0$, since $\gamma < \psi^{\varepsilon_{\Omega_1+1}}_I(\Omega_1)$.

Let $\gamma =_{\rm nf} \psi^{\alpha}_I(\kappa)$. Then, $\kappa = \Omega_1$, and $\alpha < \varepsilon_{\Omega_1+1}$ and $\alpha \in C^I(\alpha, \psi^{\alpha}_I(\Omega_1))$.

Claim 5: Every subterm β of α is less then ε_{Ω_1+1}.
(*Proof of Claim 5*: By induction on $r(\alpha) - r(\beta)$, we show $\beta < \varepsilon_{\Omega_1+1}$ and $\beta \in C^I(\alpha, \psi^{\alpha}_I(\Omega_1))$.
(i) If $\beta = \alpha$, then it is straightforward.
(ii) If there exists a substructure of α which is of the form $\alpha_1 + \alpha_2$ or $\psi\alpha_1\alpha_2$ and if β is α_1 or α_2, then it is trivial by the induction hypothesis of this claim. The case where there exists a substructure of α which is of the form Ω_β is also trivial by the same reason.
(iii) Assume there exists a substructure of α which is of the form $\psi^{\delta}_I(\pi)$ such that $\beta = \pi$ or $\beta = \delta$.
(iii-i) Let $\beta = \pi$. If $\pi > \Omega_1$, then $\psi^{\beta}_I(\pi)$ is a strongly critical ordinal larger than Ω_1 by Le.3.5. It is contradict. So, $\beta = \Omega_1$.
(iii-ii) Let $\beta = \delta$. ¿From $\psi^{\beta}_I(\pi) \in C^I(\alpha, \psi^{\alpha}_I(\Omega_1))$, it follows $(\beta \in C^I(\alpha, \psi^{\alpha}_I(\Omega_1))\ \&\ \beta < \alpha)$ or $\psi^{\beta}_I(\pi) < \psi^{\alpha}_I(\Omega_1)$. If $\beta \in C^I(\alpha, \psi^{\alpha}_I(\Omega_1))$ and $\beta < \alpha$, we have nothing to show any more. Assume $\psi^{\beta}_I(\pi) < \psi^{\alpha}_I(\Omega_1)$. Then, $\pi = \Omega_1$ and $\beta < \alpha$. On the other hand, since $\psi^{\beta}_I(\pi) \in \mathcal{T}(I)$, $\beta \in C^I(\beta, \psi^{\beta}_I(\pi))$. So, we have $\beta \in C^I(\alpha, \psi^{\alpha}_I(\Omega_1))$. □)

Claim 6: For each $\beta, \delta \in \mathcal{T}(\Omega)$, if $\beta^* \in C^I(\delta^*, \psi^{\delta^*}_I(\Omega_1))$, then $\beta \in C^{\Omega}(\delta, \psi^{\delta}_{\Omega})$. (*Proof of Claim 6*: By induction on $r(\beta)$. Only the case where $\beta =_{\rm nf} \psi^{\xi}_{\Omega}$ is not trivial. Let $\beta =_{\rm nf} \psi^{\xi}_{\Omega}$. Then, (i) $\beta^* < \psi^{\delta^*}_I(\Omega_1)$; or (ii) $(\xi^* \in C^I(\delta^*, \psi^{\delta^*}_I(\Omega_1))\ \ \&\ \xi^* < \delta^*\ \&\ \xi^* \in C^I(\xi^*, \psi^{\xi^*}_I(\Omega_1))$. (i) implies $\beta < \psi^{\delta}_{\Omega}$ by Le.4.2, and (ii) implies $(\xi \in C^{\Omega}(\delta, \psi^{\delta}_{\Omega})\ \&\ \xi < \delta\ \&\ \xi \in C^{\Omega}(\xi, \psi^{\xi}_{\Omega}))$ by the induction hypothesis and Le.4.2. So, we have the result. □)

By Claims 5 and 6, one can obtain $\eta \in \mathcal{T}(\Omega)$ with $\eta^* = \alpha$, and Claim 6 implies $\eta \in C^{\Omega}(\eta, \psi^{\eta}_{\Omega})$. Thus, we have the result. □

Theorem 4.4 $\psi^{\varepsilon_{\Omega+1}}_{\Omega} = \psi^{\varepsilon_{\Omega_1+1}}_I(\Omega_1)$, and $\psi^{\Omega}_{\Omega} = \psi^{\Omega_1}_I(\Omega_1)$.

Proof. By virtue of Le.4.2, Le.4.3, Le.2.9 and Le.3.11, the restriction of $*$ to $\psi^{\varepsilon_{\Omega+1}}_{\Omega}$ is the order-preserving bijective map from $\psi^{\varepsilon_{\Omega+1}}_{\Omega}$ to $\psi^{\varepsilon_{\Omega_1+1}}_I(\Omega_1)$, which means the results of this theorem. □

This theorem implies that the least proof-theoretically regular ordinal based on $\mathcal{T}(\Omega)$ is equal to that based on $\mathcal{T}(I)$. It also suggests the more general case: if one has two EORSs $\mathcal{T}(\kappa)$ and $\mathcal{T}(\pi)$ which are defined on suitable regular cardinals κ and π with $\kappa < \pi$, and

if one can define primitive recursive analogues of regular cardinals $\leq \kappa$ by using $\mathcal{T}(\kappa)$ and those $\leq \pi$ by using $\mathcal{T}(\pi)$, the former form an initial segment of the latter.

By employing the result by W.Pohlers in [Po98], one can also obtain the following:

Corollary 4.5 $\psi_\Omega^{\varepsilon_{\Omega+1}}$ is the proof theoretic ordinal of $\mathbf{KP}\omega$.

Such a result suggests a relationship between the least proof-theoretically regular ordinal based on $\mathcal{T}(\Omega)$ (= that based on $\mathcal{T}(I)$) and the least admissible ordinal $> \omega$ which can be characterized by $\mathbf{KP}\omega$ and the least regular cardinal, and a relationship between the least proof-theoretically inaccessible ordinal based on $\mathcal{T}(I)$ and the least recursively inaccessible ordinal which can be characterized by **KPi** and the least weakly inaccessible cardinal.

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