Scientiae Mathematicae Japonicae Online, **Vol. 7**, (2002), 61–65

ON $\mathcal{SL}(\mathcal{LG})$-TYPE SUBSEMIGROUPS OF THE FINITE FULL TRANSFORMATION SEMIGROUP

TATSUHIKO SAITO

Received January 24, 2002

Dedicated to Professor Masami Ito on his 60th birthday

ABSTRACT. A semilattice of left groups is called an $\mathcal{SL}(\mathcal{LG})$-type semigroup. For a finite set X, the semigroup $\mathcal{T}_X$ of all mappings from X into itself under composition of mappings is called the finite full transformation semigroup. In this paper, we determine all $\mathcal{SL}(\mathcal{LG})$-type subsemigroups, especially all maximal $\mathcal{SL}(\mathcal{LG})$-type subsemigroups, of $\mathcal{T}_X$

For a finite set X, let $\mathcal{T}_X$ be the full transformation semigroup on X, i.e., the semigroup of all mapping from X into itself under composition of mappings. The identity mapping on X is denoted by 1_X. The set of all subsemigroups of $\mathcal{T}_X$ is denoted by $S(\mathcal{T}_X)$. Throughout this paper, we write mapping symboles on the right and X denotes a finite set.

For $S \in S(\mathcal{T}_X)$, we define a relation ω_S on X by $x\ \omega_S\ y$ iff $x = y\alpha$ for some $\alpha \in S^1$, where $S^1 = S \cup \{1_X\}$. Then ω_S is reflexive and transitive. Define a relation σ_S on X by $x\ \sigma_S\ y$ iff $x\ \omega_S\ y$ and $y\ \omega_S\ x$. Then σ_S is an equivalence relation on X. The σ_S-class containing x is denoted by $x\sigma_S$. By defining a relation $\leq_S$ on X/σ_S by $x\sigma_S \leq_S y\sigma_S$ iff $x\ \omega_S\ y$, $(X/\sigma_S, \leq_S)$ forms an ordered set, whish is called the *characteristic ordered set of* S.

We investigate the relationship between S and its characeristic ordered set $(X/\sigma_S, \leq_S)$. For an example, S is a permutation group on X if and only if each $x\sigma_S$ is isolated in $(X/\sigma_S, \leq_S)$. In this case, σ_S is an orbit in X relative to S.

A *band* is a semigroup in which all elements are idempotent. A commutative band is called a *semilattice*. The class of semilattices is denoted by $\mathcal{SL}$. Let $\mathcal{V}$ be a class of semigroups. Then a semigroup S is called a $\mathcal{SL}(\mathcal{V})$*-type* if there exists $Y \in \mathcal{SL}$ and for each $\alpha \in Y$, there exists $S_\alpha \in V$ such that $S_\alpha S_\beta \subseteq S_{\alpha\beta}, S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$ and $S = \bigcup\{S_\alpha : \alpha \in Y\}$. A semigroup S is called a *left group* if for any $\alpha, \beta \in S$ there exists a unique $\gamma \in S$ such that $\gamma\alpha = \beta$. The class of left groups is denoted by $\mathcal{LG}$. Then a $\mathcal{SL}(\mathcal{LG})$-type semigroup is called a semilattice of left groups.

The purpose of this paper is to determine all $\mathcal{SL}(\mathcal{LG})$-type subsemigroups of $\mathcal{T}_X$, especially all maximal $\mathcal{SL}(\mathcal{LG})$-type subsemigroups of $\mathcal{T}_X$ from the point of view of the structure of their characteristic ordered sets.

For $\alpha \in \mathcal{T}_X$ and a subset Y of X, let $Y\alpha = \{y\alpha : y \in Y\}$. The *image* of α is defined as $X\alpha$ which is denoted by $\mathrm{im}\alpha$. If $Y\alpha \subseteq Y$, then the restriction $\alpha_{|Y}$ of α to Y can be defined. For $S \in S(\mathcal{T}_X)$, let $Im(S) = \{\mathrm{im}\alpha : \alpha \in S\}$. Them $Im(S)$ is a set of subsets of X.

The following facts are useful in this paper (see [4]):

Facts *Let* $S \in S(\mathcal{T}_X)$. *Then*

(1) *S is a left group if and only if* $\mathrm{im}\alpha = \mathrm{im}\beta$ *for every* $\alpha, \beta \in S$.

2000 *Mathematics Subject Classification.* 20M20.

Key words and phrases. characteristic ordered sets, $\mathcal{L}G(SL)$-type semigroups, o-ieals.

(2) $S \in \mathcal{SL}(\mathcal{LG})$ *if and only if* $\mathrm{im}\alpha\beta = \mathrm{im}\alpha \cap \mathrm{im}\beta$ *for every* $\alpha, \beta \in S$.

Let $(X, \leq)$ be an ordered set. The set of minimal elements in $(X, \leq)$ is denoted by $Min(X, \leq)$. A subset I of X is called an *o-ideal* in $(X, \leq)$ if I contains $Min(X, \leq)$ and I is convex, i.e., $x \in I$ and $y \leq x$ imply $y \in I$. The set of all o-ideals is denoted by $Id(X, \leq)$. Then $Id(X, \leq)$ forms a lattice ordered set under $\cap$ and $\cup$. For $x \in X$, the set of lower bounds is denoted by $lb(x)$, i.e., $lb(x) = \{y \in X : y \leq x\}$. Then $lb(x) \cup Min(X, \leq) \in Id(X, \leq)$ for every $x \in X$ which is called the principal o-ideal generated by x and is denoted by $\langle x \rangle$. If $(X, \leq)$ has the least element, then $lb(x) = \langle x \rangle$.

Let σ be an equivalence relation on X and let $(X/\sigma, \leq)$ be an ordered set. In what follows, S denotes $\mathcal{SL}(\mathcal{LG})$-type subsemigroups of $\mathcal{T}_X$ whose characteristic ordered sets are $(X/\sigma, \leq)$, i.e., $\sigma_S = \sigma$ and $\leq_S = \leq$.

For each $Y \in Im(S)$, let $S_Y = \{\alpha \in S : \mathrm{im}\alpha = Y\}$. Since $\mathrm{im}\alpha\beta = \mathrm{im}\alpha \cap \mathrm{im}\beta = Y$ for every $\alpha, \beta \in S_Y$, S_Y is a subsemigroup of S. From Fact (1), S_Y is a left group. For $Y, Z \in Im(S)$, let $\alpha \in S_Y$ and $\beta \in S_Z$. Since $im\alpha\beta = \mathrm{im}\alpha \cap\ im\beta = Y \cap Z$, we have $Y \cap Z \in Im(S)$. Thus $Im(S)$ is a $\cap$-semilattice, and $S_Y S_Z \subseteq S_{Y \cap Z}$.

Lemma 1. *Let* $x, y \in X$ *with* $y\sigma \leq x\sigma$ *and let* $Y \in Im(S)$. *If* $x \in Y$, *then* $y\sigma \subseteq Y$.

Proof. Let $\alpha \in S_Y$. Since $x \in Y = \mathrm{im}\alpha$, we have $z\alpha = x$ for some $z \in X$. Let $u \in y\sigma$. Since $u\sigma = y\sigma \leq x\sigma$, we have $x\beta = u$ for some $\beta \in S$, so that $u = x\beta = z\alpha\beta \in \mathrm{im}\alpha\beta \subseteq \mathrm{im}\alpha = Y$. Thus $y\sigma \subseteq Y$.

Lemma 2. *For every* $Y \in Im(S)$, $x \in Y$ *if and only if* $x\sigma \subseteq Y$. *Therefore* $Y = \bigcup\{x\sigma : x \in Y\}$.

The proof is straightforward from Lemma 1.

Lemma 3. For every $Y \in Im(S)$, let $I_Y = \{x\sigma : x \in Y\}$. Then I_Y is an o-ideal in $(X/\sigma, \leq)$.

Proof. If $x\sigma \in I_Y$ and $y\sigma \leq x\sigma$, then by Lemma 1 $y\sigma \in I_Y$, so that I_Y is convex. Let $x\sigma \in Min(X/\sigma, \leq)$ and let $\alpha \in S_Y$. Since $x\alpha\ \omega_S\ x$, we have $(x\alpha)\sigma \leq x\sigma$. From the minimality of $x\sigma$, we have $(x\alpha)\sigma = x\sigma$. Since $x\alpha \in Y$, by Lemma 2 we have $x\sigma = (x\alpha)\sigma \subseteq Y$, so that $x\sigma \in I_Y$. Thus $Min(X/\sigma. \leq) \subseteq I_Y$.

Theorem 1 *Let* σ *be an equivalence relation on* X *and let* $(X/\sigma, \leq)$ *be an ordered set. Suppose that* S *is a subsemigroup of* $\mathcal{T}_X$ *whose characteristic ordered set is* $(X/\sigma, \leq)$. *For each* $Y \in Im(S)$, *let* $S_Y = \{\alpha \in S : \mathrm{im}\alpha = Y\}$. *Then* S *is of* $\mathcal{SL}(\mathcal{LG})$*-type if and only if* S *sayisfies the following conditions*:

(1) $Y = \bigcup\{x\sigma : x \in Y\}$,

(2) $I = \{x\sigma : x \in Y\}$ *is an o-ideal in* $(X/\sigma, \leq)$,

(3) $(x\sigma)\alpha = x\sigma$ *if* $x\sigma \in I$, *otherwise* $(x\sigma)\alpha \subseteq lb(x\sigma) \cap I$ *for every* $\alpha \in S_Y$.

Proof. Suppose that S is an $\mathcal{SL}(\mathcal{LG})$-type. From lemmata 2 and 3, (1) and (2) follow.
(3) Let $\alpha \in S_Y$. Since $\mathrm{im}\alpha = \mathrm{im}\alpha^2 = Y$, we have $Y = X\alpha^2 = Y\alpha$, so that the restriction $\alpha_{|Y}$ of α to Y is a bijection for every $\alpha \in S_Y$. Thus $(S_Y)_{|Y} = \{\alpha_{|Y} : \alpha \in S\}$ is a permutation group, so that there exists $\beta \in S_Y$ such that $\beta_{|Y} = (\alpha_{|Y})^{-1}$. For $x \in Y$, let $y \in x\sigma$. Since $y \in Y$, we have $y\alpha\beta = y$, so that $y\ \omega_S\ y\alpha$. Clearly $y\alpha\ \omega_S\ y$. Thus $y\alpha\ \sigma\ y\ \sigma\ x$, so that $y\alpha \in x\sigma$, which shows $(x\sigma)\alpha \subseteq x\sigma$. Since $\alpha_{|Y}$ is a bijection, we have $(x\sigma)\alpha = x\sigma$.

Suppose that $x\sigma \notin I$. Let $y \in x\sigma$ and let $\alpha \in S_Y$ Since $y\alpha \in \mathrm{im}\alpha = Y$, by Lemma 2 $(y\alpha)\sigma \in I$. Since $y\alpha\ \omega_S\ y$, we have $(y\alpha)\sigma \leq y\sigma = x\sigma$, so that $(y\alpha)\sigma \in lb(x\sigma)$. Thus $(y\alpha)\sigma \in lb(x\sigma) \cap I$. Therefoere $y\alpha \in lb(x\sigma) \cap I$ for every $y \in x\sigma$. Consequently $(x\sigma)\alpha \subseteq lb(x\sigma) \cap I$.

Suppose that S satisfies the conditions (1), (2) and (3). Let $\alpha, \beta \in S$. Then $\alpha \in S_Y$ and $\beta \in S_Z$ for some $Y, Z \in Im(S)$. Let $x \in Y \cap Z$. By (2), $x\sigma \in I \cap J$, where $I = \{y\sigma : y \in Y\}$ and $J = \{z\sigma : z \in Z\}$. By (3) we have $x \in x\sigma = (x\sigma)\beta = (x\sigma)\alpha\beta$, so that $x \in \mathrm{im}\alpha\beta$. Thus $\mathrm{im}\alpha \cap \mathrm{im}\beta \subseteq \mathrm{im}\alpha\beta$. On the other hand, let $x \in \mathrm{im}\alpha\beta$. Then $x = z\alpha\beta$ for some $z \in X$, so that $x \ \omega_S \ z\alpha$. Thus $x\sigma \leq (z\alpha)\sigma \in I$. Since I is an o-ideal in $(X/\sigma, \leq)$, we have $x\sigma \in I$, so that $x \in x\sigma \subseteq Y = \mathrm{im}\alpha$. Clearly $x \in \mathrm{im}\beta$. Thus $\mathrm{im}\alpha\beta \subseteq \mathrm{im}\alpha \cap \mathrm{im}\beta$. Consequently $\mathrm{im}\alpha\beta = \mathrm{im}\alpha \cap \mathrm{im}\beta$. By Fact (2) S is a $\mathcal{SL}(\mathcal{LG})$-type.

By Theorem 1, if $S \in \mathcal{SL}(\mathcal{LG}) \cap S(\mathcal{T}_X)$, then, for every $Y \in Im(S)$, there exists $I \in Id(X/\sigma, \leq)$ such that $Y = \bigcup\{x\sigma : x\sigma \in I\}$, but the converse is not true. In fact, if S is a left group in $S(\mathcal{T}_X)$ which is not a group, then it is of $\mathcal{SL}(\mathcal{LG})$-type. In this case, $X/\sigma \in Id(X/\sigma, \leq)$ but $X/\sigma \notin Im(S)$. However, for every maximal $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of $\mathcal{T}_X$, the converse is also true.

Proposition 2. *Let S be an $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of $\mathcal{T}_X$. Assume that , for every $I \in Id(X/\sigma_S, \leq_S)$, there exists $Y \in Im(S)$ such that $Y = \bigcup\{x\sigma_S : x\sigma_S \in I\}$.Then $(X/\sigma_S, \leq_S$) has the least element if and only if $(X/\sigma_S, \leq_S) = (X/\sigma_T, \leq_T)$ for every $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of $\mathcal{T}_X$ with $S \subseteq T$.*

Proof. Suppose that $(X/\sigma_S. \leq_S)$ has the least element, and let T be an $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of $\mathcal{T}_X$ with $S \subseteq T$. Then clearly $\sigma_S \subseteq \sigma_T$ and $\leq_S \subseteq \leq_T$, and $lb_S(x\sigma_S) = \{z\sigma_S \in X/\sigma_S : z\sigma_S \leq_S x\sigma_S\}$ is an o-ideal in $(X/\sigma_S, \leq_S)$ for every $x \in X$. If $\sigma_S \neq \sigma_T$, then there exist $x, y \in X$ such that $x\sigma_T = y\sigma_T$ and $x\sigma_S \neq y\sigma_S$. Then $y\sigma_S \not\leq_S x\sigma_S$ or $x\sigma_S \not\leq_S y\sigma_S$. Without loss of generality, we may suppose $y\sigma_S \not\leq_S x\sigma_S$. Let $lb_S(x\sigma_S) = I$ and let $Y = \bigcup\{z\sigma \in X/\sigma_S : z\sigma_S \in I\}$. Then by the assumption $Y \in Im(S)$. In this case, $y\sigma_S \notin I$, so that $y \notin Y$. Let $\alpha \in S_Y$. Since $x \in Y = \mathrm{im}\alpha$, we have $x = u\alpha$ for some $u \in X$. Since $x\sigma_T = y\sigma_T$, we have $x\beta = y$ for some $\beta \in T$. Therefore we have $y = u\alpha\beta \in \mathrm{im}\alpha\beta \subseteq \mathrm{im}\alpha = Y$, a contradiction. Thus $\sigma_S = \sigma_T$.

Let $\sigma = \sigma_S$. Suppose that $\leq_S \neq \leq_T$. Then there exist $x\sigma, y\sigma \in X/\sigma$ such that $x\sigma \leq_T y\sigma$ and $x\sigma \not\leq_S y\sigma$. If $y\sigma \leq_S x\sigma$, then $y\sigma \leq_T x\sigma$, so that $x\sigma = y\sigma$, a contradiction. In case that $x\sigma$ and $y\sigma$ are incomparable in $(X/\sigma, \leq_S)$, let I and Y be as above. Then $y \notin Y$. The same argument as above leads to a contradiction. Thus $\leq_S = \leq_T$.

Suppose that $(X/\sigma_S, \leq_S)$ has at least two minimal elements. Let $x\sigma_S$ be a minimal element, and let $U = \{\alpha \in \mathcal{T}_X : (x\sigma_S)\alpha = x\sigma_S$ and $(y\sigma_S)\alpha \subseteq x\sigma_S$ if $y\sigma_S \neq x\sigma_S\}$. Let $T = S \cup U$. Then T is a subsemigroup of $\mathcal{T}_X$, since $SU \subseteq U$ and $US \subseteq U$. Clearly $\sigma_T = \sigma_S$ and $x\sigma_T$ is the least element in $(X/\sigma_T, \leq_T)$. It is easy to verify that T satisfies the conditions (1)-(3) in Theorem 1. Thus T is an $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of $\mathcal{T}_X$ with $S \subseteq T$. Let $z\sigma_S \in Min(X/\sigma_S, \leq_S)$ with $z\sigma_S \neq x\sigma_S$. Then $x\sigma_S$ and $z\sigma_S$ are incomparable in $(X/\sigma_S, \leq_S)$ but $x\sigma_S \leq_T y\sigma_S$. Thus $\leq_S \neq \leq_T$.

For an equivalence relation σ on X, let π_σ be the partition of X determined by σ, i.e., $\pi_\sigma = X/\sigma$.

Theorem 3. *Let σ be an equivalence relation on X and let $\pi_\sigma = \{Z_i : i \in \Lambda\}$. Suppose that $(\pi_\sigma, \leq)$ is an ordered set with the least element. For each $I \in Id(\pi_\sigma, \leq)$, let $S_I = \{\alpha \in \mathcal{T}_X : Z_i\alpha = Z_i$ if $Z_i \in I$, otherwise $Z_i\alpha \subseteq lb(Z_i) \cap I\}$. Then $S = \bigcup\{S_I : I \in Id(\pi_\sigma, \leq)\}$ is a maximal $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of $\mathcal{T}_X$. Conversely, every maximal $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of $\mathcal{T}_X$ can be constructed in this way.*

Proof. Recall that $lb(Z_i) = \langle Z_i \rangle \in Id(\pi_\sigma, \leq)$.

We first show that S_I is a left group for every $I \in Id(\pi_\sigma, \leq)$. Let $\alpha, \beta \in S_I$. If $Z_i \in I$, then $Z_i\alpha\beta = Z_i$. Suppose that $Z_i \notin I$. If $x \in Z_i$, then $x\alpha \in Z_j$ for some $Z_j \in \langle Z_i \rangle \cap I$, so that $(x\alpha)\beta \in Z_j$, since $Z_j \in I$. Thus $Z_i\alpha\beta \subseteq \langle Z_i \rangle \cap I$. Consequently $\alpha\beta \in S_I$, so that S_I is a subsemigroup of $\mathcal{T}_X$. Let $Y_I = \bigcup\{Z_i : Z_i \in I\}$. Then it is easy to see that $S_I = \{\alpha \in \mathcal{T}_X : \mathrm{im}\alpha = Y_I\}$. By Fact (1), S_I is a left group.

We next show that S is an $\mathcal{SL}(\mathcal{LG})$-type. Let $\alpha, \beta \in S$. Then $\alpha \in S_I$ and $\beta \in S_J$ for some $I, J \in Id(\pi_\sigma, \leq)$. If $Z_i \in I \cap J$, then $Z_i\alpha\beta = Z_i$. Suppose that $Z_i \notin I \cap J$. Then there are the following three cases:

Case 1. $Z_i \notin I$ and $Z_i \notin J$, Case 2. $Z_i \notin I$ and $Z_i \in J$ and

Case 3. $Z \in I$ and $Z_i \notin J$.

We show that $Z_i\alpha\beta \subseteq \langle Z_i\rangle \cap I \cap J$ in each case.

In Cases 1 and 2, let $x \in Z_i$ and let $x\alpha \in Z_j$ for some $Z_j \in \langle Z_i\rangle \cap I$.

Cace 1. If $Z_j \in J$, then $x\alpha\beta \in Z_j$, since $Z_j\beta = Z_j$, and clearly $Z_j \in \langle Z_i\rangle \cap I \cap J$. Thus $Z_i\alpha\beta \in \langle Z_i\rangle \cap I \cap J$. If $Z_j \notin J$, then $x\alpha\beta \in Z_k$ for some $Z_k \in \langle Z_j\rangle \cap J$. Since $Z_k \leq Z_j \in I$ and $Z_j \leq Z_i$, we have $\langle Z_k\rangle \in I$ and $\langle Z_j\rangle \subseteq \langle Z_i\rangle$, so that $Z_k \in \langle Z_j\rangle \cap I \cap J \subseteq \langle Z_i\rangle \cap I \cap J$. Thus $Z_i\alpha\beta \subseteq \langle Z_i\rangle \cap I \cap J$.

Case 2. Since $Z_j \leq Z_i \in J$, so that $Z_j \in J$. The proof is the same as the first part of Case 1.

Case 3. We have $Z_i\alpha\beta = Z_i\beta \in \langle Z_i\rangle \cap J = \langle Z_i\rangle \cap I \cap J$, since $\langle Z_i\rangle \in I$.

Since $I \cap J \in Id(\pi, \leq)$, we have $\alpha\beta \in S_{I\cap J}$, which shows that $S_I S_J \subseteq S_{I\cap J}$. Clearly $S_I \cap S_J = \emptyset$ if $I \neq J$, and $Id(\pi_\sigma, \leq)$ is a $\cap$-semilattice. Thus S is an $\mathcal{LG}(SL)$-type.

We last show that $\sigma_S = \sigma$ and $\leq_S = \leq$.

For every $I \in Id(\pi_\sigma, \leq)$, S_I consists of all elements α in $\mathcal{T}_X$ satisfying : if $Z_i \in I$, then $Z_i\alpha = Z_i$, otherwise $Z_j\alpha \subseteq \langle Z_j\rangle \cap I$.

Thus we obtain :

(a) for any bijective mapping $\phi : Z_i \to Z_i, x \mapsto x\phi$, there exists $\alpha \in S_I$ such that $x\alpha = x\phi$ if $Z_i \in I$, and

(b) for any mapping $\psi : Z_j \to \langle Z_j\rangle \cap I, x \mapsto x\psi$, there exists $\alpha \in S_I$ such that $x\alpha = x\psi$ if $Z_j \notin I$.

Therefore, if $Z_i \in I$, then the restriction $(S_I)_{|Z_i}$ of S_I to Z_i is a symmetric group on Z_i and the restriction of $(S_I)_{|Y_I}$ to $Y_I = \bigcup\{Z_i : Z_i \in I\}$ is isomorphic to the direct product of symmetric groups $\{(S_I)_{|Z_i} : Z_i \in I\}$.

From (a), we have that, for any $x, y \in Z_i$, there exists α, β such that $x\alpha = y$ and $y\beta = x$, so that $\sigma \subseteq \sigma_S$. On the other hand, let $x \in Z_i$ and let $y \in x\sigma_S$. Then $x\alpha = y$ and $y\alpha = x$ for some $\alpha, \beta \in S$. Let $\alpha \in S_I$ and $\beta \in S_J$ for some $I, J \in (\pi, \leq)$. Since $x\alpha\beta = x$, we have $Z_i\alpha\beta = Z_i$, and since $\alpha\beta, \alpha\beta\alpha \in S_{I\cap J}$, we have $Z_i\alpha = Z_i\alpha\beta\alpha = Z_i$, so that $y = x\alpha \in Z_i$. Thus $x\sigma_S \subseteq x\sigma$.

From (b), we have that, if $Z_j \leq Z_i$, then for any $x \in Z_i$ and $y \in Z_j$, there exists $\alpha \in S$ such that $x\alpha = y$, so that $\leq \subseteq \leq_S$. If $y\sigma \leq_S x\sigma$, then $x\alpha = y$ for some $\alpha \in S$, Let $x\sigma = Z_i$ and $y\sigma = Z_j$. Since $y \in Z_i\alpha \subseteq \langle Z_i\rangle$, we have $y\sigma = Z_j \in \langle Z_i\rangle$, so that $Z_j \leq Z_i$. Thus $\leq_S = \leq$.

Let T be an $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of T_X with $S \subseteq T$. By Proposition 2, we have $\sigma_T = \sigma$ and $\leq_T = \leq$. Clearly $Im(S) \subseteq Im(T)$ and we have $|Im(T)| \leq |Id(X/\sigma_T, \leq_T)| = |Id(\pi, \leq)| = |Im(S)|$, where $|X|$ denotes the cardinality of the set X. Thus $Im(S) = Im(T)$, so that $T = \bigcup\{T_Y : Y \in Im(S)\}$. Since each S_I consists of all elements in $\mathcal{T}_X$ satisfying the condition (3) of Theorem 1, we have $T_Y \subseteq S_I$ if $Y = \bigcup\{Z_i : Z_i \in I\}$, so that $T \subseteq S$. Thus $S = T$ which shows that S is a maximal $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of $\mathcal{T}_X$.

Let T be a maximal $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of $\mathcal{T}_X$ and let π be the partition of X determined by σ_T. i.e., $\pi = X/\sigma_T$. Then by Proposition 2 $(\pi, \leq_T)$ has the least element. Let S be the $\mathcal{SL}(\mathcal{LG})$-type subsemigroup of $\mathcal{T}_X$ constructed from $Id(X/\sigma_T, \leq_T)$ as in the former half of this theorem. Then clearly $T \subseteq S$. Thus $T = S$.

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Mukunoura 374, Innoshima
Hiroshima, 722-2321, Japan
e-mail; tatsusaito@mx4.tiki.ne.jp